9
Symmetries

When one does not know the principles or the fundamental equations that govern the phenomena one observes, one often obtains clues to them by looking at the symmetries or selection rules they exhibit. Historically, Maxwell constructed his unified theory of electromagnetism by assuming the existence of the displacement current, which was required by current conservation. Discovery of special relativity was inspired by Lorentz invariance of Maxwell's equations. History also shows that disclosed symmetries, more often than not, turn out to be approximate. Isospin and flavor $SU(3)$, which led to the discovery of the quark model, are such examples. Today what people believe to be strictly conserved are based on gauge symmetries. Energy-momentum, angular momentum and charge conservations are in this category. The conservation of baryon and lepton numbers, although phenomenologically valid, are not considered as strict laws. Their mixing is an essential feature of the grand unified theories (GUTs). Symmetries and conservation laws are intimately connected. Whether based on gauge symmetry or not, apparent symmetries and conservation laws can be powerful tools for discovering new physics.

Symmetry means one cannot tell the difference before and after a certain transformation. For the case of continuous space-time translation, this means there is no special position or absolute coordinate system in setting space-time coordinates for describing a physical phenomenon. The same experiments done at two different places (say in Tokyo and in New York) or done at different times (today or tomorrow) should produce the same result. In other words, translational invariance means the existence of certain nonobservables, i.e. absolute position and time. So far, we have elaborated on the relation between space-time translational symmetry and the resultant energy-momentum conservation. However, it is a common feature of symmetry. Similar relations can be derived for a variety of symmetry transformations. We list some of them here in Table 9.1 before proceeding further.

Symmetries come in two categories, first, those related to space-time structure, which are often referred to as external, and second, those related to internal symmetries, such as electric charge, isospin and color charge. Mathematically speaking, the latter leave the Lagrangian density invariant after the symmetry operation apart from a total derivative. It gives a surface integral that vanishes under normal circumstances. Space-time symmetries change the coordinate values, so only the

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Table 9.1 Example symmetries and conservation laws.

Nonobservables	Symmetry transformation	Conserved observables or selection rules
Space-time symmetry		
Absolute position	$\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$	Momentum
Absolute time	$t \to t + b$	Energy
Absolute direction	$\theta \to \theta + \alpha$	Angular momentum
Absolute velocity	Lorentz transformation	Lorentz invariance
Right or left	$\boldsymbol{x} \to -\boldsymbol{x}$	Parity
Internal symmetry		
Particle identity	Permutation	Bose–Einstein or Femi–Dirac statistics
Charge + or −	$Q \to -Q$	C invariance
Absolute phase	$\varphi \to e^{-iQ\alpha}\varphi$	Charge
– among quarks	$SU(3)$ gauge transformation	Color charge
Approximate symmetry		
– among leptons [a]	$\psi \to e^{-iL\alpha}\psi$	Lepton number
– among quarks	$\psi \to e^{-iB\alpha}\psi$	Baryon number
Near equal mass of u, d	Flavor $SU(2)$	Isospin
– of u, d, s	Flavor $SU(3)$	Unitary spin

a Phenomenologically, lepton and baryon numbers are conserved.

action, which is an integral of a Lagrangian over all space-time, remains invariant. In other classifications, they are either continuous or discrete. We discuss the continuous space-time symmetries first.

9.1 Continuous Symmetries

Where there is a symmetry, there exists a corresponding conservation law. What this means is that when an equation of motion or the Lagrangian is invariant under some symmetry operation, there exists a physical observable that does not change as a function of time. We have already shown some examples in Sect. 5.1.4 in the form of Noether's theorem. Here, we start by comparing how the symmetry and the invariance of the equation of motion are related in classical, quantum mechanical and quantum field theoretical treatments.

