20. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x\,\partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0$

21. $\dfrac{\partial^2 u}{\partial x^2} = 9\dfrac{\partial^2 u}{\partial x\,\partial y}$

22. $\dfrac{\partial^2 u}{\partial x\,\partial y} - \dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} = 0$

23. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} - 6\dfrac{\partial u}{\partial y} = 0$

24. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u$

25. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

26. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k > 0$

In Problems 27 and 28 show that the given partial differential equation possesses the indicated product solution.

27. $k\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) = \dfrac{\partial u}{\partial t};$
$u = e^{-k\alpha^2 t}\big(c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)\big)$

28. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0;$
$u = (c_1 \cos \alpha\theta + c_2 \sin \alpha\theta)(c_3 r^{\alpha} + c_4 r^{-\alpha})$

29. Verify that each of the products $u = XY$ in (3), (4), and (5) satisfies the second-order PDE in Example 1.

30. Definition 12.1.1 generalizes to linear PDEs with coefficients that are functions of x and y. Determine the regions in the xy-plane for which the equation

$$(xy + 1)\frac{\partial^2 u}{\partial x^2} + (x + 2y)\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} + xy^2 u = 0$$

is hyperbolic, parabolic, or elliptic.

12.2 CLASSICAL PDEs AND BOUNDARY-VALUE PROBLEMS

REVIEW MATERIAL

- Reread the material on boundary-value problems in Sections 4.1, 4.3, and 5.2.

INTRODUCTION We are not going to solve anything in this section. We are simply going to discuss the types of partial differential equations and boundary-value problems that we will be working with in the remainder of this chapter as well as in Chapters 13–15. The words *boundary-value problem* have a slightly different connotation than they did in Sections 4.1, 4.3, and 5.2. If, say, $u(x, t)$ is a solution of a PDE, where x represents a spatial dimension and t represents time, then we may be able to prescribe the value of u, or $\partial u/\partial x$, or a linear combination of u and $\partial u/\partial x$ at a specified x as well as to prescribe u and $\partial u/\partial t$ at a given time t (usually, $t = 0$). In other words, a "boundary-value problem" may consist of a PDE, along with boundary conditions *and* initial conditions.

CLASSICAL EQUATIONS We shall be concerned principally with applying the method of separation of variables to find product solutions of the following classical equations of mathematical physics:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad k > 0 \tag{1}$$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

or slight variations of these equations. The PDEs (1), (2), and (3) are known, respectively, as the **one-dimensional heat equation,** the **one-dimensional wave equation,** and the **two-dimensional form of Laplace's equation.** "One-dimensional" in the case of equations (1) and (2) refers to the fact that x denotes a spatial variable, whereas t represents time; "two-dimensional" in (3) means that x and y are both spatial variables. If you compare (1)–(3) with the linear form in Theorem 12.1.1 (with t playing the part of the symbol y), observe that the heat equation (1) is parabolic, the wave equation (2) is hyperbolic, and Laplace's equation is elliptic. This observation will be important in Chapter 15.

HEAT EQUATION Equation (1) occurs in the theory of heat flow—that is, heat transferred by conduction in a rod or in a thin wire. The function $u(x, t)$ represents temperature at a point x along the rod at some time t. Problems in mechanical vibrations often lead to the wave equation (2). For purposes of discussion, a solution $u(x, t)$ of (2) will represent the displacement of an idealized string. Finally, a solution $u(x, y)$ of Laplace's equation (3) can be interpreted as the steady-state (that is, time-independent) temperature distribution throughout a thin, two-dimensional plate.

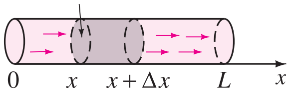


FIGURE 12.2.1 One-dimensional flow of heat

Even though we have to make many simplifying assumptions, it is worthwhile to see how equations such as (1) and (2) arise.

