$Q_k = V_k$

Question:

Show that $Q = \ln(1 + \sqrt{q}\cos p)$

$P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p$

Solution: To show that the given transform canonical, we will find its corresponding generating fun. (G.F)

$$F = F_1(q, Q, t)$$

$$p_i = \frac{\partial F_1}{\partial q_i} \quad ; \quad P_i = \frac{\partial F_1}{\partial Q_i}$$

From eq (1),

$$e^Q = 1 + \sqrt{q}\cos p$$

$$\Rightarrow \cos p = \frac{e^Q - 1}{\sqrt{q}}$$

so

$$p = \cos^{-1}\left(\frac{e^Q - 1}{\sqrt{q}}\right)$$

comparing!

$$\frac{\partial F_1}{\partial q} = \cos^{-1}\left(\frac{e^Q - 1}{\sqrt{q}}\right)$$

F_1 is not a suitable choice.

now, take $F = F_2(q, P, t)$

$$p = \frac{\partial F_2}{\partial q} \quad ; \quad Q = \frac{\partial F_2}{\partial P}$$

From eq (2),

$$P = 2e^Q\sqrt{q}\sin p$$

$$\sin p = \frac{P}{2e^Q\sqrt{q}}$$

$$p = \sin^{-1}\left(\frac{P}{2e^Q\sqrt{q}}\right)$$

$$\frac{\partial F_2}{\partial q} = \sin^{-1}\left(\frac{P}{2e^Q\sqrt{q}}\right)$$

F_2 is not suitable choice

...w, take $F = F_3(p, Q, t)$

$$q = \frac{\partial F_3}{\partial p} \quad ; \quad P = \frac{\partial F_3}{\partial Q}$$

...om eq. (1),

$$e^Q = 1 + \sqrt{q}\cos p$$

$$\left(\frac{e^Q - 1}{\cos p}\right)^2 = q$$

$$q = (e^Q - 1)^2 \sec^2 p$$

$$\frac{\partial F_3}{\partial p} = -(e^Q - 1)^2 \sec^2 p$$

...egrate!

$$F_3 = -(e^Q - 1)^2 \tan p + c$$

...ow, Differentiate w.r.t Q.

$$\frac{\partial F_3}{\partial Q} = -2(e^Q - 1)\, e^Q \tan p$$

$$= -2(1 + \sqrt{q}\cos p - 1)(1 + \sqrt{q}\cos p)\frac{\sin p}{\cos p}$$

$$= -2\sqrt{q}(1 + \sqrt{q}\cos p)\sin p$$

$$\frac{\partial F_3}{\partial Q} = P$$

Hence F_3 is required Generating fun. (G.F).

Transformation is cannonical

...how By definition the transformation

$$Q = \frac{1}{\sqrt{2i}}(q + ip) \qquad (1)$$

$$P = \frac{-1}{\sqrt{2i}}(q - ip) \qquad (2)$$

is cannonical.

To prove

$$\dot{Q} = \frac{\partial K}{\partial P} \; ; \quad \dot{P} = -\frac{\partial K}{\partial Q}$$

$$H = K$$

$$H(q, p) = K(Q, P)$$

$$q = q(Q, P)$$

$$p = p(Q, P)$$

$$\dot{q} = \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial P}\dot{P} \quad \text{—— (1)}$$

$$\dot{p} = \frac{\partial p}{\partial Q}\dot{Q} + \frac{\partial p}{\partial P}\dot{P} \quad \text{—— (2)}$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial K(Q, P)}{\partial p}$$

$$\dot{q} = \frac{\partial K}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial p} \quad \text{—— (3)}$$

$$-\dot{p} = \frac{\partial H}{\partial q} = \frac{\partial K(Q, P)}{\partial q}$$

$$\dot{p} = -\frac{\partial K}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial K}{\partial P}\frac{\partial P}{\partial q} \quad \text{—— (4)}$$

From eq. (1),

$$\frac{\partial Q}{\partial q} = \frac{1}{\sqrt{2i}} \; ; \quad \frac{\partial Q}{\partial p} = \frac{i}{\sqrt{2i}} = \sqrt{\frac{i}{2}}$$