9.1.1
Space and Time Translation

Classical Mechanics In the Lagrangian formalism, the equation of motion is given as the Euler–Lagrange formula

$$\frac{d}{dt}\left(\frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}\right) - \frac{\partial L(q_i, \dot{q}_i)}{\partial q_i} = 0 \tag{9.1}$$

If the Lagrangian is translationally invariant, namely,

$$L(q_i) = L(q_i + a_i) \tag{9.2}$$

or equivalently $\partial L/\partial q_i = 0$, the Euler–Lagrange formula Eq. (9.1) tells us that the conjugate momentum defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{9.3}$$

is a constant of time. Therefore, translational invariance in space coordinates leads to the existence of the conjugate momentum, which is time invariant. Since the Lagrangian formalism is valid in a general coordinate system, this statement is general. If q_i is an angular variable, the conservation of angular momentum follows. The time derivative of the Hamiltonian is expressed as

$$\begin{aligned}\frac{dH}{dt} = \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) &= \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}\right)\\ &= -\frac{\partial L}{\partial t}\end{aligned} \tag{9.4}$$

where the Euler–Lagrange equation was used to arrive at the last equation. The above expression tells us that the Hamiltonian is a constant of motion if the Lagrangian does not contain time variables explicitly. Namely, translational invariance in time leads to energy conservation.

Quantum Mechanics Observables in quantum mechanics are expressed as matrix elements of hermitian operators. If there is a unitary operator $\hat{U}$[1] that does not have explicit time dependence, and if the transformed wave function $\psi' = \hat{U}\psi$ obeys the same Schrödinger equation, then

$$\begin{aligned} & i\frac{\partial}{\partial t}\psi = \hat{H}\psi \rightarrow i\frac{\partial}{\partial t}\hat{U}\psi = \hat{U}\hat{H}\hat{U}^{-1}\hat{U}\psi \\ \therefore\quad & i\frac{\partial}{\partial t}\psi' = \hat{H}\psi' \rightarrow \hat{U}\hat{H}\hat{U}^{-1} = \hat{H} \quad \text{or} \quad [\hat{H}, \hat{U}] = 0 \end{aligned} \tag{9.5}$$

1) The symmetry operation has to be unitary because the transition probability of the state before and after symmetry transformation has to be the same. ($\langle\psi'|\psi'\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle = \langle\psi|\psi\rangle$). We attach a hat $\hat{}$ to differentiate operators from numbers when we discuss objects in quantum mechanics.

The operator $\hat{U}$ commutes with the Hamiltonian. If $\hat{U}$ is a continuous function of some parameter ε and $\hat{U} \to 1$ when $\varepsilon \to 0$, then for infinitesimal ε

$$\hat{U} \simeq 1 - i\varepsilon\hat{Q} \tag{9.6}$$

As $\hat{U}$ is unitary, $\hat{Q}$ is hermitian. The operator $\hat{Q}$ in $\hat{U} = e^{-i\hat{Q}\alpha}$, where α is a continuous parameter, is called the generator of the transformation. Since, the hermitian operator in quantum mechanics is an observable, $\hat{Q}$ is an observable. As $\hat{Q}$ commutes with the Hamiltonian, the time derivative of the expectation value is given by

$$\frac{d\langle\hat{Q}\rangle}{dt} = \frac{d}{dt}\langle\psi|\hat{Q}|\psi\rangle = i\langle\psi|[\hat{H}, \hat{Q}]|\psi\rangle = 0 \tag{9.7}$$

Therefore, if there is a continuous symmetry represented by a unitary operator $\hat{U}$ that leaves the equation of motion invariant, its generator $\hat{Q}$ is a constant of time.

The hamiltonian is a time translation operator. This is obvious from

$$\psi(t) = e^{-i\hat{H}t}\psi(0) \to \psi(t+a) = e^{-ia\hat{H}}\psi(t) \tag{9.8}$$

To find the corresponding space translation operator $\hat{U}(a) = e^{-ia\hat{Q}}$ that is defined by

$$\psi(x) \to \psi(x+a) = \hat{U}(a)\psi(x) \tag{9.9a}$$

we make the displacement infinitesimal:

$$\begin{aligned}\hat{U}(\epsilon)\psi(x) &= [1 - i\epsilon\hat{Q} + O(\epsilon^2)]\psi(x) = \psi(x+\epsilon)\\ &= \psi(x) + \epsilon\frac{\partial}{\partial x}\psi(x) + O(\epsilon^2)\\ \therefore \quad \hat{Q} &= i\frac{\partial}{\partial x} = -\hat{p}\end{aligned} \tag{9.9b}$$