Suppose a thin circular rod of length L has a cross-sectional area A and coincides with the x-axis on the interval $[0, L]$. See Figure 12.2.1. Let us suppose the following:

- The flow of heat within the rod takes place only in the x-direction.
- The lateral, or curved, surface of the rod is insulated; that is, no heat escapes from this surface.
- No heat is being generated within the rod.
- The rod is homogeneous; that is, its mass per unit volume ρ is a constant.
- The specific heat γ and thermal conductivity K of the material of the rod are constants.

To derive the partial differential equation satisfied by the temperature $u(x, t)$, we need two empirical laws of heat conduction:

(*i*) *The quantity of heat Q in an element of mass m is*

$$Q = \gamma m u, \tag{4}$$

where u is the temperature of the element.

(*ii*) *The rate of heat flow Q_t through the cross-section indicated in Figure 12.2.1 is proportional to the area A of the cross section and the partial derivative with respect to x of the temperature:*

$$Q_t = -KAu_x. \tag{5}$$

Since heat flows in the direction of decreasing temperature, the minus sign in (5) is used to ensure that Q_t is positive for $u_x < 0$ (heat flow to the right) and negative for $u_x > 0$ (heat flow to the left). If the circular slice of the rod shown in Figure 12.2.1 between x and $x + \Delta x$ is very thin, then $u(x, t)$ can be taken as the approximate temperature at each point in the interval. Now the mass of the slice is $m = \rho(A\,\Delta x)$, and so it follows from (4) that the quantity of heat in it is

$$Q = \gamma\rho A\,\Delta x\,u. \tag{6}$$

Furthermore, when heat flows in the positive x-direction, we see from (5) that heat builds up in the slice at the net rate

$$-KAu_x(x, t) - [-KAu_x(x + \Delta x, t)] = KA\,[u_x(x + \Delta x, t) - u_x(x, t)]. \tag{7}$$

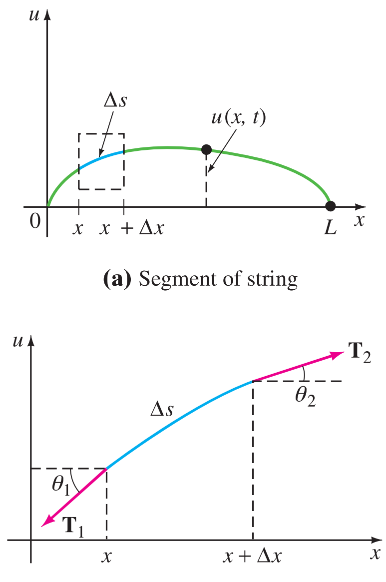
By differentiating (6) with respect to t, we see that this net rate is also given by

$$Q_t = \gamma\rho A\,\Delta x\,u_t. \tag{8}$$

Equating (7) and (8) gives

$$\frac{K}{\gamma\rho}\frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} = u_t. \tag{9}$$

Finally, by taking the limit of (9) as $\Delta x \to 0$, we obtain (1) in the form* $(K/\gamma\rho)u_{xx} = u_t$. It is customary to let $k = K/\gamma\rho$ and call this positive constant the **thermal diffusivity.**



(a) Segment of string

(b) Enlargement of segment

FIGURE 12.2.2 Flexible string anchored at $x = 0$ and $x = L$

WAVE EQUATION Consider a string of length L, such as a guitar string, stretched taut between two points on the x-axis—say, $x = 0$ and $x = L$. When the string starts to vibrate, assume that the motion takes place in the xu-plane in such a manner that each point on the string moves in a direction perpendicular to the x-axis (transverse vibrations). As is shown in Figure 12.2.2(a), let $u(x, t)$ denote the vertical displacement of any point on the string measured from the x-axis for $t > 0$. We further assume the following:

- The string is perfectly flexible.
- The string is homogeneous; that is, its mass per unit length ρ is a constant.
- The displacements u are small in comparison to the length of the string.
- The slope of the curve is small at all points.
- The tension $\mathbf{T}$ acts tangent to the string, and its magnitude T is the same at all points.
- The tension is large compared with the force of gravity.
- No other external forces act on the string.

