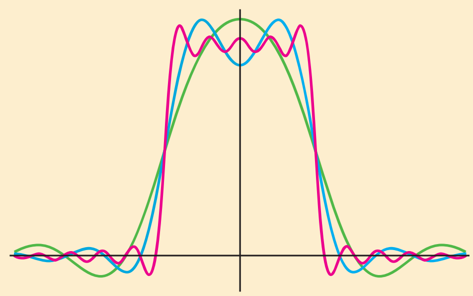
14 INTEGRAL TRANSFORMS



The method of separation of variables is a powerful but not universally applicable method for solving boundary-value problems. If the partial differential equation is nonhomogeneous, if the boundary conditions are time dependent, or if the domain of the spatial variable is an infinite interval $(-\infty, \infty)$ or a semi-infinite interval (a, ∞), we may be able to use an integral transform to solve the problem. In Section 14.2 we will solve problems that involve the heat and wave equations by means of the familiar Laplace transform. In Section 14.4 we introduce three new integral transforms—the Fourier transforms.

14.1 ERROR FUNCTION

REVIEW MATERIAL

- See (14) and Example 7 in Section 2.3.

INTRODUCTION There are many functions in mathematics that are defined in terms of an integral. For example, in many traditional calculus texts the natural logarithm is defined in the following manner: $\ln x = \int_1^x dt/t$, $x > 0$. In earlier chapters we saw, albeit briefly, the error function erf(x), the complementary error function erfc(x), the sine integral function Si(x), the Fresnel sine integral $S(x)$, and the gamma function $\Gamma(\alpha)$; all of these functions are defined by means of an integral. Before applying the Laplace transform to boundary-value problems, we need to know a little more about the error function and the complementary error function. In this section we examine the graphs and a few of the more obvious properties of erf(x) and erfc(x).

PROPERTIES AND GRAPHS The definitions of the **error function** erf(x) and **complementary error function** erfc(x) are, respectively,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du \quad \text{and} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\,du. \tag{1}$$

With the aid of polar coordinates it can be demonstrated that

$$\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2} \quad \text{or} \quad \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2}\,du = 1.$$

Thus from the additive interval property of definite integrals, $\int_0^\infty = \int_0^x + \int_x^\infty$, the last result can be written as

$$\frac{2}{\sqrt{\pi}} \left[\int_0^x e^{-u^2}\,du + \int_x^\infty e^{-u^2}\,du \right] = 1.$$

This shows that erf(x) and erfc(x) are related by the identity

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1. \tag{2}$$

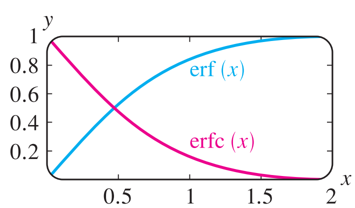


FIGURE 14.1.1 Graphs of erf(x) and erfc(x) for $x \geq 0$

The graphs of erf(x) and erfc(x) for $x \geq 0$ are given in Figure 14.1.1. Note that $\operatorname{erf}(0) = 0$, $\operatorname{erfc}(0) = 1$ and that $\operatorname{erf}(x) \to 1$, $\operatorname{erfc}(x) \to 0$ as $x \to \infty$. Other numerical values of erf(x) and erfc(x) can be obtained from a CAS or tables. In tables the error function is often referred to as the **probability integral.** The domain of erf(x) and of erfc(x) is $(-\infty, \infty)$. In Problem 11 in Exercises 14.1 you are asked to obtain the graph of each function on this interval and to deduce a few additional properties.

Table 14.1, of Laplace transforms, will be useful in the exercises in the next section. The proofs of these results are complicated and will not be given.

TABLE 14.1 Laplace Transforms

$f(t)$, $a > 0$	$\mathscr{L}\{f(t)\} = F(s)$	$f(t)$, $a > 0$	$\mathscr{L}\{f(t)\} = F(s)$
1. $\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$	**4.** $2\sqrt{\dfrac{t}{\pi}}\, e^{-a^2/4t} - a \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
2. $\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$	**5.** $e^{ab}e^{b^2 t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
3. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$	**6.** $-e^{ab}e^{b^2 t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$

EXERCISES 14.1

Answers to selected odd-numbered problems begin on page ANS-23.

1. (a) Show that $\operatorname{erf}(\sqrt{t}) = \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\,d\tau$.

(b) Use the convolution theorem and the results of Problems 41 and 42 in Exercises 7.1 to show that

$$\mathscr{L}\{\operatorname{erf}(\sqrt{t})\} = \frac{1}{s\sqrt{s+1}}.$$

2. Use the result of Problem 1 to show that

$$\mathscr{L}\{\operatorname{erfc}(\sqrt{t})\} = \frac{1}{s}\left[1 - \frac{1}{\sqrt{s+1}}\right].$$

3. Use the result of Problem 1 to show that

$$\mathscr{L}\{e^t \operatorname{erf}(\sqrt{t})\} = \frac{1}{\sqrt{s}\,(s-1)}.$$

4. Use the result of Problem 2 to show that

$$\mathscr{L}\{e^t \operatorname{erfc}(\sqrt{t})\} = \frac{1}{\sqrt{s}\,(\sqrt{s}+1)}.$$

5. Let C, G, R, and x be constants. Use Table 14.1 to show that

$$\mathscr{L}^{-1}\left\{\frac{C}{Cs+G}\left(1 - e^{-x\sqrt{RCs+RG}}\right)\right\} = e^{-Gt/C}\operatorname{erf}\left(\frac{x}{2}\sqrt{\frac{RC}{t}}\right).$$

6. Let a be a constant. Show that

$$\mathscr{L}^{-1}\left\{\frac{\sinh a\sqrt{s}}{s\sinh\sqrt{s}}\right\} = \sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+1+a}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2n+1-a}{2\sqrt{t}}\right)\right].$$

[*Hint*: Use the exponential definition of the hyperbolic sine. Expand $1/(1-e^{-2\sqrt{s}})$ in a geometric series.]

7. Use the Laplace transform and Table 14.1 to solve the integral equation

$$y(t) = 1 - \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau.$$

8. Use the third and fifth entries in Table 14.1 to derive the sixth entry.

9. Show that $\displaystyle\int_a^b e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}[\operatorname{erf}(b) - \operatorname{erf}(a)]$.

10. Show that $\displaystyle\int_{-a}^{a} e^{-u^2}\,du = \sqrt{\pi}\operatorname{erf}(a)$.

Computer Lab Assignments

11. The functions $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$ are defined for $x < 0$. Use a CAS to superimpose the graphs of $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$ on the same axes for $-10 \le x \le 10$. Do the graphs possess any symmetry? What are $\lim_{x\to-\infty}\operatorname{erf}(x)$ and $\lim_{x\to-\infty}\operatorname{erfc}(x)$?

14.2 LAPLACE TRANSFORM

REVIEW MATERIAL

- Linear second-order initial-value problems (Sections 4.3 and 4.4)
- Operational properties of the Laplace Transform (Sections 7.2–7.4)

INTRODUCTION The Laplace transform of a function $f(t)$, $t \ge 0$, is defined to be $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ whenever the improper integral converges. This integral transforms the function $f(t)$ into a function F of the transform parameter s, that is, $\mathscr{L}\{f(t)\} = F(s)$. Similar to Chapter 7, where the Laplace transform was used mainly to solve linear ordinary differential equations, in this section we use the Laplace transform to solve linear partial differential equations. But in contrast to Chapter 7, where the Laplace transform reduced a linear ODE with constant coefficients to an algebraic equation, in this section we see that a linear PDE with constant coefficients is transformed into an ODE.

TRANSFORM OF A FUNCTION OF TWO VARIABLES The boundary-value problems that we consider in this section will involve either the one-dimensional wave and heat equations or slight variations of these equations. These PDEs involve an unknown function of two independent variables $u(x, t)$, where the variable t

represents time $t \geq 0$. The Laplace transform of the function $u(x, t)$ with respect to t is defined by

$$\mathscr{L}\{u(x, t)\} = \int_0^\infty e^{-st}\, u(x, t)\, dt,$$

where x is treated as a parameter. We continue the convention of using capital letters to denote the Laplace transform of a function by writing

$$\mathscr{L}\{u(x, t)\} = U(x, s).$$

TRANSFORM OF PARTIAL DERIVATIVES The transforms of the partial derivatives $\partial u/\partial t$ and $\partial^2 u/\partial t^2$ follow analogously from (6) and (7) of Section 7.2:

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x, s) - u(x, 0), \tag{1}$$

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2U(x, s) - su(x, 0) - u_t(x, 0). \tag{2}$$

Because we are transforming with respect to t, we further suppose that it is legitimate to interchange integration and differentiation in the transform of $\partial^2 u/\partial x^2$:

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty e^{-st}\frac{\partial^2 u}{\partial x^2}\,dt = \int_0^\infty \frac{\partial^2}{\partial x^2}[e^{-st}\, u(x, t)]\, dt = \frac{d^2}{dx^2}\int_0^\infty e^{-st}\, u(x, t)\, dt = \frac{d^2}{dx^2}\mathscr{L}\{u(x, t)\};$$

that is,

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2U}{dx^2}. \tag{3}$$

In view of (1) and (2) we see that the Laplace transform is suited to problems with initial conditions—namely, those problems associated with the heat equation or the wave equation.

