EXERCISES 12.3

Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1 and 2 solve the heat equation (1) subject to the given conditions. Assume a rod of length L.

1. $u(0, t) = 0, \quad u(L, t) = 0$

$$u(x, 0) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

2. $u(0, t) = 0, \quad u(L, t) = 0$
$u(x, 0) = x(L - x)$

3. Find the temperature $u(x, t)$ in a rod of length L if the initial temperature is $f(x)$ throughout and if the ends $x = 0$ and $x = L$ are insulated.

4. Solve Problem 3 if $L = 2$ and

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

5. Suppose heat is lost from the lateral surface of a thin rod of length L into a surrounding medium at temperature zero. If the linear law of heat transfer applies, then the heat equation takes on the form

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t},$$

$0 < x < L$, $t > 0$, h a constant. Find the temperature $u(x, t)$ if the initial temperature is $f(x)$ throughout and the ends $x = 0$ and $x = L$ are insulated. See Figure 12.3.3.

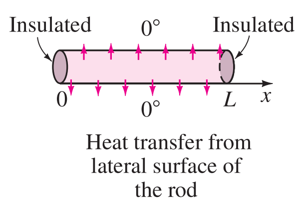


FIGURE 12.3.3 Rod losing heat in Problem 5

6. Solve Problem 5 if the ends $x = 0$ and $x = L$ are held at temperature zero.

Discussion Problems

7. Figure 12.3.2(b) shows the graphs of $u(x, t)$ for $0 \le t \le 6$ for $x = 0$, $x = \pi/12$, $x = \pi/6$, $x = \pi/4$, and $x = \pi/2$. Describe or sketch the graphs of $u(x, t)$ on the same time interval but for the fixed values $x = 3\pi/4$, $x = 5\pi/6$, $x = 11\pi/12$, and $x = \pi$.

8. Find the solution of the boundary-value problem given in (1)–(3) when $f(x) = 10\sin(5\pi x/L)$.

Computer Lab Assignments

9. **(a)** Solve the heat equation (1) subject to

$$u(0, t) = 0, \quad u(100, t) = 0, \quad t > 0$$

$$u(x, 0) = \begin{cases} 0.8x, & 0 \le x \le 50 \\ 0.8(100 - x), & 50 < x \le 100. \end{cases}$$

(b) Use the 3D-plot application of your CAS to graph the partial sum $S_5(x, t)$ consisting of the first five nonzero terms of the solution in part (a) for $0 \le x \le 100$, $0 \le t \le 200$. Assume that $k = 1.6352$. Experiment with various three-dimensional viewing perspectives of the surface (called the **ViewPoint** option in *Mathematica*).

12.4 WAVE EQUATION

REVIEW MATERIAL

- Reread pages 439–441 of Section 12.2.

INTRODUCTION We are now in a position to solve the boundary-value problem (11) that was discussed in Section 12.2. The vertical displacement $u(x, t)$ of the vibrating string of length L shown in Figure 12.2.2(a) is determined from

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < L, \quad t > 0 \tag{1}$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \tag{2}$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L. \tag{3}$$

SOLUTION OF THE BVP With the usual assumption that $u(x, t) = X(x)T(t)$, separating variables in (1) gives

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

so that

$$X'' + \lambda X = 0 \tag{4}$$

$$T'' + a^2 \lambda T = 0. \tag{5}$$

As in the preceding section, the boundary conditions (2) translate into $X(0) = 0$ and $X(L) = 0$. Equation (4) along with these boundary conditions is the regular Sturm-Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \tag{6}$$

Of the usual three possibilities for the parameter, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, only the last choice leads to nontrivial solutions. Corresponding to $\lambda = \alpha^2$, $\alpha > 0$, the general solution of (4) is

