nnonical ~~Momentum~~ Transformation:-

As we know

...t Hamilton's eq. is given by

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \quad \& \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}$$

... transformation from (q_s, p_s, t) to (Q_s, P_s, t) is called canonical or ...ontact transformation if $\exists$ some fun. $K(Q_s, P_s, t)$ s.t

$$\dot{Q}_s = \frac{\partial K}{\partial P_s} \quad \& \quad \dot{P}_s = -\frac{\partial K}{\partial Q_s}$$

Here 'K' is called Kamiltonian. K Plays the role of Hamiltonian in the new variables Q_s, P_s. As Hamilton's variational principle

$$\delta \int L\, dt = 0$$

$$\delta \int (p_i \dot{q}_i - H)\, dt = 0$$

In terms of new co-ordinates

$$\delta \int (P_i \dot{Q}_i - K)\, dt = 0$$

The integrands are differ by total time derivative of some fun (generating funF)

$$(p_i \dot{q}_i - H) - (P_i \dot{Q}_i - K) = \frac{dF}{dt} \qquad (1)$$

F be a fun. of old and new variables called generating function:

It has four types.

$F_1(q, Q; t)$, $F_2(q, P; t)$

$F_3(p, Q; t)$, $F_4(p, P; t)$

et $F = F_1(q, Q; t)$

eq. (1) becomes

$$(p_i \dot{q}_i - H) - (P_i \dot{Q}_i - K) = \frac{d}{dt} F_1(q_i, Q_i; t) \quad \text{—(2)}$$

$$= \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} \left(\frac{dt}{dt}\right)$$

On comparing,

$$p_i = \frac{\partial F_1}{\partial q_i} \quad ; \quad -P_i = \frac{\partial F_1}{\partial Q_i}$$

$$K - H = \frac{\partial F_1}{\partial t}$$

$$\Rightarrow K = H + \frac{\partial F_1}{\partial t}$$

Using

$$\frac{\partial F_1}{\partial Q_i} = -P_i$$

$$\int \frac{\partial F_1}{\partial Q_i} dQ_i = -P_i \int dQ_i$$

$$F_1 = -P_i Q_i$$

$$F_1 + P_i Q_i = F_2 \quad \text{(say)}$$

$$F_1 = F_2 - P_i Q_i$$

Use in eq. (2)

$$(p_i \dot{q}_i - H) - (P_i \dot{Q}_i - K) = \frac{d}{dt}\left[F_2 - P_i Q_i\right]$$

$$= \frac{\partial F_2}{\partial q_i} \dot{q}_i + \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t} - \dot{P}_i Q_i - P_i \dot{Q}_i$$

On comparing:

$$p_i = \frac{\partial F_2}{\partial q_i} \quad ; \quad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$K - H = \frac{\partial F_2}{\partial t}$$

...w, using $p_i = \frac{\partial F_1}{\partial q_i}$

$$\int \frac{\partial F_1}{\partial q_i} dq_i = \int p_i \, dq_i$$

$$F_1 = p_i q_i$$

$$F_1 - p_i q_i = F_3 \quad \text{(say)}$$

$$\frac{\partial F_1}{\partial q_i} - p_i = 0 = \frac{\partial F_3}{\partial q_i}$$

$$\Rightarrow F_1 - p_i q_i = F_3$$

$$F_1 = F_3 + p_i q_i$$

Use in eq (2),

$$(p_i \dot{q}_i - H) - (P_i \dot{Q}_i - K) = \frac{d}{dt}\left[F_3 + p_i q_i\right]$$

$$= \frac{\partial F_3}{\partial p_i} \dot{p}_i + \frac{\partial F_3}{\partial Q_i} \dot{Q}_i + \frac{\partial F_3}{\partial t} + \dot{p}_i q_i + \cancel{p_i \dot{q}_i}$$

On comparing!

$$-P_i = \frac{\partial F_3}{\partial Q_i} \quad \& \quad -\frac{\partial F_3}{\partial p_i} = q_i$$

$$K - H = \frac{\partial F_3}{\partial t} \Rightarrow K = H + \frac{\partial F_3}{\partial t}$$

~~Now, using~~

...ce generating functions F_4 is a double Legendre transform of F_1, thus say

$$F_3 = -P_i Q_i \Rightarrow F_1 - p_i q_i + P_i Q_i = F_4$$

∂q_i

So, $\frac{\partial F_1}{\partial Q_1} = -P_i \Rightarrow F_i = -Q_i P_i \Rightarrow F_1 + P_i Q_i = 0$ (say)

$\Rightarrow F_1 - P_i q_i + Q_i P_i = F_4$

$F_4(p, P; t) = F_1(q, Q; t) - p_i q_i + P_i Q_i$

The transformation eqs. are

$$\frac{\partial F_4}{\partial p_i} = -q_i$$

& $\frac{\partial F_4}{\partial P_i} = Q_i$

&

$$K = H + \frac{\partial F_4}{\partial t}$$

In all these transformations, time remains untransformed quantity.

Example: $F = \Sigma q_i P_i$

Solve it in the form of canonical form?

Sol: Here $F = F_2$

So $p_k = \frac{\partial F_2}{\partial q_k} = \frac{\partial}{\partial q_k}(q_i P_i)$

$$= \frac{\partial q_i}{\partial q_k} P_i + 0$$

$$= \delta_{ik} P_i$$

So $p_k = P_k$

Also, $Q_k = \frac{\partial F_2}{\partial P_k}$

$$= \frac{\partial}{\partial P_k}(\Sigma q_i P_i)$$

$$= q_i \frac{\partial P_i}{\partial P_k}$$

$$= q_i \delta_{ik}$$

So $Q_k = q_k$

Question: