eq. in terms of ...

## Special Transformation:-

Point Transformation:- If there exist a transformation from one set of generalized co-ordinates $q_1, q_2, q_3, \ldots, q_n$ to another set of generalized co-ordinates $Q_1, Q_2, \ldots, Q_n$. Such a transformation is described by a set of eqs. (called pt. transf.)

$$Q_i = Q_i(q_1, q_2, \ldots, q_n)$$

The inverse transformation is given by

$$q_i = q_i(Q_1, Q_2, \ldots, Q_n)$$

or Lagrangian Equation: As Lagrangian fn is represented by $L(q, \dot{q}; t)$ &

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L(q, \dot{q}; t)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j} L(q, \dot{q}; t) = 0$$

In terms of new co-ordinates,

$$\frac{d}{dt}\left(\frac{\partial L(Q, \dot{Q}, t)}{\partial \dot{Q}_j}\right) - \frac{\partial L}{\partial Q_j}(Q, \dot{Q}, t) = 0$$

It shows that Lagrangian eq. remains invariant under pt. transformation.

Theorem:

In a dynamical system defined by generalized co-ordinates $q_1, q_2, \ldots, q_n$ described by $H(q, p, t)$. If a new set of generalized co-ordinates $Q_1, Q_2, \ldots, Q_n$ which are related to $q$ by the pt. transformation

$$Q_p = Q_p(q, t)$$

then the corresponding momenta $P$ & the hamilton $K$ are given by

$$P_j = \Sigma p_p \frac{\partial q_p}{\partial Q_j}(Q, t)$$

&

$$K = H - \Sigma p_p \frac{\partial q_p(Q, t)}{\partial t}$$

For time independent ⇒ Lagrangian = Hamilton

Proof: For dynamical system, the Hamilton is defined as

$$H = \sum_{p=1}^{n} \dot{q}_p p_p - L$$

$$\Rightarrow L = \dot{q}_p p_p - H$$

As the Lagrangian is invariant under a pt. transformation

$$\dot{q}_p p_p - H = \dot{Q}_p P_p - K$$

$$p_p \, dq_p - H dt = P_p \, d\bar{Q}_p - K dt \quad (1)$$

As

$$dq_p = \frac{\partial q_p}{\partial Q_j} dQ_j + \frac{\partial q_p}{\partial t} dt$$

Use in eq (1), we get

$$p_p \frac{\partial q_p}{\partial Q_j} dQ_j + p_p \frac{\partial q_p}{\partial t} dt - H dt = P_p d\bar{Q}_p - K dt$$

$$= P_j d\bar{Q}_j - K dt$$

On comparing:

$$P_j = p_p \frac{\partial q_p}{\partial Q_j}$$

&

$$-K = -H + p_p \frac{\partial q_p}{\partial t}$$

i.e. $K = H - p_p \dfrac{\partial q_p}{\partial t}$ ← (Hamilton in term of pt. transf.)

If the pt. transf. from $q$ to $Q$ is independent of time '$t$' then $K = H$

Example:

A dynamical system (D.S) is

described by the generalized co-ordinates by the Hamiltonian fun. $H(q, p; t)$

$H(q, p; t) = ap^2 + bp(q + t)^2$

where 'a' & 'b' are consts. A pt transf. is made a new generalized co-ordinate $Q = q + t$

Find Hamiltonian fun $K = K(Q, P; t)$

Solution:

As Hamilton's in terms of point transformation is given by

$$K = H - p_p \frac{\partial q_p}{\partial t} \quad (1)$$

As

$$H(q, p, t) = ap^2 + bp(q + t)^2 \quad (2)$$

As $P_j = p_p \dfrac{\partial q_p}{\partial Q_j} = p_p(+1)$

$= p_p$

So eq. (2) becomes

$H = a(P_j)^2 + b(P_j)(Q)^2 \quad \because q + t = Q$

$H = aP_j^2 + bP_jQ^2$

so eq. (1) becomes,

$$K = aP_j^2 + bP_jQ^2 - P_j \frac{\partial q_p}{\partial t}$$

$K = aP_j^2 + bQ^2P_j - P_j(-1)$

So

$$K = aP_j^2 + bQ^2P_j + P_j$$

As Req.