$$\int d\phi = -\int \frac{-\operatorname{cosec}^2\theta}{\sqrt{\frac{1-c^2(1+\cot^2\theta)}{c^2}}}\,d\theta$$

$$= -\int \frac{-\operatorname{cosec}^2\theta}{\sqrt{\left(\frac{1-c^2}{c^2}\right)-\cot^2\theta}}\,d\theta$$

$$\phi = \sin^{-1}\left(\frac{c(\cot\theta)}{\sqrt{1-c^2}}\right) + \alpha \quad (\alpha \text{ constt})$$

$$\phi = \sin^{-1}(k\cot\theta) + \alpha \quad \text{where } k = \frac{c}{\sqrt{1-c^2}}$$

$$\sin^{-1}(k\cot\theta) = \alpha - \phi$$

$$k\cot\theta = \sin(\alpha - \phi)$$

$$k\left(\frac{\cos\theta}{\sin\theta}\right) = \sin\alpha\cos\phi - \cos\alpha\sin\phi$$

$$k\cos\theta = \sin\theta\sin\alpha\cos\phi - \sin\theta\cos\alpha\sin\phi$$

$$kr\cos\theta = r\sin\theta\sin\alpha\cos\phi - r\sin\theta\cos\alpha\sin\phi \quad (2)$$

where
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

so eq. (2) becomes

$$zk = x\sin\alpha - y\cos\alpha$$

which is eq of plane through the origin cutting surface of a sphere in a great circle plane

Hamilton's Equation:- Like Lagrangian $L(q, \dot{q}, t)$, generalized momentum (momenta) is given by,

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

...function is Hamiltonian / Hamilton
$(q, p; t)$. It means there is a change
basis from $(q, \dot{q}; t)$ set to $(q, p; t)$
The Hamilton function is defined as

$$g = J = \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \dot{q}_j p_j - L \quad (1)$$

where g = Legender's transformation
J = Jacobian Integral
is obtained from Lagrangian so called
Legender transformation

Physical Significance of H:
As we know that

$$H = \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \quad (A)$$

$$L = T - V$$

$$= T(q, \dot{q}) - V(q)$$

As T does not contain time so

$$T = T_1 + T_2 + T_3$$

$$= 0 + 0 + \frac{1}{2} a_{jk} \dot{q}_j \dot{q}_k$$

So

$$T = \frac{1}{2} a_{jk} \dot{q}_j \dot{q}_k \quad (2)$$

As

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

$$= \frac{1}{2} a_{jk} \left(\dot{q}_k + \dot{q}_j \delta_{kj} \right)$$

$$= \frac{1}{2} a_{jk} (\dot{q}_k + \dot{q}_k)$$

$$= \frac{1}{2} a_{jk} (2\dot{q}_k)$$

$$\frac{\partial L}{\partial \dot{q}_j} = a_{jk} \dot{q}_k$$

$$\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = a_{jk} \dot{q}_k \dot{q}_j$$

$$= 2T \quad \text{By (2),}$$

so eq (A) becomes

$$H = 2T - (T - V)$$

$$H_{\text{canonical}} = T + V \quad \text{As Req.}$$

milton's Equations of Motion:

As Hamilton is a n of generalized co-ordinates & generalized omenta, & time is given by

$$H = H(q_j, p_j, t)$$

he total differential of Hamilton is

$$dH = \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \quad \text{— (1)}$$

definition of Hamilton

$$H = \dot{q}_j p_j - L(q_j, \dot{q}_j, t)$$

o its total differential is

$$dH = \dot{q}_j dp_j + p_j d\dot{q}_j - \left(\frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \right) \quad \text{(2)}$$

As

$$\frac{\partial L}{\partial \dot{q}_j} = p_j$$

so

~~dH~~ Using $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

Cannonical ⇒ Transfromation

* Time is invariant quantity.

$$\frac{d}{dt}(p_j) = \frac{\partial L}{\partial q_j}$$

$$\dot{p}_j = \frac{\partial L}{\partial q_j}$$

which is known as generalized conjugate momentum.

eq. (2) becomes

$$dH = \dot{q}_j \, dp_j - \dot{p}_j \, dq_j - \frac{\partial L}{\partial t} dt \quad \text{——— (3)}$$

comparing eq. (1) & (3).

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \; ; \; \frac{\partial H}{\partial q_j} = -\dot{p}_j \quad \text{——— (4)}$$

$$\& \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

eq. (4) is known as "Hamilton's canonical eqs. of motion".

Example:-

Use Hamiltonian method, obtain the eq. of motion for harmonic oscillator (one - dim.)

Solution:- The Hamiltonian

$$H = \dot{q}_j p_j - L \quad \text{——— (1)}$$

As $T = \frac{1}{2} m\dot{x}^2$

$V = \frac{1}{2} kx^2$, in case of mass attached with a spring

so $L = T - V$

$$= \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

Let $q = x$ so $\dot{q} = \dot{x}$

& $p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$\Rightarrow \dot{x} = \frac{p}{m}$

So $V = \frac{1}{2} k x^2$

eq. (1) becomes

$$H = m\dot{x}^2 - \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

$$H = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = T + V$$

$$H = \frac{1}{2} m\left(\frac{p}{m}\right)^2 + \frac{1}{2} kx^2$$

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2} kx^2 \quad \text{--- (2)}$$

Using cannonical eq. of motion,

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \quad ; \quad \frac{\partial H}{\partial q_j} = -\dot{p}_j$$

i.e., $$\frac{\partial H}{\partial p} = \dot{x} \quad , \quad \frac{\partial H}{\partial x} = -\dot{p} \quad \text{--- (3)}$$

From eq. (2),

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \Rightarrow p = m\dot{x}$$

$$\frac{\partial H}{\partial x} = kx = -\dot{p}$$

So as $p = m\dot{x}$

So $\dot{p} = m\ddot{x}$

As $-\dot{p} = kx$

$\Rightarrow m\ddot{x} = -kx$

$\Rightarrow F = -kx$

Example: A mass 'm' is placed on a frictionless plane which is tangent to the surface of the earth. Determine the (Hamilton's) eq. of motion using Hamilton's method?