A Lorentz invariant expression can be obtained by combining Eqs. (9.8) and (9.9):

$$\psi(x^\mu + a^\mu) = e^{-ia^\mu P_\mu}\psi(x^\mu) \tag{9.10a}$$

$$P_\mu = i(\partial_0, \nabla) \tag{9.10b}$$

Problem 9.1

For a translational operator $\hat{p}$ where $O(x)$ is a function of $\hat{p}$ and x:

$$O(x+a) = e^{ia\hat{p}}O(x)e^{-ia\hat{p}} \tag{9.11a}$$

where $O(x)$ is a function of $\hat{p}$ and x, prove

$$\nabla O(x) = i[\hat{p}, O(x)] \tag{9.11b}$$

Translational Invariance in Quantum Field Theory

Covariant version of the Heisenberg equation of motion is expressed as

$$\partial_\mu O = i[P_\mu, O] \tag{9.12a}$$

$$O(x+a) = e^{ia^\mu P_\mu} O(x) e^{-ia^\mu P_\mu} \tag{9.12b}$$

The field theoretical version of the energy momentum operator is given by

$$P^\mu = \int d^3x\, T^{\mu 0} \tag{9.13}$$

where $T^{\mu\nu}$ is the energy-momentum tensor that can be derived from the Lagrangian using Noether's theorem. Here, P^μ and $O(x)$ are functions of the field operators. Note, corresponding to the negative sign of space variables in the Lorentz-invariant expression ($a^\mu P_\mu = a^0 P^0 - \boldsymbol{a} \cdot \boldsymbol{P}$), $P_\mu = (P^0, -\boldsymbol{P})$ is used.

We first prove that Eq. (9.12b) is equivalent to the Heisenberg equation (9.12a). To prove the equivalence, it is enough to show that one can be derived from the other and vice versa. Consider a^μ in Eq. (9.12b) as a small number ε^μ, expand both sides of the equation and compare terms of $O(\varepsilon)$:

$$(1 + i\varepsilon^\mu P_\mu) O(x) (1 - i\varepsilon^\mu P_\mu) \simeq O(x) + \varepsilon^\mu \partial_\mu O \tag{9.14a}$$

$$\therefore \quad i[P_\mu, O(x)] = \partial_\mu O(x) \tag{9.14b}$$

Conversely, to derive Eq. (9.12b) from Eq. (9.12a), we use the Baker–Campbell–Hausdorff (BCH) formula.

Baker–Campbell–Hausdorff formulae Let A, B be operators.

Theorem 9.1

$$\begin{aligned} e^A B e^{-A} &= B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots \\ &\quad + \frac{1}{n!}\overbrace{[A, [A, [A, \cdots [A}^{nA's}, B]]]] + \cdots \end{aligned} \tag{9.15}$$

Corollary 9.1

$$[B, e^{-A}] = e^{-A}\left([A, B] + \frac{1}{2}[A, [A, B]] + \cdots\right) \tag{9.16a}$$

$$[e^A, B] = \left([A, B] + \frac{1}{2}[A, [A, B]] + \cdots\right) e^A \tag{9.16b}$$

Theorem 9.2

Assuming $[A, [A, B]] = [B, [B, A]] = 0$

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{9.17a}$$

$$e^A e^B = e^{[A,B]} e^B e^A \tag{9.17b}$$

Corollary 9.2

$$e^{A/2} e^B e^{A/2} = e^{A+B} \tag{9.18}$$

Problem 9.2

Prove Theorem 9.1 and Theorem 9.2

Replace A by $i P_\mu a^\mu$, B by O, and make use of Eq. (9.12a), then

$$\begin{aligned} e^{i P_\mu a^\mu} O(x) e^{-i P_\mu a^\mu} &= O(x) + a^\mu \partial_\mu O(x) + \cdots + \frac{a^n}{n!} \partial_\mu^{(n)} O(x) + \cdots \\ &= O(x+a) \end{aligned} \tag{9.19}$$

