12.1 SEPARABLE PARTIAL DIFFERENTIAL EQUATIONS

REVIEW MATERIAL

- Sections 2.3, 4.3, and 4.4
- Reread "Two Equations Worth Knowing" on pages 135–136.

INTRODUCTION Partial differential equations (PDEs), like ordinary differential equations (ODEs), are classified as either linear or nonlinear. Analogous to a linear ODE, the dependent variable and its partial derivatives in a linear PDE are only to the first power. For the remaining chapters of this text we shall be interested in, for the most part, *linear second-order* PDEs.

LINEAR PARTIAL DIFFERENTIAL EQUATION If we let u denote the dependent variable and let x and y denote the independent variables, then the general form of a **linear second-order partial differential equation** is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \tag{1}$$

where the coefficients $A, B, C, \ldots, G$ are functions of x and y. When $G(x, y) = 0$, equation (1) is said to be **homogeneous;** otherwise, it is **nonhomogeneous.** For example, the linear equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = xy$$

are homogeneous and nonhomogeneous, respectively.

SOLUTION OF A PDE A **solution** of a linear partial differential equation (1) is a function $u(x, y)$ of two independent variables that possesses all partial derivatives occurring in the equation and that satisfies the equation in some region of the xy-plane.

It is not our intention to examine procedures for finding *general solutions* of linear partial differential equations. Not only is it often difficult to obtain a general solution of a linear second-order PDE, but a general solution is usually not all that useful in applications. Thus our focus throughout will be on finding *particular solutions* of some of the more important linear PDEs—that is, equations that appear in many applications.

SEPARATION OF VARIABLES Although there are several methods that can be tried to find particular solutions of a linear PDE, the one we are interested in at the moment is called the **method of separation of variables.** In this method we seek a particular solution of the form of a *product* of a function of x and a function of y:

$$u(x, y) = X(x)Y(y).$$

With this assumption it is *sometimes* possible to reduce a linear PDE in two variables to two ODEs. To this end we note that

$$\frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = XY', \qquad \frac{\partial^2 u}{\partial x^2} = X''Y, \qquad \frac{\partial^2 u}{\partial y^2} = XY'',$$

where the primes denote ordinary differentiation.

EXAMPLE 1 Separation of Variables

Find product solutions of $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$.

SOLUTION Substituting $u(x, y) = X(x)Y(y)$ into the partial differential equation yields

$$X''Y = 4XY'.$$

After dividing both sides by $4XY$, we have separated the variables:

$$\frac{X''}{4X} = \frac{Y'}{Y}.$$

Since the left-hand side of the last equation is independent of y and is equal to the right-hand side, which is independent of x, we conclude that both sides of the equation are independent of x *and* y. In other words, each side of the equation must be a constant. In practice it is *convenient* to write this real **separation constant** as $-\lambda$ (using λ would lead to the same solutions).

From the two equalities

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

we obtain the two linear ordinary differential equations

$$X'' + 4\lambda X = 0 \qquad \text{and} \qquad Y' + \lambda Y = 0. \tag{2}$$

Now, as in Example 1 of Section 11.4 we consider three cases for λ: zero, negative, or positive, that is, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$.

CASE I If $\lambda = 0$, then the two ODEs in (2) are

$$X'' = 0 \qquad \text{and} \qquad Y' = 0.$$

Solving each equation (by, say, integration), we find $X = c_1 + c_2x$ and $Y = c_3$. Thus a particular product solution of the given PDE is

$$u = XY = (c_1 + c_2x)c_3 = A_1 + B_1x, \tag{3}$$

where we have replaced c_1c_3 and c_2c_3 by A_1 and B_1, respectively.