EXAMPLE 1 Laplace Transform of a PDE

Find the Laplace transform of the wave equation $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $t > 0$.

SOLUTION From (2) and (3),

$$\mathscr{L}\left\{a^2\frac{\partial^2 u}{\partial x^2}\right\} = \mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

becomes

$$a^2\frac{d^2}{dx^2}\mathscr{L}\{u(x, t)\} = s^2\mathscr{L}\{u(x, t)\} - su(x, 0) - u_t(x, 0)$$

or

$$a^2\frac{d^2U}{dx^2} - s^2U = -su(x, 0) - u_t(x, 0). \tag{4}$$

■

The Laplace transform with respect to t of either the wave equation or the heat equation eliminates that variable, and for the one-dimensional equations the transformed equations are then *ordinary differential equations* in the spatial variable x. In solving a transformed equation, we treat s as a parameter.

EXAMPLE 2 Using the Laplace Transform to Solve a BVP

Solve $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0$$

subject to $$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin \pi x, \quad 0 < x < 1.$$

SOLUTION The partial differential equation is recognized as the wave equation with $a = 1$. From (4) and the given initial conditions the transformed equation is

$$\frac{d^2U}{dx^2} - s^2U = -\sin \pi x, \tag{5}$$

where $U(x, s) = \mathscr{L}\{u(x, t)\}$. Since the boundary conditions are functions of t, we must also find their Laplace transforms:

$$\mathscr{L}\{u(0, t)\} = U(0, s) = 0 \qquad \text{and} \qquad \mathscr{L}\{u(1, t)\} = U(1, s) = 0. \tag{6}$$

The results in (6) are boundary conditions for the ordinary differential equation (5). Since (5) is defined over a finite interval, its complementary function is

$$U_c(x, s) = c_1 \cosh sx + c_2 \sinh sx.$$

The method of undetermined coefficients yields a particular solution

$$U_p(x, s) = \frac{1}{s^2 + \pi^2} \sin \pi x.$$

Hence $$U(x, s) = c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2 + \pi^2} \sin \pi x.$$

But the conditions $U(0, s) = 0$ and $U(1, s) = 0$ yield, in turn, $c_1 = 0$ and $c_2 = 0$. We conclude that

$$U(x, s) = \frac{1}{s^2 + \pi^2} \sin \pi x$$

$$u(x, t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2 + \pi^2} \sin \pi x\right\} = \frac{1}{\pi} \sin \pi x \, \mathscr{L}^{-1}\left\{\frac{\pi}{s^2 + \pi^2}\right\}.$$

Therefore $$u(x, t) = \frac{1}{\pi} \sin \pi x \sin \pi t.$$ ■

EXAMPLE 3 Using the Laplace Transform to Solve a BVP

A very long string is initially at rest on the nonnegative x-axis. The string is secured at $x = 0$, and its distant right end slides down a frictionless vertical support. The string is set in motion by letting it fall under its own weight. Find the displacement $u(x, t)$.

SOLUTION Since the force of gravity is taken into consideration, it can be shown that the wave equation has the form

$$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \quad t > 0.$$

Here g represents the constant acceleration due to gravity. The boundary and initial conditions are, respectively,

$$u(0, t) = 0, \quad \lim_{x \to \infty} \frac{\partial u}{\partial x} = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x > 0.$$

The second boundary condition, $\lim_{x \to \infty} \partial u/\partial x = 0$, indicates that the string is horizontal at a great distance from the left end. Now from (2) and (3),

$$\mathscr{L}\left\{a^2 \frac{\partial^2 u}{\partial x^2}\right\} - \mathscr{L}\{g\} = \mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

becomes

$$a^2 \frac{d^2 U}{dx^2} - \frac{g}{s} = s^2 U - su(x, 0) - u_t(x, 0)$$

or, in view of the initial conditions,

$$\frac{d^2 U}{dx^2} - \frac{s^2}{a^2} U = \frac{g}{a^2 s}.$$

The transforms of the boundary conditions are

$$\mathscr{L}\{u(0, t)\} = U(0, s) = 0 \qquad \text{and} \qquad \mathscr{L}\left\{\lim_{x \to \infty} \frac{\partial u}{\partial x}\right\} = \lim_{x \to \infty} \frac{dU}{dx} = 0.$$

With the aid of undetermined coefficients, the general solution of the transformed equation is found to be

$$U(x, s) = c_1 e^{-(x/a)s} + c_2 e^{(x/a)s} - \frac{g}{s^3}.$$

The boundary condition $\lim_{x \to \infty} dU/dx = 0$ implies that $c_2 = 0$, and $U(0, s) = 0$ gives $c_1 = g/s^3$. Therefore

$$U(x, s) = \frac{g}{s^3} e^{-(x/a)s} - \frac{g}{s^3}.$$

Now by the second translation theorem we have

$$u(x, t) = \mathscr{L}^{-1}\left\{\frac{g}{s^3} e^{-(x/a)s} - \frac{g}{s^3}\right\} = \frac{1}{2} g \left(t - \frac{x}{a}\right)^2 \mathscr{U}\left(t - \frac{x}{a}\right) - \frac{1}{2} gt^2$$

or

$$u(x, t) = \begin{cases} -\dfrac{1}{2} gt^2, & 0 \le t < \dfrac{x}{a} \\[2ex] -\dfrac{g}{2a^2}(2axt - x^2), & t \ge \dfrac{x}{a}. \end{cases}$$

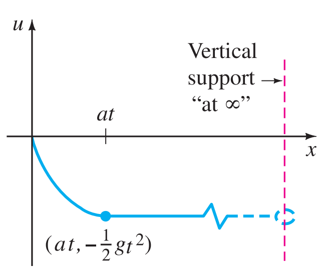


FIGURE 14.2.1 "Infinitely long" string falling under its own weight

To interpret the solution, let us suppose that $t > 0$ is fixed. For $0 \le x \le at$ the string is the shape of a parabola passing through $(0, 0)$ and $(at, -\frac{1}{2}gt^2)$. For $x > at$ the string is described by the horizontal line $u = -\frac{1}{2}gt^2$. See Figure 14.2.1. ■

Observe that the problem in the next example could be solved by the procedure in Section 12.6. The Laplace transform provides an alternative solution.

EXAMPLE 4 A Solution in Terms of erf(x)

Solve the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1, \quad t > 0$$

subject to $u(0, t) = 0, \quad u(1, t) = u_0, \quad t > 0$

$u(x, 0) = 0, \quad 0 < x < 1.$

SOLUTION From (1) and (3) and the given initial condition,

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \mathscr{L}\left\{\frac{\partial u}{\partial t}\right\}$$

becomes

$$\frac{d^2U}{dx^2} - sU = 0. \tag{7}$$

The transforms of the boundary conditions are

$$U(0, s) = 0 \quad \text{and} \quad U(1, s) = \frac{u_0}{s}. \tag{8}$$

Since we are concerned with a finite interval on the x-axis, we choose to write the general solution of (7) as

$$U(x, s) = c_1 \cosh(\sqrt{s}x) + c_2 \sinh(\sqrt{s}x).$$

Applying the two boundary conditions in (8) yields $c_1 = 0$ and $c_2 = u_0/(s \sinh\sqrt{s})$, respectively. Thus

$$U(x, s) = u_0 \frac{\sinh(\sqrt{s}x)}{s \sinh\sqrt{s}}.$$

Now the inverse transform of the latter function cannot be found in most tables. However, by writing

$$\frac{\sinh(\sqrt{s}x)}{s\sinh\sqrt{s}} = \frac{e^{\sqrt{s}x} - e^{-\sqrt{s}x}}{s(e^{\sqrt{s}} - e^{-\sqrt{s}})} = \frac{e^{(x-1)\sqrt{s}} - e^{-(x+1)\sqrt{s}}}{s(1 - e^{-2\sqrt{s}})}$$

and using the geometric series

$$\frac{1}{1 - e^{-2\sqrt{s}}} = \sum_{n=0}^{\infty} e^{-2n\sqrt{s}}$$

we find

$$\frac{\sinh(\sqrt{s}x)}{s\sinh\sqrt{s}} = \sum_{n=0}^{\infty}\left[\frac{e^{-(2n+1-x)\sqrt{s}}}{s} - \frac{e^{-(2n+1+x)\sqrt{s}}}{s}\right].$$

If we assume that the inverse Laplace transform can be done term by term, it follows from entry 3 of Table 14.1 that

$$\begin{aligned} u(x, t) &= u_0 \mathscr{L}^{-1}\left\{\frac{\sinh(\sqrt{s}x)}{s\sinh\sqrt{s}}\right\} \\ &= u_0 \sum_{n=0}^{\infty}\left[\mathscr{L}^{-1}\left\{\frac{e^{-(2n+1-x)\sqrt{s}}}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{e^{-(2n+1+x)\sqrt{s}}}{s}\right\}\right] \\ &= u_0 \sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{t}}\right)\right]. \end{aligned} \tag{9}$$

The solution (9) can be rewritten in terms of the error function using $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$:

$$u(x, t) = u_0 \sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+1+x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{t}}\right)\right]. \tag{10}$$ ■

Figure 14.2.2(a), obtained with the aid of the 3D-plot application in a CAS, shows the surface over the rectangular region $0 \le x \le 1$, $0 \le t \le 6$, defined by the partial sum $S_{10}(x, t)$ of the solution (10) with $u_0 = 100$. It is apparent from the surface and the accompanying two-dimensional graphs that at a fixed value of x (the curve of intersection of a plane slicing the surface perpendicular to the x-axis on the interval [0, 1] the temperature $u(x, t)$ increases rapidly to a constant value as time increases. See Figures 14.2.2(b) and 14.2.2(c). For a fixed time (the curve of intersection of a plane slicing the surface perpendicular to the t-axis) the temperature $u(x, t)$ naturally increases from 0 to 100. See Figures 14.2.2(d) and 14.2.2(e).

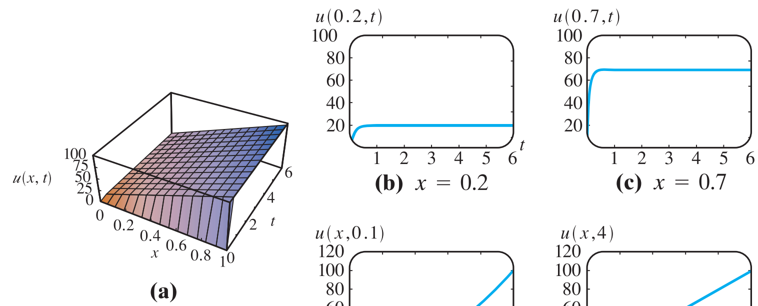


FIGURE 14.2.2 Graph of solution given in (10). In (b) and (c) x is held constant. In (d) and (e) t is held constant

EXERCISES 14.2

Answers to selected odd-numbered problems begin on page ANS-23.

1. A string is stretched along the x-axis between (0, 0) and (L, 0). Find the displacement $u(x, t)$ if the string starts from rest in the initial position $A \sin(\pi x/L)$.

2. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0, \quad u(1, t) = 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 2 \sin \pi x + 4 \sin 3\pi x.$$

3. The displacement of a semi-infinite elastic string is determined from

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \quad t > 0$$

$$u(0, t) = f(t), \quad \lim_{x\to\infty} u(x, t) = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x > 0.$$

Solve for $u(x, t)$.

4. Solve the boundary-value problem in Problem 3 when

$$f(t) = \begin{cases} \sin \pi t, & 0 \le t \le 1 \\ 0, & t > 1. \end{cases}$$

Sketch the displacement $u(x, t)$ for $t > 1$.

5. In Example 3 find the displacement $u(x, t)$ when the left end of the string at $x = 0$ is given an oscillatory motion described by $f(t) = A \sin \omega t$.

6. The displacement $u(x, t)$ of a string that is driven by an external force is determined from

$$\frac{\partial^2 u}{\partial x^2} + \sin \pi x \sin \omega t = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < 1.$$

Solve for $u(x, t)$.

7. A uniform bar is clamped at $x = 0$ and is initially at rest. If a constant force F_0 is applied to the free end at $x = L$, the longitudinal displacement $u(x, t)$ of a cross section of the bar is determined from

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < L, \quad t > 0$$

$$u(0, t) = 0, \quad E \left.\frac{\partial u}{\partial x}\right|_{x=L} = F_0, \quad E \text{ a constant}, \quad t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L.$$

Solve for $u(x, t)$. [*Hint*: Expand $1/(1 + e^{-2sL/a})$ in a geometric series.]

8. A uniform semi-infinite elastic beam moving along the x-axis with a constant velocity $-v_0$ is brought to a stop by hitting a wall at time $t = 0$. See Figure 14.2.3. The longitudinal displacement $u(x, t)$ is determined from

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \quad t > 0$$

$$u(0, t) = 0, \quad \lim_{x \to \infty} \frac{\partial u}{\partial x} = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = -v_0, \quad x > 0.$$

Solve for $u(x, t)$.

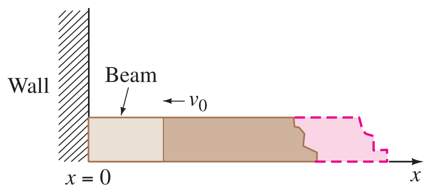


FIGURE 14.2.3 Moving elastic beam in Problem 8

9. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \quad t > 0$$

$$u(0, t) = 0, \quad \lim_{x \to \infty} u(x, t) = 0, \quad t > 0$$

$$u(x, 0) = xe^{-x}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x > 0.$$

10. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \quad t > 0$$

$$u(0, t) = 1, \quad \lim_{x \to \infty} u(x, t) = 0, \quad t > 0$$

$$u(x, 0) = e^{-x}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x > 0.$$

In Problems 11–18 use the Laplace transform to solve the heat equation $u_{xx} = u_t$, $x > 0$, $t > 0$, subject to the given conditions.

11. $u(0, t) = u_0, \quad \lim_{x \to \infty} u(x, t) = u_1, \quad u(x, 0) = u_1$

12. $u(0, t) = u_0, \quad \lim_{x \to \infty} \dfrac{u(x, t)}{x} = u_1, \quad u(x, 0) = u_1 x$

13. $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = u(0, t), \quad \lim_{x \to \infty} u(x, t) = u_0, \quad u(x, 0) = u_0$

14. $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = u(0, t) - 50, \quad \lim_{x \to \infty} u(x, t) = 0, \quad u(x, 0) = 0$

15. $u(0, t) = f(t), \quad \lim_{x \to \infty} u(x, t) = 0, \quad u(x, 0) = 0$

[*Hint*: Use the convolution theorem.]

16. $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = -f(t), \quad \lim_{x \to \infty} u(x, t) = 0, \quad u(x, 0) = 0$

17. $u(0, t) = 60 + 40\,\mathscr{U}(t - 2), \quad \lim_{x \to \infty} u(x, t) = 60,$

$u(x, 0) = 60$

18. $u(0, t) = \begin{cases} 20, & 0 < t < 1 \\ 0, & t \geq 1 \end{cases}, \quad \lim_{x \to \infty} u(x, t) = 100,$

$u(x, 0) = 100$

19. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad -\infty < x < 1, \quad t > 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = 100 - u(1, t), \quad \lim_{x \to -\infty} u(x, t) = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad -\infty < x < 1.$$

20. Show that a solution of the boundary-value problem

$$k \frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}, \qquad x > 0, \quad t > 0$$

$$u(0, t) = 0, \quad \lim_{x \to \infty} \frac{\partial u}{\partial x} = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad x > 0,$$

where r is a constant, is given by

$$u(x, t) = rt - r\int_0^t \operatorname{erfc}\left(\frac{x}{2\sqrt{k\tau}}\right) d\tau.$$

21. A rod of length L is held at a constant temperature u_0 at its ends $x = 0$ and $x = L$. If the rod's initial temperature is $u_0 + u_0 \sin(x\pi/L)$, solve the heat equation $u_{xx} = u_t$, $0 < x < L$, $t > 0$ for the temperature $u(x, t)$.

22. If there is a heat transfer from the lateral surface of a thin wire of length L into a medium at constant temperature u_m, then the heat equation takes on the form

$$k \frac{\partial^2 u}{\partial x^2} - h(u - u_m) = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \quad t > 0,$$

where h is a constant. Find the temperature $u(x, t)$ if the initial temperature is a constant u_0 throughout and the ends $x = 0$ and $x = L$ are insulated.

23. A rod of unit length is insulated at $x = 0$ and is kept at temperature zero at $x = 1$. If the initial temperature of the rod is a constant u_0, solve $ku_{xx} = u_t$, $0 < x < 1$, $t > 0$ for the temperature $u(x, t)$. [*Hint*: Expand $1/(1 + e^{-2\sqrt{s/k}})$ in a geometric series.]