It remains to prove Eqs. (9.12) with P_μ expressed as Eq. (9.13). We consider the case of a complex scalar field. Generalization to other fields is straightforward. The energy-momentum operator is given in Eq. (5.93).

$$H = \int d^3x \left[\left(\frac{\partial \varphi^\dagger}{\partial t} \right) \left(\frac{\partial \varphi}{\partial t} \right) + (\nabla \varphi^\dagger \cdot \nabla \varphi) + m^2 \varphi^\dagger \varphi \right] \tag{9.20a}$$

$$\boldsymbol{P} = -\int d^3x \left[\frac{\partial \varphi^\dagger}{\partial t} \nabla \varphi + \nabla \varphi^\dagger \frac{\partial \varphi}{\partial t} \right] \tag{9.20b}$$

Using the equal time commutation relation

$$[\varphi(x), \pi(y)]_{t_x = t_y} = [\varphi(x), \dot{\varphi}^\dagger(y)]_{t_x = t_y} = i \delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{9.21}$$

it is easy to verify Eq. (9.12a) if $O = \varphi(x)$ or $O = \pi(x)$.

Then for any operators A, B that satisfy

$$\partial_\mu A = i[P_\mu, A], \quad \partial_\mu B = i[P_\mu, B] \tag{9.22a}$$

$$i[P_\mu, AB] = i[P_\mu, A]B + Ai[P_\mu, B] = (\partial_\mu A)B + A\partial_\mu B = \partial_\mu(AB) \tag{9.22b}$$

it follows that the equation is valid if (AB) is replaced with any polynomials of A and B. Since P_μ is a conserved operator that does not include any space-time coordinates, applying any function of space-time differentiation $g(\partial)$ on both sides of the equation gives

$$i[P_\mu, \{g(\partial)O(A,B)\}] = \partial_\mu\{g(\partial)O(A,B)\} \tag{9.23}$$

Therefore, if O is a local operator, i.e. a polynomial of $\varphi(x)$, $\pi(x)$ and their differentials, it satisfies

$$\partial_\mu O = i[P_\mu\ O] \tag{9.24}$$

q.e.d.

In summary, the notion that for the space-time translational symmetry there exists a corresponding conserved quantity, the energy-momentum, is valid in quantum field theory as it was in classical field theory. In quantum field theory, the energy-momentum operators are the generators of the translational transformation. Note also that the Heisenberg equation holds even if the fields are interacting, because the equal time commutation relation, which was the basis of the proof, is also valid for interacting fields. The validity is ensured by the fact that they are connected by a unitary transformation [see Eq. (6.7)].

9.1.2 Rotational Invariance in the Two-Body System

Discussions in the previous section have shown that energy-momentum conservation is a result of translational invariance in Cartesian coordinates. As the Lagrangian formalism is valid for a general coordinate system, translational invariance in polar coordinates $\phi \to \phi + \alpha$ results in another conserved quantity, the angular momentum. Expressed in the original Cartesian coordinates

$$p_\phi = -i\partial_\phi = -i(x\partial_y - y\partial_x) = \boldsymbol{r} \times \frac{\nabla}{i} = \boldsymbol{l}_z \tag{9.25}$$

In Chap. 3 we have shown that Lorentz invariance leads to the existence of spin angular momentum, which acts on spin components of the field and is an intrinsic property of particles, the quanta of the field. Determining the spin is an essential step in the study of elementary particle physics and the method is based on the consideration of rotational invariance. Experimentally, the most frequently used reactions are two-body scatterings. If rotational invariance holds, angular momentum is conserved. Then the scattering amplitude can be decomposed into partial waves of definite angular momentum, and by analyzing the angular distributions of the two-body scattering state the spin of the system can be determined. We have already learned in quantum mechanics that the scattering amplitude in the center of mass (CM) frame can be expanded in partial waves:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{9.26a}$$