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

$X(0) = 0$ and $X(L) = 0$ indicate that $c_1 = 0$ and $c_2 \sin \alpha L = 0$. The last equation again implies that $\alpha L = n\pi$ or $\alpha = n\pi/L$. The eigenvalues and corresponding eigenfunctions of (6) are $\lambda_n = n^2\pi^2/L^2$ and $X(x) = c_2 \sin \frac{n\pi}{L} x$, $n = 1, 2, 3, \ldots$. The general solution of the second-order equation (5) is then

$$T(t) = c_3 \cos \frac{n\pi a}{L} t + c_4 \sin \frac{n\pi a}{L} t.$$

By rewriting c_2c_3 as A_n and c_2c_4 as B_n, solutions that satisfy both the wave equation (1) and boundary conditions (2) are

$$u_n = \left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x \tag{7}$$

and

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x. \tag{8}$$

Setting $t = 0$ in (8) and using the initial condition $u(x, 0) = f(x)$ gives

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x.$$

Since the last series is a half-range expansion for f in a sine series, we can write $A_n = b_n$:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx. \tag{9}$$

To determine B_n, we differentiate (8) with respect to t and then set $t = 0$:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi a}{L} \sin \frac{n\pi a}{L} t + B_n \frac{n\pi a}{L} \cos \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty} \left(B_n \frac{n\pi a}{L}\right) \sin \frac{n\pi}{L} x.$$

For this last series to be the half-range sine expansion of the initial velocity g on the interval, the *total* coefficient $B_n n\pi a/L$ must be given by the form b_n in (5) of Section 11.3, that is,

$$B_n \frac{n\pi a}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

from which we obtain

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx. \tag{10}$$

The solution of the boundary-value problem (1)–(3) consists of the series (8) with coefficients A_n and B_n defined by (9) and (10), respectively.

We note that when the string is released from *rest,* then $g(x) = 0$ for every x in the interval $[0, L]$, and consequently, $B_n = 0$.

PLUCKED STRING A special case of the boundary-value problem in (1)–(3) is the model of the **plucked string.** We can see the motion of the string by plotting the solution or displacement $u(x, t)$ for increasing values of time t and using the animation feature of a CAS. Some frames of a "movie" generated in this manner are given in Figure 12.4.1; the initial shape of the string is given in Figure 12.4.1(a). You are asked to emulate the results given in the figure plotting a sequence of partial sums of (8). See Problems 7 and 22 in Exercises 12.4.

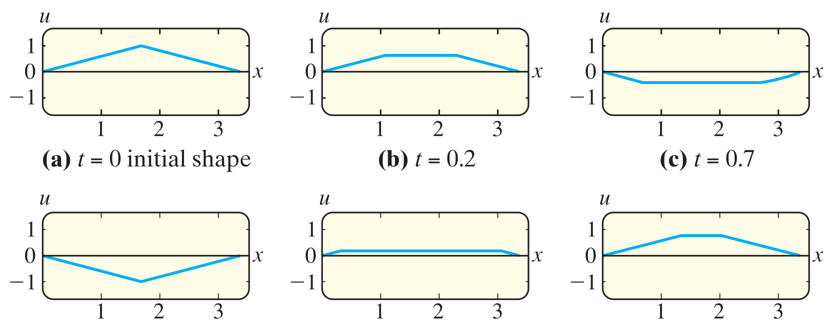


FIGURE 12.4.1 Frames of a CAS "movie"

STANDING WAVES Recall from the derivation of the one-dimensional wave equation in Section 12.2 that the constant a appearing in the solution of the boundary-value problem in (1), (2), and (3) is given by $\sqrt{T/\rho}$, where ρ is mass per unit length and T is the magnitude of the tension in the string. When T is large enough, the vibrating string produces a musical sound. This sound is the result of standing waves. The solution (8) is a superposition of product solutions called **standing waves** or **normal modes:**

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots.$$

In view of (6) and (7) of Section 5.1 the product solutions (7) can be written as