24. An infinite porous slab of unit width is immersed in a solution of constant concentration c_0. A dissolved substance in the solution diffuses into the slab. The concentration $c(x, t)$ in the slab is determined from

$$D\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}, \qquad 0 < x < 1, \quad t > 0$$

$$c(0, t) = c_0, \quad c(1, t) = c_0, \qquad t > 0$$

$$c(x, 0) = 0, \quad 0 < x < 1,$$

where D is a constant. Solve for $c(x, t)$.

25. A very long telephone transmission line is initially at a constant potential u_0. If the line is grounded at $x = 0$ and insulated at the distant right end, then the potential $u(x, t)$ at a point x along the line at time t is determined from

$$\frac{\partial^2 u}{\partial x^2} - RC\frac{\partial u}{\partial t} - RGu = 0, \qquad x > 0, \quad t > 0$$

$$u(0, t) = 0, \qquad \lim_{x\to\infty}\frac{\partial u}{\partial x} = 0, \qquad t > 0$$

$$u(x, 0) = u_0, \quad x > 0,$$

where R, C, and G are constants known as resistance, capacitance, and conductance, respectively. Solve for $u(x, t)$. [*Hint*: See Problem 5 in Exercises 14.1.]

26. Show that a solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \qquad x > 0, \quad t > 0, \quad h \text{ constant}$$

$$u(0, t) = u_0, \qquad \lim_{x\to\infty} u(x, t) = 0, \qquad t > 0$$

$$u(x, 0) = 0, \qquad x > 0$$

is $$u(x, t) = \frac{u_0 x}{2\sqrt{\pi}}\int_0^t \frac{e^{-h\tau - x^2/4\tau}}{\tau^{3/2}}\,d\tau.$$

27. Starting at $t = 0$, a concentrated load of magnitude F_0 moves with a constant velocity v_0 along a semi-infinite string. In this case the wave equation becomes

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + F_0\delta\left(t - \frac{x}{v_0}\right),$$

where $\delta(t - x/v_0)$ is the Dirac delta function. Solve the above PDE subject to

$$u(0, t) = 0, \qquad \lim_{x\to\infty} u(x, t) = 0, \qquad t > 0$$

$$u(x, 0) = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \qquad x > 0$$

(a) when $v_0 \neq a$ **(b)** when $v_0 = a$.

Computer Lab Assignments

28. (a) The temperature in a semi-infinite solid is modeled by the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad x > 0, \quad t > 0$$

$$u(0, t) = u_0, \qquad \lim_{x\to\infty} u(x, t) = 0, \qquad t > 0$$

$$u(x, 0) = 0, \qquad x > 0.$$

Solve for $u(x, t)$. Use the solution to determine analytically the value of $\lim_{t\to\infty} u(x, t)$, $x > 0$.

(b) Use a CAS to graph $u(x, t)$ over the rectangular region defined by $0 \le x \le 10$, $0 \le t \le 15$. Assume that $u_0 = 100$ and $k = 1$. Indicate the two boundary conditions and initial condition on your graph. Use 2D and 3D plots of $u(x, t)$ to verify your answer to part (a).

29. (a) In Problem 28 if there is a constant flux of heat into the solid at its left-hand boundary, then the boundary condition is $\left.\frac{\partial u}{\partial x}\right|_{x=0} = -A$, $A > 0$, $t > 0$. Solve for $u(x, t)$. Use the solution to determine analytically the value of $\lim_{t\to\infty} u(x, t)$, $x > 0$.

(b) Use a CAS to graph $u(x, t)$ over the rectangular region defined by $0 \le x \le 10$, $0 \le t \le 15$. Assume that $u_0 = 100$ and $k = 1$. Use 2D and 3D plots of $u(x, t)$ to verify your answer to part (a).

30. Humans gather most of our information on the outside world through sight and sound. But many creatures use chemical signals as their primary means of communication; for example, honeybees, when alarmed, emit a substance and fan their wings feverishly to relay the warning signal to the bees that attend to the queen. These molecular messages between members of the same species are called pheromones. The signals may be carried by moving air or water or by a *diffusion process* in which the random movement of gas molecules transports the chemical away from its source. Figure 14.2.4 shows an ant emitting an alarm chemical into the still air of a tunnel. If $c(x, t)$ denotes the concentration of the chemical x centimeters from the source at time t, then $c(x, t)$ satisfies

$$k\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}, \qquad x > 0, \quad t > 0$$

and k is a positive constant. The emission of pheromones as a discrete pulse gives rise to a boundary condition of the form

$$\left.\frac{\partial c}{\partial x}\right|_{x=0} = -A\delta(t),$$

where $\delta(t)$ is the Dirac delta function.

(a) Solve the boundary-value problem if it is further known that

$$c(x, 0) = 0, \quad x > 0 \quad \text{and} \quad \lim_{x\to\infty} c(x, t) = 0, \quad t > 0.$$

(b) Use a CAS to graph the solution in part (a) for $x \geq 0$ at the fixed times $t = 0.1$, $t = 0.5$, $t = 1$, $t = 2$, and $t = 5$.

(c) For any fixed time t, show that $\int_0^\infty c(x, t)\,dx = Ak$. Thus Ak represents the total amount of chemical discharged.

FIGURE 14.2.4 Ant responding to chemical signal in Problem 30

14.3 FOURIER INTEGRAL

REVIEW MATERIAL

- The Fourier integral has different forms that are analogous to the four forms of Fourier series given in Definitions 11.2.1 and 11.3.1 and Problem 21 in Exercises 14.2. A review of these various forms is recommended.

INTRODUCTION In Chapters 11–13 we used Fourier series to represent a function f defined on a finite interval such as $(-p, p)$ or $(0, L)$. When f and f' are piecewise continuous on such an interval, a Fourier series represents the function on the interval and converges to the periodic extension of f outside the interval. In this way we are justified in saying that Fourier series are associated only with *periodic functions*. We shall now derive, in a nonrigorous fashion, a means of representing certain kinds of *nonperiodic functions* that are defined on either an infinite interval $(-\infty, \infty)$ or a semi-infinite interval $(0, \infty)$.

FOURIER SERIES TO FOURIER INTEGRAL Suppose a function f is defined on the interval $(-p, p)$. If we use the integral definitions of the coefficients (9), (10), and (11) of Section 11.2 in (8) of that section, then the Fourier series of f on the interval is

$$f(x) = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt + \frac{1}{p}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}t\,dt\right)\cos\frac{n\pi}{p}x + \left(\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}t\,dt\right)\sin\frac{n\pi}{p}x\right]. \quad (1)$$

If we let $\alpha_n = n\pi/p$, $\Delta\alpha = \alpha_{n+1} - \alpha_n = \pi/p$, then (1) becomes

$$f(x) = \frac{1}{2\pi}\left(\int_{-p}^{p} f(t)\,dt\right)\Delta\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\alpha_n t\,dt\right)\cos\alpha_n x + \left(\int_{-p}^{p} f(t)\sin\alpha_n t\,dt\right)\sin\alpha_n x\right]\Delta\alpha. \quad (2)$$

We now expand the interval $(-p, p)$ by letting $p \to \infty$. Since $p \to \infty$ implies that $\Delta\alpha \to 0$, the limit of (2) has the form $\lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty} F(\alpha_n)\,\Delta\alpha$, which is suggestive of the definition of the integral $\int_0^\infty F(\alpha)\,d\alpha$. Thus if $\int_{-\infty}^{\infty} f(t)\,dt$ exists, the limit of the first term in (2) is zero, and the limit of the sum becomes

$$f(x) = \frac{1}{\pi}\int_0^\infty\left[\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\right]d\alpha. \quad (3)$$

The result given in (3) is called the **Fourier integral** of f on $(-\infty, \infty)$. As the following summary shows, the basic structure of the Fourier integral is reminiscent of that of a Fourier series.

DEFINITION 14.3.1 Fourier Integral

The **Fourier integral** of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha, \tag{4}$$

where
$$A(\alpha) = \int_{-\infty}^\infty f(x)\cos\alpha x\,dx \tag{5}$$

$$B(\alpha) = \int_{-\infty}^\infty f(x)\sin\alpha x\,dx. \tag{6}$$

CONVERGENCE OF A FOURIER INTEGRAL Sufficient conditions under which a Fourier integral converges to $f(x)$ are similar to, but slightly more restrictive than, the conditions for a Fourier series.