$$f(\theta) = \frac{1}{2ip}\sum_J (2J+1)[S^J(E) - 1]P_J(\theta) \tag{9.26b}$$

where p is the momentum in CM, $S^J(E)$ is the partial element of the scattering matrix with angular momentum J and $P_J(\theta)$ is the Legendre function of $\cos\theta$. The above formula is valid for a spinless particle. It can be extended to particles having spin [217]. For the treatment, it is convenient to work in the eigenstates where the incoming and outgoing states are specified by their helicity

$$h = \frac{\boldsymbol{J}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \tag{9.27}$$

instead of J_z, which is a component of the angular momentum along the fixed z-axis. The helicity is by its definition rotationally invariant.

What we are dealing with is a two-particle system having momentum and helicity $\boldsymbol{p}_1$, λ_1 and $\boldsymbol{p}_2$, λ_2, respectively:

$$|\Psi\rangle = |\boldsymbol{p}_1\lambda_1; \boldsymbol{p}_2\lambda_2\rangle = \psi_1(\boldsymbol{p}_1, \lambda_1)\psi_2(\boldsymbol{p}_2, \lambda_2) \tag{9.28}$$

Let us first separate the motion of the two-body reaction

$$A(p_1) + B(p_2) \to C(p_3) + D(p_4) \tag{9.29}$$

into that of the CM frame itself and the relative motion in it. Defining the momentum of the CM system $\boldsymbol{P}$ and the relative momentum $\boldsymbol{p}$ in the initial and final states by

$$\boldsymbol{P}_i = \quad \boldsymbol{p}_1 + \boldsymbol{p}_2\,, \quad \boldsymbol{P}_f = \quad \boldsymbol{p}_3 + \boldsymbol{p}_4 \tag{9.30a}$$

$$\boldsymbol{p}_i = \frac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\,, \quad \boldsymbol{p}_f = \frac{1}{2}(\boldsymbol{p}_3 - \boldsymbol{p}_4) \tag{9.30b}$$

and working in the CM frame, in which $P_i = P_f = (E, \boldsymbol{0})$, state vectors in CM can be expressed by their relative momentum in polar coordinates. We rewrite the two-particle state as

$$|p\,\theta\,\phi; \lambda_1\lambda_2\rangle = \psi_1(p\lambda_1)\psi_2(p\lambda_2) \tag{9.31}$$

where particle 1 has momentum $\boldsymbol{p} = (p, \theta, \phi)$ in polar coordinates and helicity λ_1 and particle 2 has $-\boldsymbol{p}$ and helicity λ_2. Sometimes we will use the total helicity

$$\lambda = \lambda_1 - \lambda_2\,, \quad \mu = \lambda_3 - \lambda_4 \tag{9.32}$$

to denote the two-particle helicity state, i.e. $|\lambda\rangle = |\lambda_1\lambda_2\rangle$, when there is no danger of confusion. Expanding the plane-wave state in terms of those that have angular momentum J, M is the center of discussion.

If rotational invariance holds, angular momentum is conserved and the scattering matrix elements can be expressed as

$$\langle J'M'\lambda_3\lambda_4|S|JM\lambda_1\lambda_2\rangle = \delta_{JJ'}\delta_{MM'}\langle\lambda_3\lambda_4|S^J|\lambda_1\lambda_2\rangle \tag{9.33}$$

where $|JM\lambda_1\lambda_2\rangle$ is a state of definite angular momentum constructed out of states containing particles of definite helicity. The essence of the partial-wave expansion is condensed in this equality.