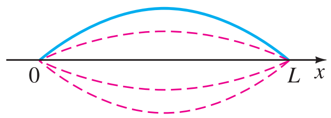
$$u_n(x, t) = C_n \sin\left(\frac{n\pi a}{L}t + \phi_n\right)\sin\frac{n\pi}{L}x, \tag{11}$$

where $C_n = \sqrt{A_n^2 + B_n^2}$ and ϕ_n is defined by $\sin\phi_n = A_n/C_n$ and $\cos\phi_n = B_n/C_n$. For $n = 1, 2, 3, \ldots$ the standing waves are essentially the graphs of $\sin(n\pi x/L)$, with a time-varying amplitude given by

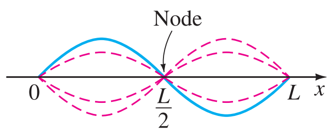
$$C_n \sin\left(\frac{n\pi a}{L}t + \phi_n\right).$$

Alternatively, we see from (11) that at a fixed value of x each product function $u_n(x, t)$ represents simple harmonic motion with amplitude $C_n|\sin(n\pi x/L)|$ and frequency $f_n = na/2L$. In other words, each point on a standing wave vibrates with a different amplitude but with the same frequency. When $n = 1$,

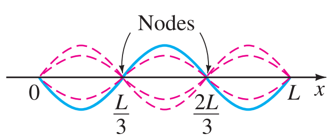
$$u_1(x, t) = C_1 \sin\left(\frac{\pi a}{L}t + \phi_1\right)\sin\frac{\pi}{L}x$$



(a) First standing wave



(b) Second standing wave



(c) Third standing wave

FIGURE 12.4.2 First three standing waves

is called the **first standing wave,** the **first normal mode,** or the **fundamental mode of vibration.** The first three standing waves, or normal modes, are shown in Figure 12.4.2. The dashed graphs represent the standing waves at various values of time. The points in the interval $(0, L)$, for which $\sin(n\pi/L)x = 0$, correspond to points on a standing wave where there is no motion. These points are called **nodes.** For example, in Figures 12.4.2(b) and 12.4.2(c) we see that the second standing wave has one node at $L/2$ and the third standing wave has two nodes at $L/3$ and $2L/3$. In general, the nth normal mode of vibration has $n - 1$ nodes.

The frequency

$$f_1 = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

of the first normal mode is called the **fundamental frequency** or **first harmonic** and is directly related to the pitch produced by a stringed instrument. It is apparent that the greater the tension on the string, the higher the pitch of the sound. The frequencies f_n of the other normal modes, which are integer multiples of the fundamental frequency, are called **overtones.** The second harmonic is the first overtone, and so on.

EXERCISES 12.4

Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1–8 solve the wave equation (1) subject to the given conditions.

1. $u(0, t) = 0, \quad u(L, t) = 0$

$u(x, 0) = \frac{1}{4}x(L - x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

2. $u(0, t) = 0, \quad u(L, t) = 0$

$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = x(L - x)$

3. $u(0, t) = 0, \quad u(L, t) = 0$

$u(x, 0)$, given in Figure 12.4.3, $\quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

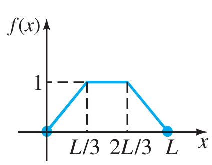


FIGURE 12.4.3 Initial displacement in Problem 3

4. $u(0, t) = 0, \quad u(\pi, t) = 0$

$u(x, 0) = \frac{1}{6}x(\pi^2 - x^2), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

5. $u(0, t) = 0, \quad u(\pi, t) = 0$

$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin x$

6. $u(0, t) = 0, \quad u(1, t) = 0$

$u(x, 0) = 0.01 \sin 3\pi x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

7. $u(0, t) = 0, \quad u(L, t) = 0$

$$u(x, 0) = \begin{cases} \dfrac{2hx}{L}, & 0 < x < \dfrac{L}{2} \\ 2h\left(1 - \dfrac{x}{L}\right), & \dfrac{L}{2} \le x < L \end{cases}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

8. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

$u(x, 0) = x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

This problem could describe the longitudinal displacement $u(x, t)$ of a vibrating elastic bar. The boundary conditions at $x = 0$ and $x = L$ are called **free-end conditions.** See Figure 12.4.4.