Now in Figure 12.2.2(b) the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ are tangent to the ends of the curve on the interval $[x, x + \Delta x]$. For small θ_1 and θ_2 the net vertical force acting on the corresponding element Δs of the string is then

$$\begin{aligned} T\sin\theta_2 - T\sin\theta_1 &\approx T\tan\theta_2 - T\tan\theta_1 \\ &= T\,[u_x(x + \Delta x, t) - u_x(x, t)],^{\dagger} \end{aligned}$$

where $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Now $\rho\,\Delta s \approx \rho\,\Delta x$ is the mass of the string on $[x, x + \Delta x]$, so Newton's second law gives

$$T[u_x(x + \Delta x, t) - u_x(x, t)] = \rho\,\Delta x\,u_{tt}$$

or

$$\frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} = \frac{\rho}{T}u_{tt}.$$

If the limit is taken as $\Delta x \to 0$, the last equation becomes $u_{xx} = (\rho/T)u_{tt}$. This of course is (2) with $a^2 = T/\rho$.

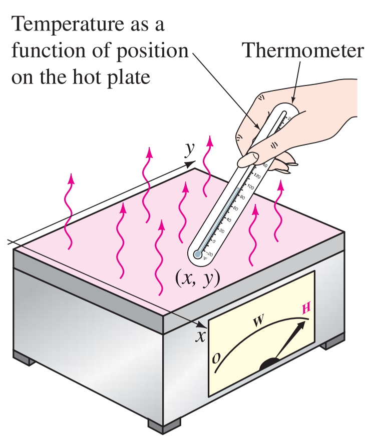


FIGURE 12.2.3 Steady-state temperatures in a rectangular plate

LAPLACE'S EQUATION Although we shall not present its derivation, Laplace's equation in two and three dimensions occurs in time-independent problems involving potentials such as electrostatic, gravitational, and velocity in fluid mechanics. Moreover, a solution of Laplace's equation can also be interpreted as a steady-state temperature distribution. As illustrated in Figure 12.2.3, a solution $u(x, y)$ of (3) could represent the temperature that varies from point to point—but not with time—of a rectangular plate. Laplace's equation in two dimensions and in three dimensions is abbreviated as $\nabla^2 u = 0$, where

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{and} \qquad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

are called the two-dimensional **Laplacian** and the three-dimensional Laplacian, respectively, of a function u.

*The definition of the second partial derivative is $u_{xx} = \lim_{\Delta x \to 0} \dfrac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x}$.

†$\tan\theta_2 = u_x(x + \Delta x, t)$ and $\tan\theta_1 = u_x(x, t)$ are equivalent expressions for slope.

We often wish to find solutions of equations (1), (2), and (3) that satisfy certain side conditions.

INITIAL CONDITIONS Since solutions of (1) and (2) depend on time t, we can prescribe what happens at $t = 0$; that is, we can give **initial conditions (IC).** If $f(x)$ denotes the initial temperature distribution throughout the rod in Figure 12.2.1, then a solution $u(x, t)$ of (1) must satisfy the single initial condition $u(x, 0) = f(x)$, $0 < x < L$. On the other hand, for a vibrating string we can specify its initial displacement (or shape) $f(x)$ as well as its initial velocity $g(x)$. In mathematical terms we seek a function $u(x, t)$ that satisfies (2) and the two initial conditions:

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \qquad 0 < x < L. \tag{10}$$

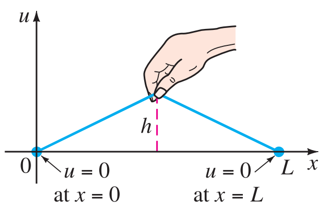


FIGURE 12.2.4 Plucked string

For example, the string could be plucked, as shown in Figure 12.2.4, and released from rest ($g(x) = 0$).

