General

Now for $q_2 = z$

$$\frac{\partial H}{\partial p_z} = \dot{z} = \frac{p_z}{m}$$

$$\Rightarrow p_z = m\dot{z}$$

$$\Rightarrow \dot{p}_z = m\ddot{z} \quad \text{—— (4)}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz$$

So eq (4) becomes

$$m\ddot{z} = -kz$$

$$m\ddot{z} + kz = 0$$

$$\ddot{z} = -\frac{k}{m} z$$

$$\ddot{z} = -\omega^2 z$$

It shows that motion of a particle is S.H.M along z-axis.

Natural Motion:-

A motion is said to be natural motion if the Hamiltonian of the system satisfies the following.

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \quad ; \quad \frac{\partial H}{\partial q_j} = -\dot{p}_j$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

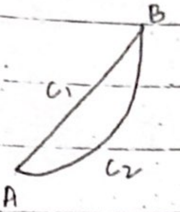
Question:- Hamilton's principle

Show that the integral of action has a stationary value for the natural motion with the adjacent motions during the same end pts.

$$H = p_j \dot{q}_j - L$$

so $$L = p_j \dot{q}_j - H$$

$$\int L\,dt = \int_A^B \left(p_j \frac{dq_j}{dt} - H\right) dt$$



$$S = \int_A^B p_j\, dq_j - H\, dt$$

which is known as integral of action

$\delta S = ?$ variation

$$\delta S = \delta \int_A^B \left(p_j\, dq_j - H dt\right)$$

$$= \int_A^B \delta p_j\, dq_j + p_j\, \delta(dq_j) - \delta H dt - H\,\delta(dt)$$

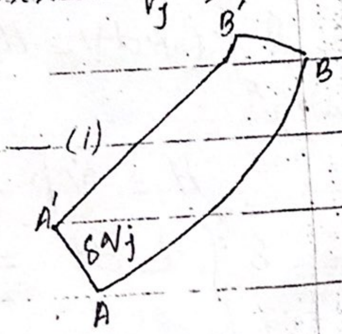
$$= \int_A^B \delta p_j\, dq_j + \int_A^B \underset{\mathrm{I}}{p_j}\, \underset{\mathrm{II}}{d(\delta q_j)} - \int_A^B \delta H dt - \int_A^B \underset{\mathrm{I}}{H}\, \underset{\mathrm{II}}{d(\delta t)}$$

$$= \int_A^B \delta p_j\, dq_j + \left(p_j\,\delta q_j \Big|_A^B - \int_A^B \delta q_j\, dp_j\right) - \int_A^B \delta H dt$$

$$- \left(H\,\delta t \Big|_A^B - \int_A^B \delta t\, dH\right)$$

$$= \left| p_j\,\delta q_j - H\,\delta t \right|_A^B + \int_A^B \left(\delta p_j\, dq_j - \delta q_j\, dp_j - \delta H dt + \delta t\, dH\right)$$

As the two curves with the same end pts A & B. If $\delta t \to 0$ then $\delta q_j \to 0$



so

$$\delta S = \int_A^B \delta p\, dq - \delta q\, dp - \delta H dt + \delta t\, dH \quad \text{---(1)}$$

As

$$H = H(q, p, t)$$

$$\delta H = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial t}\delta t$$

Using this eq. in (1) →

$$\delta S = \int_A^B \delta p\left(dq - \frac{\partial H}{\partial p}dt\right) - \delta q\left(dp + \frac{\partial H}{\partial q}dt\right)$$

$$+ \delta t\left(dH - \frac{\partial H}{\partial t}dt\right)$$

$$\delta S = \int_A^B \left[\delta p\left(\dot{q} - \frac{\partial H}{\partial p}\right) - \delta q\left(\dot{p} + \frac{\partial H}{\partial q}\right) + \delta t\left(\dot{H} - \frac{\partial H}{\partial t}\right)\right]dt$$

As Hamiltonian eq. of motion is

$$\frac{\partial H}{\partial p} = \dot{q} \quad ; \quad \frac{\partial H}{\partial q} = -\dot{p}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

so:

$$\delta S = \delta \int_A^B (p\,dq - H\,dt) = 0$$

Thus, 'S' has a stationary value for a natural motion when compared with adjacent motions for the same end points.

Using Hamilton's Principle Prove Lagrange's EQ. of Motion:-

As Hamilton's Principle States

$$\delta S = \delta \int_A^B (p\,dq - H\,dt) = 0.$$

By using

$$H = \dot{q}p - L$$

$$\delta S = \delta \int_{t_1}^{t_2} L\,dt = 0$$

$$L = T - V = L(\dot{q}, q)$$

$$\delta S = \delta \int_{t_1}^{t_2} L(\dot{q}, q)\, dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right) dt$$

$$= \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta \left(\frac{dq}{dt} \right) dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial q} \delta q\, dt$$

$$= \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta (dq) + \int_{t_1}^{t_2} \frac{\partial L}{\partial q} \delta q\, dt$$

$$= \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} d(\delta q) + \int_{t_1}^{t_2} \frac{\partial L}{\partial q} \delta q\, dt$$

$$= \frac{\partial L}{\partial \dot{q}} \delta q \Big|_A^B - \int_A^B \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt + \int_A^B \frac{\partial L}{\partial q} \delta q\, dt$$

As "A & B" are same end pts so

as $\delta t \to 0$, $\delta q \to 0$

$$0 + \int_A^B \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q\, dt = 0$$

$$\Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$