THEOREM 14.3.1 Conditions for Convergence

Let f and f' be piecewise continuous on every finite interval and let f be absolutely integrable on $(-\infty, \infty)$.* Then the Fourier integral of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

EXAMPLE 1 Fourier Integral Representation

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

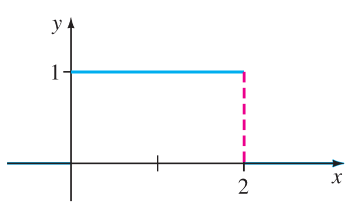


FIGURE 14.3.1 Piecewise-continuous function defined on $(-\infty, \infty)$

SOLUTION The function, whose graph is shown in Figure 14.3.1, satisfies the hypotheses of Theorem 14.3.1. Hence from (5) and (6) we have at once

$$\begin{aligned} A(\alpha) &= \int_{-\infty}^\infty f(x)\cos\alpha x\,dx \\ &= \int_{-\infty}^0 f(x)\cos\alpha x\,dx + \int_0^2 f(x)\cos\alpha x\,dx + \int_2^\infty f(x)\cos\alpha x\,dx \\ &= \int_0^2 \cos\alpha x\,dx = \frac{\sin 2\alpha}{\alpha} \\ B(\alpha) &= \int_{-\infty}^\infty f(x)\sin\alpha x\,dx = \int_0^2 \sin\alpha x\,dx = \frac{1 - \cos 2\alpha}{\alpha}. \end{aligned}$$

*This means that the integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges.

Substituting these coefficients into (4) then gives

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\left(\frac{\sin 2\alpha}{\alpha}\right)\cos \alpha x + \left(\frac{1-\cos 2\alpha}{\alpha}\right)\sin \alpha x\right] d\alpha.$$

When we use trigonometric identities, the last integral simplifies to

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin \alpha \cos \alpha(x-1)}{\alpha}\, d\alpha. \tag{7}$$

The Fourier integral can be used to evaluate integrals. For example, it follows from Theorem 14.3.1 that (7) converges to $f(1) = 1$; that is,

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \alpha}{\alpha}\, d\alpha = 1 \qquad \text{and so} \qquad \int_0^\infty \frac{\sin \alpha}{\alpha}\, d\alpha = \frac{\pi}{2}.$$

The latter result is worthy of special note, since it cannot be obtained in the "usual" manner; the integrand $(\sin x)/x$ does not possess an antiderivative that is an elementary function.

COSINE AND SINE INTEGRALS When f is an even function on the interval $(-\infty, \infty)$, then the product $f(x)\cos \alpha x$ is also an even function, whereas $f(x)\sin \alpha x$ is an odd function. As a consequence of property (g) of Theorem 11.3.1, $B(\alpha) = 0$, and so (4) becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos \alpha t\, dt\right)\cos \alpha x\, d\alpha.$$

Here we have also used property (f) of Theorem 11.3.1 to write

$$\int_{-\infty}^\infty f(t)\cos \alpha t\, dt = 2\int_0^\infty f(t)\cos \alpha t\, dt.$$

Similarly, when f is an odd function on $(-\infty, \infty)$, products $f(x)\cos \alpha x$ and $f(x)\sin \alpha x$ are odd and even functions, respectively. Therefore $A(\alpha) = 0$, and

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\sin \alpha t\, dt\right)\sin \alpha x\, d\alpha.$$

We summarize in the following definition.

DEFINITION 14.3.2 Fourier Cosine and Sine Integrals

(*i*) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the **cosine integral**

$$f(x) = \frac{2}{\pi}\int_0^\infty A(\alpha)\cos \alpha x\, d\alpha, \tag{8}$$

where

$$A(\alpha) = \int_0^\infty f(x)\cos \alpha x\, dx. \tag{9}$$

(*ii*) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the **sine integral**

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\sin \alpha x\, d\alpha, \tag{10}$$

where

$$B(\alpha) = \int_0^\infty f(x)\sin \alpha x\, dx. \tag{11}$$

EXAMPLE 2 Cosine Integral Representation

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a. \end{cases}$$

SOLUTION It is apparent from Figure 14.3.2 that f is an even function. Hence we represent f by the Fourier cosine integral (8). From (9) we obtain

$$A(\alpha) = \int_0^\infty f(x) \cos \alpha x \, dx = \int_0^a f(x) \cos \alpha x \, dx + \int_a^\infty f(x) \cos \alpha x \, dx = \int_0^a \cos \alpha x \, dx = \frac{\sin a\alpha}{\alpha},$$

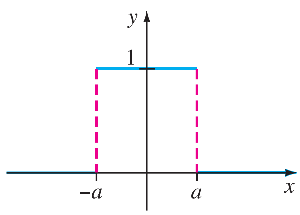


FIGURE 14.3.2 Piecewise-continuous even function defined on $(-\infty, \infty)$

so $$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin a\alpha \cos \alpha x}{\alpha} \, d\alpha. \qquad (12)$$ ■

The integrals (8) and (10) can be used when f is neither odd nor even and defined only on the half-line $(0, \infty)$. In this case (8) represents f on the interval $(0, \infty)$ and its even (but not periodic) extension to $(-\infty, 0)$, whereas (10) represents f on $(0, \infty)$ and its odd extension to the interval $(-\infty, 0)$. The next example illustrates this concept.

EXAMPLE 3 Cosine and Sine Integral Representations

Represent $f(x) = e^{-x}$, $x > 0$
(a) by a cosine integral **(b)** by a sine integral.

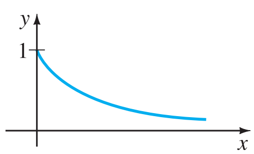


FIGURE 14.3.3 Function defined on $(0, \infty)$

SOLUTION The graph of the function is given in Figure 14.3.3.

(a) Using integration by parts, we find

$$A(\alpha) = \int_0^\infty e^{-x} \cos \alpha x \, dx = \frac{1}{1 + \alpha^2}.$$

Therefore the cosine integral of f is

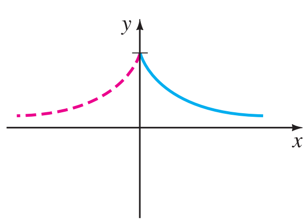
$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos \alpha x}{1 + \alpha^2} \, d\alpha. \qquad (13)$$

(b) Similarly, we have

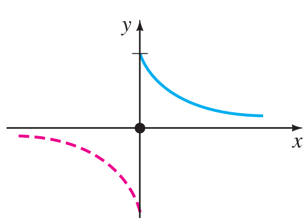
$$B(\alpha) = \int_0^\infty e^{-x} \sin \alpha x \, dx = \frac{\alpha}{1 + \alpha^2}.$$

The sine integral of f is then

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{1 + \alpha^2} \, d\alpha. \qquad (14)$$



(a) Cosine integral



(b) Sine integral

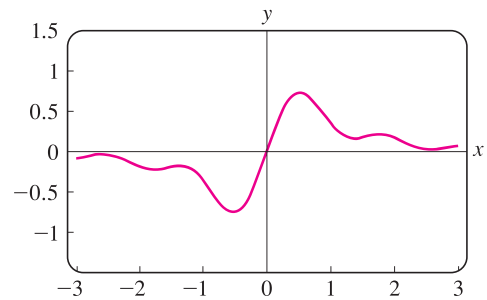
FIGURE 14.3.4 (a) is the even extension of f; (b) is the odd extension of f

Figure 14.3.4 shows the graphs of the functions and their extensions represented by the two integrals. ■

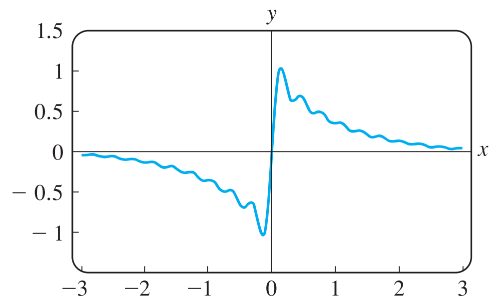
USE OF COMPUTERS We can examine the convergence of a Fourier integral in a manner similar to graphing partial sums of a Fourier series. To illustrate, let's use part (b) of Example 3. Then by definition of an improper integral the Fourier sine integral representation (14) of $f(x) = e^{-x}$, $x > 0$, can be written as $f(x) = \lim_{b\to\infty} F_b(x)$, where x is considered a parameter in

$$F_b(x) = \frac{2}{\pi} \int_0^b \frac{\alpha \sin \alpha x}{1 + \alpha^2} \, d\alpha. \qquad (15)$$

Now the idea is this: Since the Fourier sine integral (14) converges, for a specified value of $b > 0$ the graph of the **partial integral** $F_b(x)$ in (15) will be an approximation to the graph of f in Figure 14.3.4(b). The graphs of $F_b(x)$ for $b = 5$ and $b = 20$ given in Figure 14.3.5 were obtained by using *Mathematica* and its **NIntegrate** application. See Problem 21 in Exercises 14.3.



