prove

$\sum (\delta p_i\, dq_i - \delta q_i\, dp_i) = \sum_i (\delta P_i\, dQ_i - \delta Q_i\, dP_i)$

the Bilinear covariant and show that transformation is cannonical i77

$(\delta p_i\, dq_i - \delta q_i\, dp_i)$ —— (1)

invariant under this transformation using L.B

Answer :- The relation

$\sum_i (\delta p_i\, dq_i - \delta q_i\, dp_i)$ is linear in δp_i & dp_i and also linear δq_i & dq_i

Therefore, it is called bilinear covariant or invariant.

Consider $p_i = p_i (Q_k, P_k)$

$q_i = q_i (Q_k, P_k)$

$\sum_i (\delta p_i\, dq_i - \delta q_i\, dp_i)$

$$\left(\frac{\partial p_i}{\partial Q_k}\delta Q_k + \frac{\partial p_i}{\partial P_k}\delta P_k\right) * \left(\frac{\partial q_i}{\partial Q_m}\, dQ_m + \frac{\partial q_i}{\partial P_m}\, dP_m\right) -$$

$$\left(\frac{\partial q_i}{\partial Q_k}\delta Q_k + \frac{\partial q_i}{\partial P_k}\delta P_k\right)\left(\frac{\partial p_i}{\partial Q_m}\, dQ_m + \frac{\partial p_i}{\partial P_m}\, dP_m\right)$$

$$\left(\frac{\partial p_i}{\partial Q_k}\cdot\frac{\partial q_i}{\partial Q_m} - \frac{\partial q_i}{\partial Q_k}\cdot\frac{\partial p_i}{\partial Q_m}\right)\delta Q_k\, dQ_m$$

$$+\left(\frac{\partial p_i}{\partial P_k}\cdot\frac{\partial q_i}{\partial P_m} - \frac{\partial q_i}{\partial P_k}\cdot\frac{\partial p_i}{\partial P_m}\right)\delta P_k\, dP_m$$

$$+\left(\frac{\partial p_i}{\partial Q_k}\cdot\frac{\partial q_i}{\partial P_m} - \frac{\partial q_i}{\partial Q_k}\cdot\frac{\partial p_i}{\partial P_m}\right)\delta Q_k\, dP_m$$

$$+\left(\frac{\partial p_i}{\partial P_k}\cdot\frac{\partial q_i}{\partial Q_m} - \frac{\partial q_i}{\partial P_k}\cdot\frac{\partial p_i}{\partial Q_m}\right)\delta P_k\, dQ_m$$

By using definition of "Lagrange Brackets"

$[Q_m, Q_k]\, \delta Q_k\, dQ_m + [P_m, P_k]\, \delta P_k\, dP_m$

$+ [P_m, Q_k]\, \delta Q_k\, dP_m + [Q_m, P_k]\, \cancel{\delta Q_m}\, \delta P_k\, dQ_m$

$= 0 + 0 + (-\delta_{km}\, \delta Q_k)dP_m) + (\delta_{mk}\, \delta P_k)\, dQ_m$

$= -\delta Q_m\, dP_m + \delta P_m\, dQ_m$

$= \sum_i (\delta P_i\, dQ_i - \delta Q_i\, dP_i)$

Now, conversely,

Consider $P_i = P_i\,(q_k, p_k)$

$Q_i = Q_i\,(q_k, p_k)$

$\sum_i (\delta P_i\, dQ_i - \delta Q_i\, dP_i)$

$= \left(\frac{\partial P_i}{\partial q_k}\delta q_k + \frac{\partial P_i}{\partial p_k}\delta p_k\right)\left(\frac{\partial Q_i}{\partial q_m}dq_m + \frac{\partial Q_i}{\partial p_m}dp_m\right)$

$- \left(\frac{\partial Q_i}{\partial q_k}\delta q_k + \frac{\partial Q_i}{\partial p_k}\delta p_k\right)\left(\frac{\partial P_i}{\partial q_m}dq_m + \frac{\partial P_i}{\partial p_m}dp_m\right)$

$= \left(\frac{\partial P_i}{\partial q_k}\frac{\partial Q_i}{\partial q_m} - \frac{\partial Q_i}{\partial q_k}\cdot\frac{\partial P_i}{\partial q_m}\right)\delta q_k\, dq_m$

$+ \left(\frac{\partial P_i}{\partial q_k}\cdot\frac{\partial Q_i}{\partial p_m} - \frac{\partial Q_i}{\partial q_k}\cdot\frac{\partial P_i}{\partial p_m}\right)\delta q_k\, dp_m$

$+ \left(\frac{\partial P_i}{\partial p_k}\cdot\frac{\partial Q_i}{\partial q_m} - \frac{\partial Q_i}{\partial p_k}\cdot\frac{\partial P_i}{\partial q_m}\right)\delta p_k\, dq_m$

$+ \left(\frac{\partial P_i}{\partial p_k}\cdot\frac{\partial Q_i}{\partial p_m} - \frac{\partial Q_i}{\partial p_k}\cdot\frac{\partial P_i}{\partial p_m}\right)\delta p_k\, dp_m$

using def of "Lagrange Brackets"

$$[q_m, q_k]\,\delta q_k\, dq_m + [p_m, q_k]\,\delta q_k\, dp_m$$
$$+ [q_m, p_k]\,\delta p_k\, dq_m + [p_m, p_k]\,\delta p_k\, dp_m$$

$$= 0 - (\delta_{km}\,\delta q_k)\, dp_m + (\delta_{mk}\,\delta p_k)\, dq_m + 0$$

$$= -\delta q_m\, dp_m + \delta p_m\, dq_m$$
$$= \sum_i (\delta p_i\, dq_i - \delta q_i\, dp_i)$$

Hence Proved.

