WHITTAKER'S EQUATION:

Using energy integral,

$$L - \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} = h \quad \text{---(1)}$$

$s = 1, 2, \ldots, n$

or $L = L(q_s, \dot{q}_s)$

Whittaker's replace $\dot{q}_s$ into $\dot{q}_1$ &

$\dot{q}_2, \dot{q}_3, \ldots \ldots, \dot{q}_n$

and Lagrangian denoted by

$$L(q_s, \dot{q}_s) = \Omega(q_s, \dot{q}_1, \dot{q}_r) \quad \text{---(2)}$$

$r = 2, 3, \ldots, n$

Now,

$$\dot{q}_r = \frac{dq_r}{dt} = \frac{dq_r}{dq_1} \cdot \frac{dq_1}{dt}$$

$$\dot{q}_r = q_r' \dot{q}_1$$

so

$$q_r' = \frac{\dot{q}_r}{\dot{q}_1}$$

$$\frac{\partial q_r'}{\partial \dot{q}_r} = \frac{1}{\dot{q}_1} \quad \& \quad \frac{\partial q_r'}{\partial \dot{q}_1} = -\frac{\dot{q}_r}{\dot{q}_1^2}$$

As $L(q_s, \dot{q}_s) = \Omega(q_s, \dot{q}_1, q_r')$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial \Omega}{\partial \dot{q}_1} + \frac{\partial \Omega}{\partial q_r'} \cdot \frac{\partial q_r'}{\partial \dot{q}_1}$$

$$= \frac{\partial \Omega}{\partial \dot{q}_1} - \frac{\dot{q}_r}{\dot{q}_1^2} \cdot \frac{\partial \Omega}{\partial q_r'} \quad \text{——(3)}$$

ω,

$$\frac{\partial L}{\partial \dot{q}_r} = \frac{\partial \Omega}{\partial q_r'} \cdot \frac{\partial q_r'}{\partial \dot{q}_r} = \frac{1}{\dot{q}_1} \frac{\partial \Omega}{\partial q_r'} \quad \text{——(4)}$$

Eq (3) $\Rightarrow$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial \Omega}{\partial \dot{q}_1} - \frac{\dot{q}_r}{\dot{q}_1}\left(\frac{1}{\dot{q}_1} \frac{\partial \Omega}{\partial q_r'}\right)$$

$$= \frac{\partial \Omega}{\partial \dot{q}_1} - \frac{\dot{q}_r}{\dot{q}_1} \frac{\partial L}{\partial \dot{q}_r} \quad \text{(Use (4))}$$

$$\dot{q}_1 \frac{\partial L}{\partial \dot{q}_1} + \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} = \dot{q}_1 \frac{\partial \Omega}{\partial \dot{q}_1}$$

$$\dot{q}_s \frac{\partial L}{\partial \dot{q}_s} = \dot{q}_1 \frac{\partial \Omega}{\partial \dot{q}_1} \quad \text{——(5)}$$

So eq (1) $\Rightarrow$

$$\Omega - \dot{q}_1 \frac{\partial \Omega}{\partial \dot{q}_1} = h \quad \text{——(6)}$$

Differentiate w.r.t q_r',

$$\cancel{\frac{\partial \Omega}{\partial \dot{q}_1} \cdot \frac{\partial \dot{q}_1}{\partial q_r'}} + \frac{\partial \Omega}{\partial q_r'} - \dot{q}_1 \left[\left(\frac{\partial}{\partial \dot{q}_1}\left(\frac{\partial \Omega}{\partial \dot{q}_1}\right)\frac{\partial \dot{q}_1}{\partial q_r'}\right.\right.$$

$$\left.\left. + \frac{\partial}{\partial q_r'}\left(\frac{\partial \Omega}{\partial \dot{q}_1}\right)\right)\right] - \cancel{\frac{\partial \Omega}{\partial \dot{q}_1} \frac{\partial \dot{q}_1}{\partial q_r'}} = 0$$

$$\frac{\partial \Omega}{\partial q_r'} = \frac{\partial^2 \Omega}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_r'} + \frac{\partial^2 \Omega}{\partial q_r' \partial \dot{q}_1} \quad \text{--- (7)}$$

eq (6) w.r.t q_r,

$$\frac{\partial \Omega}{\partial q_r} = \frac{\partial^2 \Omega}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_r} + \frac{\partial^2 \Omega}{\partial q_r \partial \dot{q}_1} \quad \text{--- (8)}$$

define $L' = \frac{\partial \Omega}{\partial \dot{q}_1}$ (say)

$$\frac{\partial}{\partial q_r'}\left(\frac{\partial \Omega}{\partial \dot{q}_1}\right)$$

$$= \frac{\partial^2 \Omega}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_r'} + \frac{\partial^2 \Omega}{\partial \dot{q}_1 \partial q_r'}$$

(Using (7))

$$= \frac{1}{\dot{q}_1}\frac{\partial \Omega}{\partial q_r'}$$

(Using (4))

$$= \frac{\partial L}{\partial \dot{q}_r} \quad \text{--- (9)}$$

$$= \frac{\partial^2 \Omega}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_r} + \frac{\partial^2 \Omega}{\partial \dot{q}_1 \partial q_r}$$

(Using (8))

$$= \frac{1}{\dot{q}_1}\frac{\partial \Omega}{\partial q_r}$$

$$= \frac{1}{\dot{q}_r}\frac{\partial L}{\partial q_r}$$

$$\text{--- (10)}$$

$$\frac{\partial L}{\partial q_r} = \dot{q}_1 \frac{\partial L'}{\partial q_r}$$

Lagrange's eq. of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$$