(a) $F_5(x)$



(b) $F_{20}(x)$

FIGURE 14.3.5 Convergence of $F_b(x)$ to $f(x)$ in Example 3(b) as $b \to \infty$

COMPLEX FORM The Fourier integral (4) also possesses an equivalent **complex form,** or **exponential form,** that is analogous to the complex form of a Fourier series (see Problem 21 in Exercises 11.2). If (5) and (6) are substituted into (4), then

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\,[\cos \alpha t \cos \alpha x + \sin \alpha t \sin \alpha x]\,dt\,d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos \alpha(t - x)\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos \alpha(t - x)\,dt\,d\alpha \tag{16}$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)[\cos \alpha(t - x) + i \sin \alpha(t - x)]\,dt\,d\alpha \tag{17}$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)e^{i\alpha(t-x)}\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(t)e^{i\alpha t}\,dt\right)e^{-i\alpha x}\,d\alpha. \tag{18}$$

We note that (16) follows from the fact that the integrand is an even function of α. In (17) we have simply added zero to the integrand;

$$i\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \sin \alpha(t - x)\,dt\,d\alpha = 0$$

because the integrand is an odd function of α. The integral in (18) can be expressed as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty C(\alpha)e^{-i\alpha x}\,d\alpha, \tag{19}$$

where

$$C(\alpha) = \int_{-\infty}^\infty f(x)e^{i\alpha x}\,dx. \tag{20}$$

This latter form of the Fourier integral will be put to use in the next section when we return to the solution of boundary-value problems.

EXERCISES 14.3

Answers to selected odd-numbered problems begin on page ANS-24.

In Problems 1–6 find the Fourier integral representation of the given function.

1. $f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 < x < 0 \\ 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

2. $f(x) = \begin{cases} 0, & x < \pi \\ 4, & \pi < x < 2\pi \\ 0, & x > 2\pi \end{cases}$

3. $f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 3 \\ 0, & x > 3 \end{cases}$

4. $f(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

5. $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$

6. $f(x) = \begin{cases} e^{x}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

In Problems 7–12 represent the given function by an appropriate cosine or sine integral.

7. $f(x) = \begin{cases} 0, & x < -1 \\ -5, & -1 < x < 0 \\ 5, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

8. $f(x) = \begin{cases} 0, & |x| < 1 \\ \pi, & 1 < |x| < 2 \\ 0, & |x| > 2 \end{cases}$

9. $f(x) = \begin{cases} |x|, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

10. $f(x) = \begin{cases} x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

11. $f(x) = e^{-|x|} \sin x$

12. $f(x) = xe^{-|x|}$

In Problems 13–16 find the cosine and sine integral representations of the given function.

13. $f(x) = e^{-kx}, \quad k > 0, \quad x > 0$

14. $f(x) = e^{-x} - e^{-3x}, \quad x > 0$

15. $f(x) = xe^{-2x}, \quad x > 0$

16. $f(x) = e^{-x} \cos x, \quad x > 0$

In Problems 17 and 18 solve the given integral equation for the function *f*.

17. $\displaystyle\int_0^\infty f(x) \cos \alpha x \, dx = e^{-\alpha}$

18. $\displaystyle\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1, & 0 < \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$

19. **(a)** Use (7) to show that

$$\int_0^\infty \frac{\sin 2x}{x} \, dx = \frac{\pi}{2}.$$

[*Hint*: α is a dummy variable of integration.]

(b) Show in general that for $k > 0$,

$$\int_0^\infty \frac{\sin kx}{x} \, dx = \frac{\pi}{2}.$$

20. Use the complex form (19) to find the Fourier integral representation of $f(x) = e^{-|x|}$. Show that the result is the same as that obtained from (8).

Computer Lab Assignments

21. While the integral (12) can be graphed in the same manner discussed on page 501 to obtain Figure 14.3.5, it can also be expressed in terms of a special function that is built into a CAS.

(a) Use a trigonometric identity to show that an alternative form of the Fourier integral representation (12) of the function f in Example 2 (with $a = 1$) is

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\sin \alpha(x + 1) - \sin \alpha(x - 1)}{\alpha} \, d\alpha.$$

(b) As a consequence of part (a), $f(x) = \lim_{b \to \infty} F_b(x)$, where

$$F_b(x) = \frac{1}{\pi} \int_0^b \frac{\sin \alpha(x + 1) - \sin \alpha(x - 1)}{\alpha} \, d\alpha.$$

Show that the last integral can be written as

$$F_b(x) = \frac{1}{\pi}[\text{Si}(b(x + 1)) - \text{Si}(b(x - 1))],$$

where $\text{Si}(x)$ is the **sine integral function.** See Problem 49 in Exercises 2.3.

(c) Use a CAS and the sine integral form of $F_b(x)$ in part (b) to obtain the graphs on the interval $[-3, 3]$ for $b = 4$, 6, and 15. Then graph $F_b(x)$ for larger values of $b > 0$.

14.4 FOURIER TRANSFORMS

REVIEW MATERIAL

- Definition 14.3.2
- Equations (19) and (20) in Section 14.3

INTRODUCTION So far in this text we have studied and used only one integral transform: the Laplace transform. But in Section 14.3 we saw that the Fourier integral had three alternative forms: the cosine integral, the sine integral, and the complex or exponential form. In the present section we shall take these three forms of the Fourier integral and develop them into three new integral transforms, not surprisingly called **Fourier transforms.** In addition, we shall expand on the concept of a transform pair, that is, an integral transform and its inverse. We shall also see that the inverse of an integral transform is itself another integral transform.

TRANSFORM PAIRS The Laplace transform $F(s)$ of a function $f(t)$ is defined by an integral, but up to now we have been using the symbolic representation $f(t) = \mathscr{L}^{-1}\{F(s)\}$ to denote the inverse Laplace transform of $F(s)$. Actually, the inverse Laplace transform is also an integral transform.

If $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$, then the inverse Laplace transform is

$$\mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}F(s)\,ds = f(t).$$

The last integral is called a **contour integral;** its evaluation requires the use of complex variables and is beyond the scope of this text. The point here is this: Integral transforms appear in **transform pairs.** If $f(x)$ is transformed into $F(\alpha)$ by an **integral transform**

$$F(\alpha) = \int_a^b f(x)K(\alpha, x)\,dx,$$

then the function f can be recovered by another integral transform

$$f(x) = \int_c^d F(\alpha)H(\alpha, x)\,d\alpha,$$

called the **inverse transform.** The functions K and H in the integrands are called the **kernels** of their respective transforms. We identify $K(s, t) = e^{-st}$ as the kernel of the Laplace transform and $H(s, t) = e^{st}/2\pi i$ as the kernel of the inverse Laplace transform.

FOURIER TRANSFORM PAIRS The Fourier integral is the source of three new integral transforms. From (20)–(19), (11)–(10), and (9)–(8) of Section 14.3 we are prompted to define the following **Fourier transform pairs.**

DEFINITION 14.4.1 Fourier Transform Pairs

(*i*) Fourier transform:
$$\mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = F(\alpha) \qquad (1)$$

Inverse Fourier transform:
$$\mathscr{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}\,d\alpha = f(x) \qquad (2)$$

(*ii*) Fourier sine transform:		$\mathscr{F}_s\{f(x)\} = \int_0^\infty f(x) \sin \alpha x \, dx = F(\alpha)$	(3)
Inverse Fourier sine transform:		$\mathscr{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha) \sin \alpha x \, d\alpha = f(x)$	(4)
(*iii*) Fourier cosine transform:		$\mathscr{F}_c\{f(x)\} = \int_0^\infty f(x) \cos \alpha x \, dx = F(\alpha)$	(5)
Inverse Fourier cosine transform:		$\mathscr{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha) \cos \alpha x \, d\alpha = f(x)$	(6)

EXISTENCE The conditions under which (1), (3), and (5) exist are more stringent than those for the Laplace transform. For example, you should verify that $\mathscr{F}\{1\}$, $\mathscr{F}_s\{1\}$, and $\mathscr{F}_c\{1\}$ do not exist. Sufficient conditions for existence are that f be absolutely integrable on the appropriate interval and that f and f' be piecewise continuous on every finite interval.

OPERATIONAL PROPERTIES Since our immediate goal is to apply these new transforms to boundary-value problems, we need to examine the transforms of derivatives.