From eq. (2),

$$\frac{\partial P}{\partial q} = \frac{-1}{\sqrt{2i}} \; ; \quad \frac{\partial P}{\partial p} = \frac{i}{\sqrt{2i}} = \sqrt{\frac{i}{2}}$$

in eq. (3) & (4),

$$\dot{q} = \sqrt{\frac{i}{2}}\frac{\partial K}{\partial Q} + \sqrt{\frac{i}{2}}\frac{\partial K}{\partial P} \quad \text{--- (5)}$$

$$\dot{p} = -\frac{1}{\sqrt{2i}}\frac{\partial K}{\partial Q} + \frac{1}{\sqrt{2i}}\frac{\partial K}{\partial P} \quad \text{--- (6)}$$

As $Q = \frac{1}{\sqrt{2i}}(q + ip)$ & $P = -\frac{1}{\sqrt{2i}}(q - ip)$

$dQ = \frac{1}{\sqrt{2i}}(dq + idp)$ & $dP = -\frac{1}{\sqrt{2i}}(dq - idp)$

On adding, we have

$$\frac{i2\,dp}{\sqrt{2i}} = dQ + dP$$

$$dp = \frac{1}{\sqrt{2i}}dQ + \frac{1}{\sqrt{2i}}dP \quad \text{--- (7)}$$

On subtracting,

$$\frac{2}{\sqrt{2i}}dq = dQ \mp dP$$

So

$$dq = \sqrt{\frac{i}{2}}dQ \mp \sqrt{\frac{i}{2}}dP \quad \text{--- (8)}$$

As

$$q = q(Q, P)$$

$$dq = \frac{\partial q}{\partial Q}dQ + \frac{\partial q}{\partial P}dP \quad \text{--- (9)}$$

As

$$p = p(Q, P)$$

$$dp = \frac{\partial p}{\partial Q}dQ + \frac{\partial p}{\partial P}dP \quad \text{--- (10)}$$

On comparing (8) & (9),

$$\frac{\partial q}{\partial Q} = \sqrt{\frac{i}{2}} \quad \& \quad \frac{\partial q}{\partial P} = -\sqrt{\frac{i}{2}}$$

On comparing (7) & (10),

$$\frac{\partial p}{\partial Q} = \frac{1}{\sqrt{2i}} \quad \& \quad \frac{\partial p}{\partial P} = \frac{1}{\sqrt{2i}}.$$

sing these values in eq. (1) & (2),

$$\dot{q} = \sqrt{\frac{i}{2}}\, \dot{Q} - \sqrt{\frac{i}{2}}\, \dot{P} \qquad (11)^*$$

$$\dot{p} = \frac{1}{\sqrt{2i}}\, \dot{Q} + \frac{1}{\sqrt{2i}}\, \dot{P} \qquad (12)^*$$

m eq. (5) & (11)*,

$$\sqrt{\frac{i}{2}}\frac{\partial K}{\partial Q} + \sqrt{\frac{i}{2}}\frac{\partial K}{\partial P} = \sqrt{\frac{i}{2}}\, \dot{Q} - \sqrt{\frac{i}{2}}\, \dot{P}$$

$$\frac{\partial K}{\partial Q} - \left(-\frac{\partial K}{\partial P}\right) = \dot{Q} - \dot{P}$$

$$\frac{\partial K}{\partial P} + \frac{\partial K}{\partial Q} = \dot{Q} - \dot{P} \qquad (11)$$

w,

om eq. (6) & (12)*,

$$-\frac{1}{\sqrt{2i}}\frac{\partial K}{\partial Q} + \frac{1}{\sqrt{2i}}\frac{\partial K}{\partial P} = \frac{1}{\sqrt{2i}}\, \dot{Q} + \frac{1}{\sqrt{2i}}\, \dot{P}$$

$$-\frac{\partial K}{\partial Q} + \frac{\partial K}{\partial P} = \dot{Q} + \dot{P}$$

$$\frac{\partial K}{\partial P} - \frac{\partial K}{\partial Q} = \dot{Q} + \dot{P}. \qquad (12)$$

adding eq. (11) & (12),

$$\dot{Q} = \frac{\partial K}{\partial P}$$

Subtracting,

$$\dot{P} = -\frac{\partial K}{\partial Q}$$

ce the given transf. is cannonical transf.