Question:-

Show by method of bilinear covariant the transf. defined by

$$Q = \sqrt{e^{-2q} - p^2}$$

& $$P = \cos^{-1}(pe^{q})$$

is cannonical or contact transf.

Prove:- $\delta P dQ - \delta Q dP = \delta p dq - \delta q dp$

Solution:- As

$$Q = \sqrt{e^{-2q} - p^2}$$

$$\frac{\partial Q}{\partial q} = \frac{-e^{-2q}}{\sqrt{e^{-2q} - p^2}}$$

$$\frac{\partial Q}{\partial p} = \frac{-p}{\sqrt{e^{-2q} - p^2}}$$

Also, as $P = \cos^{-1}(pe^{q})$

so $$\frac{\partial P}{\partial q} = \frac{-1}{\sqrt{1 - p^2 e^{2q}}}\,(pe^{q})$$

& $\dfrac{\partial P}{\partial p} = \dfrac{-1}{\sqrt{1-p^2e^{2q}}}\, e^{q}$

Now, $\delta P\, dQ - \delta Q\, dP$

$$= \left(\frac{\partial P}{\partial q}\delta q + \frac{\partial P}{\partial p}\delta p\right)\left(\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp\right)$$

$$- \left(\frac{\partial Q}{\partial q}\delta q + \frac{\partial Q}{\partial p}\delta p\right)\left(\frac{\partial P}{\partial q}dq + \frac{\partial P}{\partial p}dp\right)$$

By putting values, we get

$$= \left(\frac{-pe^{q}}{\sqrt{1-p^2e^{2q}}}\delta q - \frac{e^{q}}{\sqrt{1-p^2e^{2q}}}\delta p\right)\left(\frac{-e^{-2q}}{\sqrt{e^{-2q}-p^2}}dq\right.$$

$$\left. + \frac{p}{\sqrt{e^{-2q}-p^2}}dp\right) - \left(\frac{-e^{-2q}}{\sqrt{e^{-2q}-p^2}}\delta q\right.$$

$$\left. + \frac{p}{\sqrt{e^{-2q}-p^2}}\delta p\right)\left(\frac{-pe^{q}}{\sqrt{1-p^2e^{2q}}}dq - \frac{e^{q}}{\sqrt{1-p^2e^{2q}}}dp\right)$$

$$\cancel{\frac{+pe^{-q}}{e^{2q}(e^{-2q}-p^2)}}$$

$$\left(\frac{pe^{-q}}{(\sqrt{1-p^2e^{2q}})(\sqrt{e^{-2q}-p^2})} - \frac{pe^{-q}}{(\sqrt{1-p^2e^{2q}})(\sqrt{e^{-2q}-p^2})}\right)\delta q\, dq$$

$$+\left(\frac{p^2e^{q}}{(\sqrt{1-p^2e^{2q}})(\sqrt{e^{-2q}-p^2})} - \frac{e^{-q}}{(\sqrt{1-p^2e^{2q}})(\sqrt{e^{-2q}-p^2})}\right)\delta q\, dp$$

$$+\left(\frac{e^{-q}}{(\sqrt{1-p^2e^{2q}})(\sqrt{e^{-2q}-p^2})} - \frac{p^2e^{q}}{(\sqrt{1-p^2e^{2q}})(\sqrt{e^{-2q}-p^2})}\right)\delta p\, dq$$

$$+\left(\frac{pe^{q}}{(\sqrt{1-p^2e^{2q}})(\sqrt{e^{-2q}-p^2})} - \frac{pe^{q}}{(\sqrt{1-p^2e^{2q}})(\sqrt{e^{-2q}-p^2})}\right)\delta p\, dp$$

$$\frac{e^{-q}(1-p^2e^{2q})}{e^{-q}(\sqrt{1-p^2e^{2q}})(\sqrt{1-p^2e^{2q}})}\delta q\,dp$$

$$+\frac{e^{-q}(1-p^2e^{2q})}{e^{-q}(\sqrt{1-p^2e^{2q}})(\sqrt{1-p^2e^{2q}})}\delta p\,dq$$

$$\delta p\,dq - \delta q\,dp$$

Hence Proved!

stion:- Show by method of bilinear *iant* that the transf.

$$Q = \ln\left(\frac{\sin p}{q}\right)$$

&

$$P = q\cot p$$

canonical or contact transf.

: $\delta P dQ - \delta Q dP = \delta p\,dq - \delta q\,dp$

ion:- As $Q = \ln\left(\frac{\sin p}{q}\right)$

$$= \ln(\sin p) - \ln q$$

$$\frac{\partial Q}{\partial q} = -\frac{1}{q}$$

$$\frac{\partial Q}{\partial p} = \frac{1}{\sin p}\cdot\cos p = \cot p$$

$$P = q\cot p$$

$$\frac{\partial P}{\partial q} = \cot p$$

$$\frac{\partial P}{\partial p} = q(-\operatorname{cosec}^2 p)$$

$$= -q\operatorname{cosec}^2 p$$

Now, $\cancel{\delta P d}\ \frac{\partial P}{\partial p} = -q(1+\cot^2 p)$