We note that the final state, where the particle is scattered by angle θ, can be obtained by rotation from the initial state, where the momentum is along the z-axis ($\theta = \phi = 0$). The rotation in this case is around the y-axis by θ and then around the z-axis by ϕ:

$$|p\,\theta\,\phi;\lambda\rangle = e^{-iJ_Z\phi}e^{-iJ_y\theta}|p00;\lambda\rangle \equiv R(\theta,\phi)|p00;\lambda\rangle \tag{9.34}$$

where $R(\theta,\phi)$ denotes the rotation operator. Conventionally the alternative form

$$R(\theta) = e^{-iJ_Z\phi}e^{-iJ_y\theta}e^{iJ_Z\phi} \tag{9.35}$$

is often used, which we will adopt here. It differs from Eq. (9.34) only in the phase. Now we define the rotation matrix by

$$\begin{aligned}\langle J'M'\lambda|R(\theta,\phi)|JM\lambda\rangle &= \delta_{JJ'}e^{-i(M'-M)\phi}\langle JM'\lambda|e^{-i\theta J_y}|JM\lambda\rangle \\ &= \delta_{JJ'}e^{-i(M'-M)\phi}d^J_{M'M}(\theta)\end{aligned} \tag{9.36}$$

The two states have the same helicity, as it is conserved by rotation. The rotation matrix $d^J_{MM'}(\theta)$ satisfies the orthogonality condition and is normalized by

$$\int_{-1}^{1} d\cos\theta\, d^J_{MM'}(\theta)d^{J'}_{MM'}(\theta) = \delta_{JJ'}\frac{2}{2J+1} \tag{9.37}$$

General properties and expressions for $d^J_{M',M}(\theta)$ are given in the Appendix E. Multiplying Eq. (9.34) by $\langle JM;\lambda|$ from the left gives

$$\begin{aligned}\langle JM\lambda|p\,\theta\,\phi;\lambda\rangle &= \sum_{J'M'}\langle JM\lambda|R(\theta,\phi)|J'M'\rangle\langle J'M'|p00;\lambda\rangle \\ &= \sum_{M'} e^{i(M'-M)\phi}d^J_{MM'}(\theta)\langle JM'\lambda|p00;\lambda\rangle \\ &= e^{i(\lambda-M)\phi}d^J_{M\lambda}(\theta)\langle JM\lambda|p00;\lambda\rangle \\ &\equiv N_J e^{i(\lambda-M)\phi}d^J_{M\lambda}(\theta)\end{aligned} \tag{9.38}$$

The penultimate equality follows because the momentum of the state $|p00;\lambda\rangle$ is along the z axis and its total helicity $\lambda = \lambda_1 - \lambda_2$ is exactly the same as $M = J_z$, hence

$$\langle JM'\lambda|p00;\lambda\rangle = \delta_{M'\lambda}N_J \tag{9.39}$$

N_J is calculated to be

$$N_J = \sqrt{\frac{2J+1}{4\pi}} \tag{9.40}$$

To derive N_J, we use the normalization conditions

$$\langle J'M'\lambda'|JM\lambda\rangle = \delta_{JJ'}\delta_{MM'}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} \tag{9.41a}$$

$$\langle p'\theta'\phi';\lambda'|p\theta\phi;\lambda\rangle = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi')\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} \tag{9.41b}$$

and apply them to

$$\begin{aligned}1 = \langle JM\lambda|JM\lambda\rangle &= \sum_\mu \int d\cos\theta\, d\phi \langle JM\lambda|p\theta\phi;\mu\rangle\langle p\theta\phi;\mu|JM\lambda\rangle \\ &\overset{(9.38)(9.37)}{=} \frac{4\pi}{2J+1}|\langle JM\lambda|p00;\lambda\rangle|^2 = \frac{4\pi}{2J+1}N_J^2\end{aligned} \tag{9.41c}$$

Next we expand the state Eq. (9.34) in terms of angular momentum eigenstates:

$$|p\theta\phi\lambda\rangle = \sum_{J,M}|JM\lambda\rangle\langle JM\lambda|p\theta\phi;\lambda\rangle = \sum_{JM} N_J d^J_{M\lambda} e^{i(\lambda-M)\phi}|JM\lambda\rangle \tag{9.42}$$

