AFTER MID TERM

Lagrange Bracket:

This is defined as

$$[u, v]_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right)$$

$$= \sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix}$$

$$= \sum_i \frac{\partial (q_i, p_i)}{\partial (u, v)}$$

Properties of Lagrange Bracket:

1. Lagrange brackets are invariant under cannonical transformation i.e

$$[u, v]_{q,p} = [u, v]_{Q,P} \quad \text{(By pt. trans}$$

2. Lagrange bracket is non commutative

i.e $[u, v] = -[v, u]$

L.H.S.

$$[u, v] = \frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v}$$

$$= -\left[\frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} - \frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} \right]$$

$$= -\left[\frac{\partial q_i}{\partial v} \frac{\partial p_i}{\partial u} - \frac{\partial p_i}{\partial v} \frac{\partial q_i}{\partial u} \right]$$

$$= -[v, u] = R.H.S$$

$[q_i, q_j]_{q,p} = 0$

L.H.S.

$[q_i, q_j]_{q,p} = \frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial q_j} - \frac{\partial p_k}{\partial q_i}\frac{\partial q_k}{\partial q_j}$

$= \delta_{ki}(0) - (0)\delta_{kj}$

$= 0 \quad = R.H.S$

$[p_i, p_j]_{q,p} = 0$

L.H.S.

$[p_i, p_j]_{q,p} = \frac{\partial q_i}{\partial p_i}\frac{\partial p_i}{\partial p_j} - \frac{\partial p_i}{\partial p_i}\frac{\partial q_i}{\partial p_j}$

$= 0$

$[q_i, p_j] = \delta_{ij}$

L.H.S.

$\underbrace{[q_i, p_j]}_{\text{fun}}{}_{\underbrace{q,p}_{\text{index}}} = \frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial p_j} - \frac{\partial p_k}{\partial q_i}\frac{\partial q_k}{\partial p_j}$

$= \delta_{ki}\,\delta_{kj} - 0$

$= \delta_{ij}$ while $[p_j, q_i] = -\delta_{ij}$

If the index & fun. are same then ans. is '1' otherwise zero

Poisson Bracket:- It is defined as

$[u, v]_{q,p} = \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} \right)$

Properties Of Poisson Bracket:-

$[u, v]_{q,p} = -[v, u]_{q,p}$

$[u,u] = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial u}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial u}{\partial q_i} \right) = 0$

$[u,c]_{q,p} = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial c}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial c}{\partial q_i} \right) = 0$

$[q_i, q_j] = \frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k}$

$= 0$

$[q_i, p_j]_{q,p} = \frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k}$

$= \delta_{ik} \delta_{jk} = \delta_{ij}$

ESTION:- Show that Lagrange & Poisson Brackets are reciprocal to each other.

R :

u_l where $l = 1, 2, \ldots, 2n$ be a fun. of $2n$ variables

$q_1, q_2, \ldots, q_n \quad p_1, p_2, \ldots, p_n$

then $\sum_{l=1}^{2n} [u_l, u_i]\{u_l, u_j\} = \delta_{ij}$

olution:-

Using definition of two brackets,

$\sum_{l=1}^{2n} [u_l, u_i]\{u_l, u_j\}$

$\sum_{l=1}^{2n} \left(\frac{\partial q_k}{\partial u_l} \frac{\partial p_k}{\partial u_i} - \frac{\partial p_k}{\partial u_l} \frac{\partial q_k}{\partial u_i} \right) \left(\frac{\partial u_l}{\partial q_m} \frac{\partial u_j}{\partial p_m} - \frac{\partial u_l}{\partial p_m} \frac{\partial u_j}{\partial q_m} \right)$

where $k = m = 1, 2, \ldots n$

$$\sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_j}\cdot\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_l}{\partial q_m}\frac{\partial u_j}{\partial p_m}-\frac{\partial q_k}{\partial u_i}\frac{\partial p_k}{\partial u_i}\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right.$$

$$\left.-\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_l}{\partial q_m}\cdot\frac{\partial u_j}{\partial p_m}+\frac{\partial p_k}{\partial u_l}\frac{\partial q_k}{\partial u_i}\frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right)$$