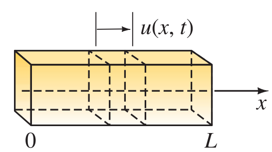


FIGURE 12.4.4 Vibrating elastic bar in Problem 8

9. A string is stretched and secured on the x-axis at $x = 0$ and $x = \pi$ for $t > 0$. If the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity, then the wave equation takes on the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta\frac{\partial u}{\partial t}, \qquad 0 < \beta < 1, \quad t > 0.$$

Find the displacement $u(x, t)$ if the string starts from rest from the initial displacement $f(x)$.

10. Show that a solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + u, \qquad 0 < x < \pi, \quad t > 0$$

$$u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0$$

$$u(x, 0) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \le x < \pi \end{cases}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$$

is

$$u(x, t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin(2k-1)x \cos \sqrt{(2k-1)^2 + 1}\, t.$$

11. The transverse displacement $u(x, t)$ of a vibrating beam of length L is determined from a fourth-order partial differential equation

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \qquad 0 < x < L, \quad t > 0.$$

If the beam is **simply supported,** as shown in Figure 12.4.5, the boundary and initial conditions are

$$u(0, t) = 0, \qquad u(L, t) = 0, \quad t > 0$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0, \quad t > 0$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L.$$

Solve for $u(x, t)$. [*Hint*: For convenience use $\lambda = \alpha^4$ when separating variables.]

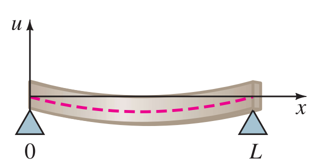


FIGURE 12.4.5 Simply supported beam in Problem 11

12. If the ends of the beam in Problem 11 are **embedded** at $x = 0$ and $x = L$, the boundary conditions become, for $t > 0$,

$$u(0, t) = 0, \quad u(L, t) = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

(a) Show that the eigenvalues of the problem are $\lambda_n = x_n^2/L^2$, where x_n, $n = 1, 2, 3, \ldots,$ are the positive roots of the equation

$$\cosh x \cos x = 1.$$

(b) Show graphically that the equation in part (a) has an infinite number of roots.

(c) Use a calculator or a CAS to find approximations to the first four eigenvalues. Use four decimal places.

13. Consider the boundary-value problem given in (1), (2), and (3) of this section. If $g(x) = 0$ for $0 < x < L$, show that the solution of the problem can be written as

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)].$$

[*Hint*: Use the identity
$2 \sin \theta_1 \cos \theta_2 = \sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)$.]

14. The vertical displacement $u(x, t)$ of an infinitely long string is determined from the initial-value problem

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x). \tag{12}$$

This problem can be solved without separating variables.

(a) Show that the wave equation can be put into the form $\partial^2 u/\partial\eta\partial\xi = 0$ by means of the substitutions $\xi = x + at$ and $\eta = x - at$.

(b) Integrate the partial differential equation in part (a), first with respect to η and then with respect to ξ, to show that $u(x, t) = F(x + at) + G(x - at)$, where F and G are arbitrary twice differentiable functions, is a solution of the wave equation. Use this solution and the given initial conditions to show that

$$F(x) = \frac{1}{2} f(x) + \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds + c$$

and $$G(x) = \frac{1}{2} f(x) - \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds - c,$$

where x_0 is arbitrary and c is a constant of integration.