Then the scattering matrix between the two states expressed in polar coordinates can be written

$$\begin{aligned}&\langle p_f\theta\phi;\mu|S|p_i00;\lambda\rangle \\ &\overset{(9.33)}{=} \sum_{JM}\frac{2J+1}{4\pi}\left(e^{i(\mu-M)\phi}d^J_{M\mu}\right)^* \langle\mu|S^J(E)|\lambda\rangle \left(e^{i(\lambda-M)\phi}d^J_{M\lambda}\right)|_{\theta=\phi=0} \\ &= \sum_{JM}\frac{2J+1}{4\pi}\langle\mu|S^J(E)|\lambda\rangle d^J_{\lambda\mu}(\theta)e^{i(\lambda-\mu)\phi} \\ &\langle\mu|S^J(E)|\lambda\rangle \equiv \langle\lambda_3\lambda_4|S^J(E)|\lambda_1\lambda_2\rangle\,, \quad \lambda=\lambda_1-\lambda_2\,, \quad \mu=\lambda_3-\lambda_4\end{aligned} \tag{9.43}$$

To normalize the scattering amplitude, we compare the scattering matrix in linear momentum space and in polar coordinates in the CM system:

$$\begin{aligned}\langle p_3p_4;\mu|S|p_1p_2;\lambda\rangle &= \delta_{if} - (2\pi)^4 i\delta^4\left(\sum p_i - \sum p_f\right)\mathcal{M}_{fi} \\ &= (2\pi)^4\delta^4(P_i-P_f)\times(4\pi)^2\sqrt{\frac{s}{p_ip_f}} \\ &\quad\times\langle p_f\theta\phi;\mu|S|p_i00;\lambda\rangle\end{aligned} \tag{9.44}$$

where $s = (p_1 + p_2)^2$. The factor in the second line results from the normalization conditions Eq. (9.41). To prove it, set $S = 1$ and use

$$\langle p_3 p_4 | p_1 p_2 \rangle = (2\pi)^6 (16 E_1 E_2 E_3 E_4)^{1/2} \delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_3)\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_4)$$
$$\delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_3)\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_4) d^3 p_3 d^3 p_4 = \delta^3(\boldsymbol{P}_i - \boldsymbol{P}_f)\delta^3(\boldsymbol{p}_i - \boldsymbol{p}_f) d^3 P_f d^3 p_f$$
$$= \delta^4(P_i - P_f)\delta(\cos\theta - \cos\theta')\delta(\phi - \phi')\left(\frac{\partial E_f}{\partial p_f}\right)^{-1} {p_f}^2 d^4 P_f \, d\cos\theta \, d\phi \tag{9.45}$$

and symmetrize between the initial and the final state. We define the scattering matrix $T^J(E)$ by[2)]

$$\langle \lambda_3 \lambda_4 | S^J(E) | \lambda_1 \lambda_2 \rangle = \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} + i \langle \lambda_3 \lambda_4 | T^J(E) | \lambda_1 \lambda_2 \rangle \tag{9.46}$$

Then using Eq. (9.44) and comparing with the cross section formula (6.90) for $d\sigma/d\Omega|_{\text{CM}}$, we obtain

$$\frac{d\sigma}{d\Omega} = |f_{\lambda_3\lambda_4,\lambda_1\lambda_2}|^2 \tag{9.47a}$$

$$f_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \sqrt{\frac{p_f}{p_i}} \frac{\mathcal{M}_{fi}}{8\pi\sqrt{s}} \tag{9.47b}$$

$$= \frac{1}{2ip_i} \sum_J (2J+1) \langle \lambda_3 \lambda_4 | \left(S^J(E) - 1\right) | \lambda_1 \lambda_2 \rangle d^J_{\lambda\mu}(\theta) e^{i(\lambda-\mu)\phi} \tag{9.47c}$$

The equation is an extension of Eq. (9.26). In fact, referring to Eq. (E.21) in Appendix E,

$$d^J_{00} = P_J(\theta) \tag{9.48}$$

This agrees with Eq. (9.26) when the particles have no spin.

Unitarity of S-matrix

From the unitarity of the scattering matrix we have

$$S S^\dagger = S^\dagger S = 1 \tag{9.49}$$

Sandwiching it between an initial and final state gives

$$\delta_{fi} = \sum_n S^*_{nf} S_{ni} \tag{9.50}$$

In terms of the T matrix defined in Eq. (9.46)

$$i(T^*_{if} - T_{fi}) = \sum_n T^*_{nf} T_{ni} \tag{9.51}$$

2) $T^J = (S^J - 1)/2i$ is also often used in the literature.