$$\left(\frac{\partial q_k}{\partial q_m}\cdot\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial p_m}\right)-\sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial p_m}\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}\right)$$

$$\left(\frac{\partial p_k}{\partial q_m}\cdot\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial p_m}\right)+\sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial p_m}\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial q_m}\right)$$

$$\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial p_m}-0-0+\sum_{k,m=1}^{n}\delta_{km}\frac{\partial q_k}{\partial u_i}\frac{\partial u_j}{\partial q_m}$$

$$\left(\frac{\partial p_k}{\partial u_i}\frac{\partial u_j}{\partial p_k}+\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial q_k}\right)\qquad\left(\delta_{ij}A_i=A_j\right)$$

using $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial x}$

$$\frac{\partial u_j}{\partial u_i}+\frac{\partial u_j}{\partial u_i}=\frac{\partial u_j}{\partial u_i}=\delta_{ij}$$

As Req.

...utive Property In Case Of Poisson Bracket:

$$[u, v+w]=[u,v]+[u,w]$$

s.

Using Poisson Bracket

$$[u, v+w]=\sum_{k=1}^{n}\left(\frac{\partial u}{\partial q_k}\cdot\frac{\partial}{\partial p_k}(v+w)-\frac{\partial}{\partial q_k}(v+w)\frac{\partial u}{\partial p_k}\right)$$

$$=\sum\left(\frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k}+\frac{\partial u}{\partial q_k}\frac{\partial w}{\partial p_k}-\frac{\partial v}{\partial q_k}\frac{\partial u}{\partial p_k}\right.$$

$$\left.-\frac{\partial w}{\partial q_k}\frac{\partial u}{\partial p_k}\right)$$

$$\sum_i \left(\frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial v}{\partial q_k} \frac{\partial u}{\partial p_k} \right) + \sum_i \left(\frac{\partial u}{\partial q_k} \frac{\partial w}{\partial p_k} - \frac{\partial w}{\partial q_k} \frac{\partial u}{\partial p_k} \right)$$

$$= [u, v] + [u, w]$$

Show that

$$\frac{\partial}{\partial t}[u, v] = \left[\frac{\partial u}{\partial t}, v \right] + \left[u, \frac{\partial v}{\partial t} \right]$$

Sol. L.H.S.

$$\frac{\partial}{\partial t}[u, v] = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right)$$

$$= \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial q_k} \right) \frac{\partial v}{\partial p_k} + \frac{\partial u}{\partial q_k} \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial p_k} \right)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial p_k} \right) \frac{\partial v}{\partial q_k} + - \frac{\partial u}{\partial p_k} \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial q_k} \right)$$

$$= \frac{\partial}{\partial q_k} \left(\frac{\partial u}{\partial t} \right) \frac{\partial v}{\partial p_k} + \frac{\partial}{\partial p_k} \left(\frac{\partial v}{\partial t} \right) \frac{\partial u}{\partial q_k} - \frac{\partial}{\partial p_k} \left(\frac{\partial u}{\partial t} \right) \left(\frac{\partial v}{\partial q_k} \right)$$

$$\left(\frac{\partial u}{\partial p_k} \right) \left(\frac{\partial}{\partial q_k} \left(\frac{\partial v}{\partial t} \right) \right)$$

$$= \left(\frac{\partial}{\partial q_k} \left(\frac{\partial u}{\partial t} \right) \frac{\partial v}{\partial p_k} - \frac{\partial}{\partial p_k} \left(\frac{\partial u}{\partial t} \right) \frac{\partial v}{\partial q_k} \right) + \left(\frac{\partial}{\partial p_k} \left(\frac{\partial v}{\partial t} \right) \frac{\partial u}{\partial q_k} \right.$$

$$\left. - \frac{\partial}{\partial q_k} \left(\frac{\partial v}{\partial t} \right) \frac{\partial u}{\partial p_k} \right)$$

$$= \left[\frac{\partial u}{\partial t}, v \right] + \left[u, \frac{\partial v}{\partial t} \right]$$

$$= R.H.S.$$

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