BOUNDARY CONDITIONS The string in Figure 12.2.4 is secured to the x-axis at $x = 0$ and $x = L$ for all time. We interpret this by the two **boundary conditions (BC):**

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad t > 0.$$

Note that in this context the function f in (10) is continuous, and consequently, $f(0) = 0$ and $f(L) = 0$. In general, there are three types of boundary conditions associated with equations (1), (2), and (3). On a boundary we can specify the values of *one* of the following:

$$(i) \quad u, \qquad (ii) \quad \frac{\partial u}{\partial n}, \qquad \text{or} \qquad (iii) \quad \frac{\partial u}{\partial n} + hu, \qquad h \text{ a constant.}$$

Here $\partial u/\partial n$ denotes the normal derivative of u (the directional derivative of u in the direction perpendicular to the boundary). A boundary condition of the first type (i) is called a **Dirichlet condition;** a boundary condition of the second type (ii) is called a **Neumann condition;** and a boundary condition of the third type (iii) is known as a **Robin condition.** For example, for $t > 0$ a typical condition at the right-hand end of the rod in Figure 12.2.1 can be

$$(i)' \quad u(L, t) = u_0, \qquad u_0 \text{ a constant,}$$

$$(ii)' \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \qquad \text{or}$$

$$(iii)' \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = -h(u(L, t) - u_m), \qquad h > 0 \text{ and } u_m \text{ constants.}$$

Condition $(i)'$ simply states that the boundary $x = L$ is held by some means at a constant *temperature* u_0 for all time $t > 0$. Condition $(ii)'$ indicates that the boundary $x = L$ is *insulated*. From the empirical law of heat transfer, the flux of heat across a boundary (that is, the amount of heat per unit area per unit time conducted across the boundary) is proportional to the value of the normal derivative $\partial u/\partial n$ of the temperature u. Thus when the boundary $x = L$ is thermally insulated, no heat flows into or out of the rod, so

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

We can interpret $(iii)'$ to mean that *heat is lost* from the right-hand end of the rod by being in contact with a medium, such as air or water, that is held at a constant temperature. From Newton's law of cooling, the outward flux of heat from the rod is proportional to the difference between the temperature $u(L, t)$ at the boundary and the temperature u_m of the surrounding medium. We note that if heat is lost from the left-hand end of the rod, the boundary condition is

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = h(u(0, t) - u_m).$$

The change in algebraic sign is consistent with the assumption that the rod is at a higher temperature than the medium surrounding the ends so that $u(0, t) > u_m$ and $u(L, t) > u_m$. At $x = 0$ and $x = L$ the slopes $u_x(0, t)$ and $u_x(L, t)$ must be positive and negative, respectively.

Of course, at the ends of the rod we can specify different conditions at the same time. For example, we could have

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad \text{and} \qquad u(L, t) = u_0, \qquad t > 0.$$

We note that the boundary condition in $(i)'$ is homogeneous if $u_0 = 0$; if $u_0 \neq 0$, the boundary condition is nonhomogeneous. The boundary condition $(ii)'$ is homogeneous; $(iii)'$ is homogeneous if $u_m = 0$ and nonhomogeneous if $u_m \neq 0$.

BOUNDARY-VALUE PROBLEMS Problems such as

$$\begin{aligned}
&\textit{Solve:} && a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < L, \quad t > 0 \\
&\textit{Subject to:} && \text{(BC)} \quad u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \\
& && \text{(IC)} \quad u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\textit{Solve:} && \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b \\
&\textit{Subject to:} && \text{(BC)} \begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=a} = 0, & 0 < y < b \\ u(x, 0) = 0, \quad u(x, b) = f(x), & 0 < x < a \end{cases}
\end{aligned} \tag{12}$$

are called **boundary-value problems.**

MODIFICATIONS The partial differential equations (1), (2), and (3) must be modified to take into consideration internal or external influences acting on the physical system. More general forms of the one-dimensional heat and wave equations are, respectively,

$$k\frac{\partial^2 u}{\partial x^2} + G(x, t, u, u_x) = \frac{\partial u}{\partial t} \tag{13}$$

and

$$a^2\frac{\partial^2 u}{\partial x^2} + F(x, t, u, u_t) = \frac{\partial^2 u}{\partial t^2}. \tag{14}$$