FOURIER TRANSFORM Suppose that f is continuous and absolutely integrable on the interval $(-\infty, \infty)$ and f' is piecewise continuous on every finite interval. If $f(x) \to 0$ as $x \to \pm\infty$, then integration by parts gives

$$\begin{aligned}\mathscr{F}\{f'(x)\} &= \int_{-\infty}^{\infty} f'(x)e^{i\alpha x}\,dx \\ &= f(x)\,e^{i\alpha x}\Big|_{-\infty}^{\infty} - i\alpha\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx \\ &= -i\alpha\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx,\end{aligned}$$

that is,
$$\mathscr{F}\{f'(x)\} = -i\alpha\,F(\alpha). \tag{7}$$

Similarly, under the added assumptions that f' is continuous on $(-\infty, \infty)$, $f''(x)$ is piecewise continuous on every finite interval and $f'(x) \to 0$ as $x \to \pm\infty$, we have

$$\mathscr{F}\{f''(x)\} = (-i\alpha)^2\,\mathscr{F}\{f(x)\} = -\alpha^2 F(\alpha). \tag{8}$$

It is important to be aware that the sine and cosine transforms are not suitable for transforming the first derivative (or, for that matter, any derivative of *odd* order). It is readily shown that

$$\mathscr{F}_s\{f'(x)\} = -\alpha\mathscr{F}_c\{f(x)\} \qquad \text{and} \qquad \mathscr{F}_c\{f'(x)\} = \alpha\mathscr{F}_s\{f(x)\} - f(0).$$

The difficulty is apparent; the transform of $f'(x)$ is not expressed in terms of the original integral transform.

FOURIER SINE TRANSFORM Suppose that f and f' are continuous, f is absolutely integrable on the interval $[0, \infty)$, and f'' is piecewise continuous on every finite interval. If $f \to 0$ and $f' \to 0$ as $x \to \infty$, then

$$\begin{aligned}\mathscr{F}_s\{f''(x)\} &= \int_0^\infty f''(x)\sin\alpha x\,dx \\ &= f'(x)\sin\alpha x\Big|_0^\infty - \alpha\int_0^\infty f'(x)\cos\alpha x\,dx \\ &= -\alpha\left[f(x)\cos\alpha x\Big|_0^\infty + \alpha\int_0^\infty f(x)\sin\alpha x\,dx\right] \\ &= \alpha f(0) - \alpha^2\mathscr{F}_s\{f(x)\},\end{aligned}$$

that is,

$$\mathscr{F}_s\{f''(x)\} = -\alpha^2F(\alpha) + \alpha f(0). \tag{9}$$

FOURIER COSINE TRANSFORM Under the same assumptions that lead to (9) we find the Fourier cosine transform of $f''(x)$ to be

$$\mathscr{F}_c\{f''(x)\} = -\alpha^2F(\alpha) - f'(0). \tag{10}$$

A natural question is "How do we know which transform to use on a given boundary-value problem?" Clearly, to use a Fourier transform, the domain of the variable to be eliminated must be $(-\infty, \infty)$. To utilize a sine or cosine transform, the domain of at least one of the variables in the problem must be $[0, \infty)$. But the determining factor in choosing between the sine transform and the cosine transform is the type of boundary condition specified at zero.

■ Remember this when working Exercises 14.4.

In the examples that follow, we shall assume without further mention that both u and $\partial u/\partial x$ (or $\partial u/\partial y$) approach zero as $x \to \pm\infty$. This is not a major restriction, since these conditions hold in most applications.

EXAMPLE 1 Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $-\infty < x < \infty$, $t > 0$, subject to

$$u(x, 0) = f(x), \qquad \text{where} \qquad f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1. \end{cases}$$

SOLUTION The problem can be interpreted as finding the temperature $u(x, t)$ in an infinite rod. Because the domain of x is the infinite interval $(-\infty, \infty)$, we use the Fourier transform (1) and define

$$\mathscr{F}\{u(x,t)\} = \int_{-\infty}^\infty u(x,t)e^{i\alpha x}\,dx = U(\alpha, t).$$

If we transform the partial differential equation and use (8),

$$\mathscr{F}\left\{k\frac{\partial^2 u}{\partial x^2}\right\} = \mathscr{F}\left\{\frac{\partial u}{\partial t}\right\}$$

yields

$$-k\alpha^2U(\alpha, t) = \frac{dU}{dt} \qquad \text{or} \qquad \frac{dU}{dt} + k\alpha^2U(\alpha, t) = 0.$$

Solving the last equation gives $U(\alpha, t) = ce^{-k\alpha^2 t}$. Now the transform of the initial condition is

$$\mathscr{F}\{u(x,0)\} = \int_{-\infty}^\infty f(x)e^{i\alpha x}\,dx = \int_{-1}^1 u_0e^{i\alpha x}\,dx = u_0\frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha}.$$

This result is the same as $U(\alpha, 0) = 2u_0 \dfrac{\sin \alpha}{\alpha}$. Applying this condition to the solution $U(\alpha, t)$ gives $U(\alpha, 0) = c = (2u_0 \sin \alpha)/\alpha$, so

$$U(\alpha, t) = 2u_0 \frac{\sin \alpha}{\alpha} e^{-k\alpha^2 t}.$$

It then follows from the inversion integral (2) that

$$u(x, t) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha} e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha.$$

The last expression can be simplified somewhat by using Euler's formula $e^{-i\alpha x} = \cos \alpha x - i \sin \alpha x$ and noting that

$$\int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha} e^{-k\alpha^2 t} \sin \alpha x \; d\alpha = 0,$$

since the integrand is an odd function of α. Hence we finally have

$$u(x, t) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} e^{-k\alpha^2 t}\, d\alpha. \qquad (11)$$ ■

It is left to the reader to show that the solution (11) can be expressed in terms of the error function. See Problem 23 in Exercises 14.4.

EXAMPLE 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < \pi, \quad y > 0$$

$$u(0, y) = 0, \quad u(\pi, y) = e^{-y}, \quad y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0 < x < \pi.$$

Solve for $u(x, y)$.

SOLUTION The domain of the variable y and the prescribed condition at $y = 0$ indicate that the Fourier cosine transform is suitable for the problem. We define

$$\mathscr{F}_c\{u(x, y)\} = \int_0^{\infty} u(x, y) \cos \alpha y \, dy = U(x, \alpha).$$

In view of (10), $$\mathscr{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathscr{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \mathscr{F}_c\{0\}$$

becomes $$\frac{d^2 U}{dx^2} - \alpha^2 U(x, \alpha) - u_y(x, 0) = 0 \qquad \text{or} \qquad \frac{d^2 U}{dx^2} - \alpha^2 U = 0.$$

Since the domain of x is a finite interval, we choose to write the solution of the ordinary differential equation as

$$U(x, \alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x. \qquad (12)$$

Now $\mathscr{F}_c\{u(0, y)\} = \mathscr{F}_c\{0\}$ and $\mathscr{F}_c\{u(\pi, y)\} = \mathscr{F}_c\{e^{-y}\}$ are in turn equivalent to

$$U(0, \alpha) = 0 \qquad \text{and} \qquad U(\pi, \alpha) = \frac{1}{1 + \alpha^2}.$$

When we apply these latter conditions, the solution (12) gives $c_1 = 0$ and $c_2 = 1/[(1 + \alpha^2)\sinh \alpha\pi]$. Therefore

$$U(x, \alpha) = \frac{\sinh \alpha x}{(1 + \alpha^2)\sinh \alpha\pi},$$

so from (6) we arrive at

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \frac{\sinh \alpha x}{(1 + \alpha^2)\sinh \alpha\pi}\cos \alpha y \, d\alpha. \qquad (13)$$ ■

Had $u(x, 0)$ been given in Example 2 rather than $u_y(x, 0)$, then the sine transform would have been appropriate.

EXERCISES 14.4

Answers to selected odd-numbered problems begin on page ANS-24.

In Problems 1–21 use the Fourier integral transforms of this section to solve the given boundary-value problem. Make assumptions about boundedness where necessary.

1. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0$

$u(x, 0) = e^{-|x|}, \quad -\infty < x < \infty$

2. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0$

$$u(x, 0) = \begin{cases} 0, & x < -1 \\ -100, & -1 < x < 0 \\ 100, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

3. Find the temperature $u(x, t)$ in a semi-infinite rod if $u(0, t) = u_0$, $t > 0$ and $u(x, 0) = 0$, $x > 0$.

4. Use the result $\displaystyle\int_0^\infty \frac{\sin \alpha x}{\alpha}\,d\alpha = \frac{\pi}{2}$, $x > 0$, to show that the solution of Problem 3 can be written as

$$u(x, t) = u_0 - \frac{2u_0}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha} e^{-k\alpha^2 t}\,d\alpha.$$

5. Find the temperature $u(x, t)$ in a semi-infinite rod if $u(0, t) = 0$, $t > 0$, and

$$u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1. \end{cases}$$

6. Solve Problem 3 if the condition at the left boundary is

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -A, \quad t > 0,$$

where A is a constant.