CASE II If $\lambda = -\alpha^2$, then the DEs in (2) are

$$X'' - 4\alpha^2X = 0 \qquad \text{and} \qquad Y' - \alpha^2Y = 0.$$

From their general solutions

$$X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \qquad \text{and} \qquad Y = c_6e^{\alpha^2 y}$$

we obtain another particular product solution of the PDE,

$$u = XY = (c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x)c_6e^{\alpha^2 y}$$

or

$$u = A_2e^{\alpha^2 y} \cosh 2\alpha x + B_2e^{\alpha^2 y} \sinh 2\alpha x, \tag{4}$$

where $A_2 = c_4c_6$ and $B_2 = c_5c_6$.

CASE III If $\lambda = \alpha^2$, then the DEs

$$X'' + 4\alpha^2X = 0 \qquad \text{and} \qquad Y' + \alpha^2Y = 0$$

and their general solutions

$$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x \qquad \text{and} \qquad Y = c_9e^{-\alpha^2 y}$$

give yet another particular solution

$$u = A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_3 e^{-\alpha^2 y} \sin 2\alpha x, \qquad (5)$$

where $A_3 = c_7 c_9$ and $B_2 = c_8 c_9$. ■

It is left as an exercise to verify that (3), (4), and (5) satisfy the given PDE. See Problem 29 in Exercises 12.1.

SUPERPOSITION PRINCIPLE The following theorem is analogous to Theorem 4.1.2 and is known as the **superposition principle.**

THEOREM 12.1.1 Superposition Principle

If $u_1, u_2, \ldots, u_k$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k,$$

where the c_i, $i = 1, 2, \ldots, k$, are constants, is also a solution.

Throughout the remainder of the chapter we shall assume that whenever we have an infinite set $u_1, u_2, u_3, \ldots$ of solutions of a homogeneous linear equation, we can construct yet another solution u by forming the infinite series

$$u = \sum_{k=1}^{\infty} c_k u_k,$$

where the c_i, $i = 1, 2, \ldots$ are constants.

CLASSIFICATION OF EQUATIONS A linear second-order partial differential equation in two independent variables with constant coefficients can be classified as one of three types. This classification depends only on the coefficients of the second-order derivatives. Of course, we assume that at least one of the coefficients A, B, and C is not zero.

DEFINITION 12.1.1 Classification of Equations

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0,$$

where A, B, C, D, E, and F are real constants, is said to be

hyperbolic if $B^2 - 4AC > 0$,
parabolic if $B^2 - 4AC = 0$,
elliptic if $B^2 - 4AC < 0$.

EXAMPLE 2 Classifying Linear Second-Order PDEs

Classify the following equations:

(a) $3\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y}$ **(b)** $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$ **(c)** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

SOLUTION **(a)** By rewriting the given equation as

$$3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0,$$

we can make the identifications $A = 3$, $B = 0$, and $C = 0$. Since $B^2 - 4AC = 0$, the equation is parabolic.

(b) By rewriting the equation as

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

we see that $A = 1$, $B = 0$, $C = -1$, and $B^2 - 4AC = -4(1)(-1) > 0$. The equation is hyperbolic.

(c) With $A = 1$, $B = 0$, $C = 1$, and $B^2 - 4AC = -4(1)(1) < 0$ the equation is elliptic. ■

REMARKS

(*i*) In case you are wondering, separation of variables is not a general method for finding particular solutions; some linear partial differential equations are simply *not* separable. You are encouraged to verify that the assumption $u = XY$ does not lead to a solution for the linear PDE $\partial^2 u/\partial x^2 - \partial u/\partial y = x$.

(*ii*) A detailed explanation of why we would want to classify a linear second-order PDE as hyperbolic, parabolic, or elliptic is beyond the scope of this text, but you should at least be aware that this classification is of practical importance. We are going to solve some PDEs subject to only boundary conditions and others subject to both boundary and initial conditions; the kinds of side conditions that are appropriate for a given equation depend on whether the equation is hyperbolic, parabolic, or elliptic. On a related matter, we shall see in Chapter 15 that numerical-solution methods for linear second-order PDEs differ in conformity with the classification of the equation.

EXERCISES 12.1

Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1–16 use separation of variables to find, if possible, product solutions for the given partial differential equation.

1. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y}$

2. $\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} = 0$

3. $u_x + u_y = u$

4. $u_x = u_y + u$

5. $x\dfrac{\partial u}{\partial x} = y\dfrac{\partial u}{\partial y}$

6. $y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0$

7. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

8. $y\dfrac{\partial^2 u}{\partial x\,\partial y} + u = 0$

9. $k\dfrac{\partial^2 u}{\partial x^2} - u = \dfrac{\partial u}{\partial t}, \quad k > 0$

10. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k > 0$

11. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

12. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} + 2k\dfrac{\partial u}{\partial t}, \quad k > 0$

13. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

14. $x^2\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

15. $u_{xx} + u_{yy} = u$

16. $a^2 u_{xx} - g = u_{tt}, \quad g$ a constant

In Problems 17–26 classify the given partial differential equation as hyperbolic, parabolic, or elliptic.

17. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

18. $3\dfrac{\partial^2 u}{\partial x^2} + 5\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

19. $\dfrac{\partial^2 u}{\partial x^2} + 6\dfrac{\partial^2 u}{\partial x\,\partial y} + 9\dfrac{\partial^2 u}{\partial y^2} = 0$

20. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x\,\partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0$

21. $\dfrac{\partial^2 u}{\partial x^2} = 9\dfrac{\partial^2 u}{\partial x\,\partial y}$

22. $\dfrac{\partial^2 u}{\partial x\,\partial y} - \dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} = 0$

23. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} - 6\dfrac{\partial u}{\partial y} = 0$

24. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u$

25. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

26. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k > 0$

In Problems 27 and 28 show that the given partial differential equation possesses the indicated product solution.

27. $k\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) = \dfrac{\partial u}{\partial t};$
$u = e^{-k\alpha^2 t}\big(c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)\big)$

28. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0;$
$u = (c_1 \cos \alpha\theta + c_2 \sin \alpha\theta)(c_3 r^{\alpha} + c_4 r^{-\alpha})$

29. Verify that each of the products $u = XY$ in (3), (4), and (5) satisfies the second-order PDE in Example 1.

30. Definition 12.1.1 generalizes to linear PDEs with coefficients that are functions of x and y. Determine the regions in the xy-plane for which the equation

$$(xy + 1)\frac{\partial^2 u}{\partial x^2} + (x + 2y)\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} + xy^2 u = 0$$

is hyperbolic, parabolic, or elliptic.

12.2 CLASSICAL PDEs AND BOUNDARY-VALUE PROBLEMS

REVIEW MATERIAL

- Reread the material on boundary-value problems in Sections 4.1, 4.3, and 5.2.

INTRODUCTION We are not going to solve anything in this section. We are simply going to discuss the types of partial differential equations and boundary-value problems that we will be working with in the remainder of this chapter as well as in Chapters 13–15. The words *boundary-value problem* have a slightly different connotation than they did in Sections 4.1, 4.3, and 5.2. If, say, $u(x, t)$ is a solution of a PDE, where x represents a spatial dimension and t represents time, then we may be able to prescribe the value of u, or $\partial u/\partial x$, or a linear combination of u and $\partial u/\partial x$ at a specified x as well as to prescribe u and $\partial u/\partial t$ at a given time t (usually, $t = 0$). In other words, a "boundary-value problem" may consist of a PDE, along with boundary conditions *and* initial conditions.

CLASSICAL EQUATIONS We shall be concerned principally with applying the method of separation of variables to find product solutions of the following classical equations of mathematical physics:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad k > 0 \tag{1}$$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

or slight variations of these equations. The PDEs (1), (2), and (3) are known, respectively, as the **one-dimensional heat equation,** the **one-dimensional wave equation,** and the **two-dimensional form of Laplace's equation.** "One-dimensional" in the case of equations (1) and (2) refers to the fact that x denotes a spatial variable, whereas t represents time; "two-dimensional" in (3) means that x and y are both spatial variables. If you compare (1)–(3) with the linear form in Theorem 12.1.1 (with t playing