For example, if there is heat transfer from the lateral surface of a rod into a surrounding medium that is held at a constant temperature u_m, then the heat equation (13) is

$$k\frac{\partial^2 u}{\partial x^2} - h(u - u_m) = \frac{\partial u}{\partial t}.$$

In (14) the function F could represent the various forces acting on the string. For example, when external, damping, and elastic restoring forces are taken into account,

(14) assumes the form

$$a^2 \frac{\partial^2 u}{\partial x^2} + \underbrace{f(x, t)}_{\text{External force}} = \frac{\partial^2 u}{\partial t^2} + \underbrace{c\,\frac{\partial u}{\partial t}}_{\text{Damping force}} + \underbrace{ku}_{\text{Restoring force}} \tag{15}$$

REMARKS

The analysis of a wide variety of diverse phenomena yields mathematical models (1), (2), or (3) or their generalizations involving a greater number of spatial variables. For example, (1) is sometimes called the **diffusion equation,** since the diffusion of dissolved substances in solution is analogous to the flow of heat in a solid. The function $u(x, t)$ satisfying the partial differential equation in this case represents the concentration of the dissolved substance. Similarly, equation (2) arises in the study of the flow of electricity in a long cable or transmission line. In this setting (2) is known as the **telegraph equation.** It can be shown that under certain assumptions the current and the voltage in the line are functions satisfying two equations identical with (2). The wave equation (2) also appears in the theory of high-frequency transmission lines, fluid mechanics, acoustics, and elasticity. Laplace's equation (3) is encountered in the static displacement of membranes.

EXERCISES 12.2

Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1–4 a rod of length L coincides with the interval $[0, L]$ on the x-axis. Set up the boundary-value problem for the temperature $u(x, t)$.

1. The left end is held at temperature zero, and the right end is insulated. The initial temperature is $f(x)$ throughout.

2. The left end is held at temperature u_0, and the right end is held at temperature u_1. The initial temperature is zero throughout.

3. The left end is held at temperature 100, and there is heat transfer from the right end into the surrounding medium at temperature zero. The initial temperature is $f(x)$ throughout.

4. The ends are insulated, and there is heat transfer from the lateral surface into the surrounding medium at temperature 50. The initial temperature is 100 throughout.

In Problems 5–8 a string of length L coincides with the interval $[0, L]$ on the x-axis. Set up the boundary-value problem for the displacement $u(x, t)$.

5. The ends are secured to the x-axis. The string is released from rest from the initial displacement $x(L - x)$.

6. The ends are secured to the x-axis. Initially, the string is undisplaced but has the initial velocity $\sin(\pi x/L)$.

7. The left end is secured to the x-axis, but the right end moves in a transverse manner according to $\sin \pi t$. The string is released from rest from the initial displacement $f(x)$. For $t > 0$ the transverse vibrations are damped with a force proportional to the instantaneous velocity.

8. The ends are secured to the x-axis, and the string is initially at rest on that axis. An external vertical force proportional to the horizontal distance from the left end acts on the string for $t > 0$.

In Problems 9 and 10 set up the boundary-value problem for the steady-state temperature $u(x, y)$.

9. A thin rectangular plate coincides with the region defined by $0 \le x \le 4$, $0 \le y \le 2$. The left end and the bottom of the plate are insulated. The top of the plate is held at temperature zero, and the right end of the plate is held at temperature $f(y)$.

10. A semi-infinite plate coincides with the region defined by $0 \le x \le \pi$, $y \ge 0$. The left end is held at temperature e^{-y}, and the right end is held at temperature 100 for $0 < y \le 1$ and temperature zero for $y > 1$. The bottom of the plate is held at temperature $f(x)$.