7. Solve Problem 5 if the end $x = 0$ is insulated.

8. Find the temperature $u(x, t)$ in a semi-infinite rod if $u(0, t) = 1$, $t > 0$, and $u(x, 0) = e^{-x}$, $x > 0$.

9. (a) $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad -\infty < x < \infty$$

(b) If $g(x) = 0$, show that the solution of part (a) can be written as $u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)]$.

10. Find the displacement $u(x, t)$ of a semi-infinite string if

$$u(0, t) = 0, \quad t > 0$$

$$u(x, 0) = xe^{-x}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x > 0$$

11. Solve the problem in Example 2 if the boundary conditions at $x = 0$ and $x = \pi$ are reversed: $u(0, y) = e^{-y}$, $u(\pi, y) = 0$, $y > 0$.

12. Solve the problem in Example 2 if the boundary condition at $y = 0$ is $u(x, 0) = 1$, $0 < x < \pi$.

13. Find the steady-state temperature $u(x, y)$ in a plate defined by $x \ge 0$, $y \ge 0$ if the boundary $x = 0$ is insulated and, at $y = 0$,

$$u(x, 0) = \begin{cases} 50, & 0 < x < 1 \\ 0, & x > 1. \end{cases}$$

14. Solve Problem 13 if the boundary condition at $x = 0$ is $u(0, y) = 0$, $y > 0$.

15. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad x > 0, \quad 0 < y < 2$

$u(0, y) = 0, \quad 0 < y < 2$

$u(x, 0) = f(x), \quad u(x, 2) = 0, \quad x > 0$

16. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad y > 0$

$u(0, y) = f(y), \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=\pi} = 0, \quad y > 0$

$\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0 < x < \pi$

In Problems 17 and 18 find the steady-state temperature in the plate given in the figure. [*Hint*: One way of proceeding is to express Problems 17 and 18 as two- and three-boundary-value problems, respectively. Use the superposition principle. See Section 12.5.]

17.

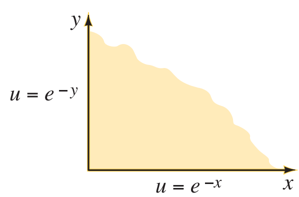


FIGURE 14.4.1 Plate in Problem 17

18.

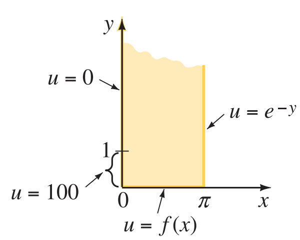


FIGURE 14.4.2 Plate in Problem 18

19. Use the result $\mathscr{F}\{e^{-x^2/4p^2}\} = 2\sqrt{\pi}pe^{-p^2\alpha^2}$ to solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = e^{-x^2}, \quad -\infty < x < \infty.$$

20. If $\mathscr{F}\{f(x)\} = F(\alpha)$ and $\mathscr{F}\{g(x)\} = G(\alpha)$, then the **convolution theorem** for the Fourier transform is given by

$$\int_{-\infty}^{\infty} f(\tau)g(x - \tau)\, d\tau = \mathscr{F}^{-1}\{F(\alpha)G(\alpha)\}.$$

Use this result and $\mathscr{F}\{e^{-x^2/4p^2}\} = 2\sqrt{\pi}pe^{-p^2\alpha^2}$ to show that a solution of the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty$$

is $$u(x, t) = \frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(\tau)e^{-(x-\tau)^2/4kt}\, d\tau.$$

21. Use the transform $\mathscr{F}\{e^{-x^2/4p^2}\}$ given in Problem 19 to find the steady-state temperature in the infinite strip shown in Figure 14.4.3.

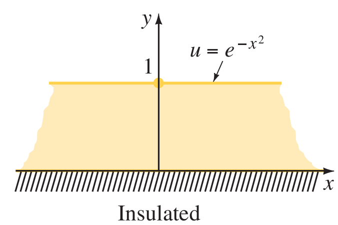


FIGURE 14.4.3 Infinite strip in Problem 21

22. The solution of Problem 14 can be integrated. Use entries 42 and 43 of the table in Appendix III to show that

$$u(x, y) = \frac{100}{\pi}\left[\arctan\frac{x}{y} - \frac{1}{2}\arctan\frac{x+1}{y} - \frac{1}{2}\arctan\frac{x-1}{y}\right].$$

23. Use the solution given in Problem 20 to rewrite the solution of Example 1 in an alternative integral form. Then use the change of variables $v = (x - \tau)/2\sqrt{kt}$ and the results of Problem 9 in Exercises 14.1 to show that the solution of Example 1 can be expressed as

$$u(x, t) = \frac{u_0}{2}\left[\operatorname{erf}\left(\frac{x+1}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{kt}}\right)\right].$$

Computer Lab Assignments

24. Assume that $u_0 = 100$ and $k = 1$ in the solution in Problem 23. Use a CAS to graph $u(x, t)$ over the rectangular region defined by $-4 \le x \le 4$, $0 \le t \le 6$. Use a 2D plot to superimpose the graphs of $u(x, t)$ for $t = 0.05, 0.125, 0.5, 1, 2, 4, 6$, and 15 on the interval $[-4, 4]$. Use the graphs to conjecture the values of $\lim_{t\to\infty} u(x, t)$ and $\lim_{x\to\infty} u(x, t)$. Then prove these results analytically using the properties of $\operatorname{erf}(x)$.

CHAPTER 14 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-24.

In Problems 1–16 solve the given boundary-value problem by an appropriate integral transform. Make assumptions about boundedness where necessary.

1. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad x > 0, \quad 0 < y < \pi$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad 0 < y < \pi$

$u(x, 0) = 0, \quad \left.\dfrac{\partial u}{\partial y}\right|_{y=\pi} = e^{-x}, \quad x > 0$

2. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$

$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$

$u(x, 0) = 50 \sin 2\pi x, \quad 0 < x < 1$

3. $\dfrac{\partial^2 u}{\partial x^2} - hu = \dfrac{\partial u}{\partial t}, \quad h > 0, \quad x > 0, \quad t > 0$

$u(0, t) = 0, \quad \lim_{x\to\infty} \dfrac{\partial u}{\partial x} = 0, \quad t > 0$

$u(x, 0) = u_0, \quad x > 0$

4. $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = e^{-|x|}, \quad -\infty < x < \infty, \quad t > 0$

$u(x, 0) = 0, \quad -\infty < x < \infty$

5. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad x > 0, \quad t > 0$

$u(0, t) = t, \quad \lim_{x\to\infty} u(x, t) = 0$

$u(x, 0) = 0, \quad x > 0$ [*Hint*: Use Theorem 7.4.2.]

6. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0$

$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$

$u(x, 0) = \sin \pi x, \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = -\sin \pi x, \quad 0 < x < 1$

7. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0$

$u(x, 0) = \begin{cases} 0, & x < 0 \\ u_0, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

8. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad y > 0$

$u(0, y) = 0, \quad u(\pi, y) = \begin{cases} 0, & 0 < y < 1 \\ 1, & 1 < y < 2 \\ 0, & y > 2 \end{cases}$

$\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0 < x < \pi$

9. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad x > 0, \quad y > 0$

$u(0, y) = \begin{cases} 50, & 0 < y < 1 \\ 0, & y > 1 \end{cases}$

$u(x, 0) = \begin{cases} 100, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

10. $\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(1, t) = 0, \quad t > 0$

$u(x, 0) = 0, \quad 0 < x < 1$

11. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad x > 0, \quad 0 < y < \pi$

$u(0, y) = A, \quad 0 < y < \pi$

$\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\dfrac{\partial u}{\partial y}\right|_{y=\pi} = Be^{-x}, \quad x > 0$

12. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$

$u(0, t) = u_0, \quad u(1, t) = u_0, \quad t > 0$

$u(x, 0) = 0, \quad 0 < x < 1$

[*Hint*: Use the identity

$$\sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y,$$

and then use Problem 6 in Exercises 14.1.]

13. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0$

$u(x, 0) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$

14. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad x > 0, \quad t > 0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = -50, \quad \lim_{x\to\infty} u(x, t) = 100, \quad t > 0$

$u(x, 0) = 100, \quad x > 0$

15. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad x > 0, \quad t > 0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad t > 0$

$u(x, 0) = e^{-x}, \quad x > 0$

16. Show that a solution of the BVP

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad -\infty < x < \infty, \quad 0 < y < 1$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad u(x, 1) = f(x), \quad -\infty < x < \infty$$

is $u(x, y) = \dfrac{1}{\pi}\displaystyle\int_0^\infty \int_{-\infty}^\infty f(t)\, \frac{\cosh \alpha y \cos \alpha(t - x)}{\cosh \alpha}\, dt\, d\alpha.$