(c) Use the results in part (b) to show that

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds. \tag{13}$$

Note that when the initial velocity $g(x) = 0$, we obtain

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)], \qquad -\infty < x < \infty.$$

This last solution can be interpreted as a superposition of two **traveling waves,** one moving to the right (that is, $\frac{1}{2} f(x - at)$) and one moving to the left ($\frac{1}{2}f(x + at)$). Both waves travel with speed a and have the same basic shape as the initial displacement $f(x)$. The form of $u(x, t)$ given in (13) is called **d'Alembert's solution.**

In Problems 15–18 use d'Alembert's solution (13) to solve the initial-value problem in Problem 14 subject to the given initial conditions.

15. $f(x) = \sin x, \quad g(x) = 1$

16. $f(x) = \sin x, \quad g(x) = \cos x$

17. $f(x) = 0, \quad g(x) = \sin 2x$

18. $f(x) = e^{-x^2}, \quad g(x) = 0$

Computer Lab Assignments

19. (a) Use a CAS to plot d'Alembert's solution in Problem 18 on the interval $[-5, 5]$ at the times $t = 0$, $t = 1$, $t = 2$, $t = 3$, and $t = 4$. Superimpose the graphs on one coordinate system. Assume that $a = 1$.

(b) Use the 3D-plot application of your CAS to plot d'Alembert's solution $u(x, t)$ in Problem 18 for $-5 \le x \le 5$, $0 \le t \le 4$. Experiment with various three-dimensional viewing perspectives of this surface. Choose the perspective of the surface for which you feel the graphs in part (a) are most apparent.

20. A model for an infinitely long string that is initially held at the three points $(-1, 0)$, $(1, 0)$, and $(0, 1)$ and then simultaneously released at all three points at time $t = 0$ is given by (12) with

$$f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \quad \text{and} \quad g(x) = 0.$$

(a) Plot the initial position of the string on the interval $[-6, 6]$.

(b) Use a CAS to plot d'Alembert's solution (13) on $[-6, 6]$ for $t = 0.2k$, $k = 0, 1, 2, \ldots, 25$. Assume that $a = 1$.

(c) Use the animation feature of your computer algebra system to make a movie of the solution. Describe the motion of the string over time.

21. An infinitely long string coinciding with the x-axis is struck at the origin with a hammer whose head is 0.2 inch in diameter. A model for the motion of the string is given by (12) with

$$f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} 1, & |x| \le 0.1 \\ 0, & |x| \ge 0.1. \end{cases}$$

(a) Use a CAS to plot d'Alembert's solution (13) on $[-6, 6]$ for $t = 0.2k$, $k = 0, 1, 2, \ldots, 25$. Assume that $a = 1$.

(b) Use the animation feature of your computer algebra system to make a movie of the solution. Describe the motion of the string over time.

22. The model of the vibrating string in Problem 7 is called the **plucked string.** The string is tied to the x-axis at $x = 0$ and $x = L$ and is held at $x = L/2$ at h units above the x-axis. See Figure 12.2.4. Starting at $t = 0$ the string is released from rest.

(a) Use a CAS to plot the partial sum $S_6(x, t)$—that is, the first six nonzero terms of your solution—for $t = 0.1k$, $k = 0, 1, 2, \ldots, 20$. Assume that $a = 1$, $h = 1$, and $L = \pi$.

(b) Use the animation feature of your computer algebra system to make a movie of the solution to Problem 7.

12.5 LAPLACE'S EQUATION

REVIEW MATERIAL

- Reread page 438 of Section 12.2 and Example 1 in Section 11.4.

INTRODUCTION Suppose we wish to find the steady-state temperature $u(x, y)$ in a rectangular plate whose vertical edges $x = 0$ and $x = a$ are insulated, as shown in Figure 12.5.1. When no heat escapes from the lateral faces of the plate, we solve the following boundary-value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b \tag{1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0, \quad 0 < y < b \tag{2}$$

$$u(x, 0) = 0, \quad u(x, b) = f(x), \quad 0 < x < a. \tag{3}$$