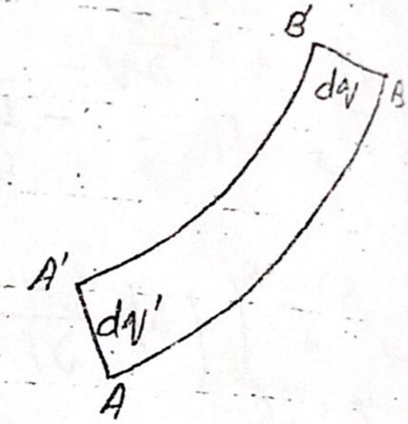
Hamilton – Jacobi Equation

Using integral action

$$S = \int_A^B (p\,dq - H\,dt)$$



then

distribution from A to $A' = \delta q'$

" " B to $B' = \delta q$

b/w two different curves.

Change in $S = \delta S = \delta \int_A^B (p\,dq - H\,dt)$

$$= \int_A^B \left[\delta p\,dq + p\,\delta(dq) - \delta H\,dt - H\,\delta(dt)\right]$$

$$= \int_A^B \left[\delta p\,dq + p\,d(\delta q) - \delta H\,dt - H\,d(\delta t)\right]$$

$$= \left| p\,\delta q - H\,\delta t \right|_A^B + \int_A^B (\delta p\,dq - \delta q\,dp - \delta H\,dt + \delta t\,dH)$$

As

$$\delta H = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial t}\delta t$$

Use in (1),

$$\delta S = \left| p\,\delta q - H\,\delta t \right|_A^B + \int_A^B \left[\delta p\left(\dot q - \frac{\partial H}{\partial p}\right) - \delta q\left(\dot p + \frac{\partial H}{\partial q}\right) + \delta t\left(\dot H - \frac{\partial H}{\partial t}\right)\right] dt$$

ng cannonical eqs., we get

$$s = \left| p\delta q - H\delta t \right|_A^B$$

$$= p\delta q - H\delta t - p'\delta q' + H'\delta t' \quad \text{---} (2)$$

$$\delta s = \delta s\,(\delta q, \delta t, \delta q', \delta t')$$

$$s = s\,(q, t, q', t')$$

h is known as Hamilton's characteristic

. Eq. (2) implies

$$\frac{\partial s}{\partial q} = p \quad ; \quad \frac{\partial s}{\partial t} = -H$$

$$\frac{\partial s}{\partial q'} = -p' \quad ; \quad \frac{\partial s}{\partial t'} = H'$$

g

$$\frac{\partial s}{\partial t} + H = 0$$

$$\frac{\partial s}{\partial t} + H\,(q, p, t) = 0$$

$$\frac{\partial s}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0$$

$$+ H\left(q_1, q_2, \ldots, q_n, \frac{\partial s}{\partial q_1}, \ldots, \frac{\partial s}{\partial q_n}, t\right) = 0$$

this eq. is called Hamilton's Jacobi

— in terms of 's' (characteristic fun.)

ial Transformation:-

t Transformation:- If there exist a

formation from one set of generalized

rdinates $q_1, q_2, q_3, \ldots, q_n$ to another

of generalized co-ordinates $Q_1, Q_2, \ldots, Q_n$

h a transformation is described by (called pt. transf. s)

set of eqs.

$$Q_i = Q_i\,(q_1, q_2, \ldots, q_n)$$

e inverse transformation is given by

$$q_i = q_i\,(Q_1, Q_2, \ldots, Q_n)$$

For Lagrangian Equation: As Lagrangian is represented by $L(q, \dot{q}; t)$ &

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L(q, \dot{q}; t)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j} L(q, \dot{q}; t) = 0$$

In terms of new co-ordinates,

$$\frac{d}{dt}\left(\frac{\partial L(Q, \dot{Q}, t)}{\partial \dot{Q}_j}\right) - \frac{\partial L}{\partial Q_j}(Q, \dot{Q}, t) = 0$$

It shows that Lagrangian eq. remains invariant under pt. transformation.

Theorem:

In a dynamical system defined by generalized co-ordinates $q_1, q_2, \ldots, q_n$ described by $H(q, p, t)$. If a new set of generalized co-ordinates $Q_1, Q_2, \ldots$ which are related to 'q' by the pt. transformation

$$Q_p = Q_p(q, t)$$

then the corresponding momenta P & the hamilton K are given by

$$P_j = \sum p_p \frac{\partial q_p}{\partial Q_j}(Q, t)$$

&

$$K = H - \sum p_p \frac{\partial q_p}{\partial t}(Q, t)$$