If it is a helicity amplitude, time reversal invariance requires $S_{fi} = S_{if}$ [Eq. (9.121)],

$$2\,\mathrm{Im}\,(T_{fi}) = \sum_n T^*_{nf} T_{ni} \tag{9.52}$$

For forward scattering ($i = f$),

$$2\,\mathrm{Im}\,(T_{ii}) = \sum_n |T_{ni}|^2 \tag{9.53}$$

The right hand side (rhs) is proportional to the total cross section and the left hand side (lhs) to the imaginary part of the forward scattering amplitude. The relation is general in the sense that it is valid for inelastic as well as elastic scattering. Namely,

$$\sigma_{\mathrm{TOT}} = k\,\mathrm{Im}\left[f(\theta = 0)\right] \tag{9.54}$$

To obtain the proportionality constant, we use Eq. (9.47) and

$$T_{fi} = -(2\pi)^4 \delta^4(P_i - P_f) \mathcal{M}_{fi} \tag{9.55}$$

Referring to the cross section formula in terms of $\mathcal{M}_{fi}$ [see (6.86)]

$$\begin{aligned} \sigma_{\mathrm{TOT}} &= \sum_f (2\pi)^4 \delta^4(P_i - P_f) \frac{|\mathcal{M}_{fi}|^2}{2s\lambda(1, x_1, x_2)} = \frac{2\,\mathrm{Im}\,(\mathcal{M}_{fi})}{4p\sqrt{s}} \\ &= \frac{4\pi}{p}\,\mathrm{Im}\,\{f(0)\} \end{aligned} \tag{9.56}$$

where we have used the relation $p_{\mathrm{CM}} = \lambda(s, m_1^2, m_2^2)/(2\sqrt{s}) = \sqrt{s}\lambda(1, x_1 x_2)/2$ [Eq. (6.56)].

Problem 9.3

Using Eq. (9.47), calculate the total cross section for the elastic scattering

$$\sigma_{\mathrm{TOT}} = \int d\Omega \sum_{\lambda_3\lambda_4} |f_{\lambda_3\lambda_4,\lambda_1\lambda_2}|^2 \tag{9.57}$$

and then prove that

$$\sigma_{\mathrm{TOT}}(\lambda_1, \lambda_2) = \frac{4\pi}{p}\,\mathrm{Im}\,(f_{\lambda_3\lambda_4,\lambda_1\lambda_2}) \tag{9.58}$$

assuming there is no inelastic scattering.

9.2 Discrete Symmetries

Discrete symmetries are not functions of a continuous parameter and no infinitesimal variation or differentiation is possible. Parity transformation, time reversal and charge conjugation to exchange particles and antiparticles are such examples.

9.2.1 Parity Transformation

General Properties Parity, or P operation for short, is often referred to as "mirroring" (reflection) of an object, but the exact definition is reversal of signs on all space coordinates:

$$\boldsymbol{x} \xrightarrow{\mathrm{P}} -\boldsymbol{x} \tag{9.59}$$

Performing the parity transformation again brings the state back to the original one, that is $P^2 = 1$. Therefore the corresponding unitary transformation U_P satisfies

$$U_P U_P = 1 \qquad \therefore \quad U_P = U_P^{-1} = U_P^\dagger \tag{9.60}$$

The parity transformation is at the same time unitary and hermitian, hence an observable by itself. Its eigenvalue is ± 1. It is known that parity is violated maximally in the weak interaction, but that it is conserved in the strong and electromagnetic interactions, at least within experimental limits.

Momentum and Angular Momentum There are many observables that have their own parity. Classically, the momentum $\boldsymbol{p}$ is $m\boldsymbol{v} = m d\boldsymbol{x}/dt$ and quantum mechanically $-i\nabla$; it changes its sign under parity transformation. Orbital angular momentum, on the other hand, has positive parity since it is given by $\boldsymbol{x} \times \boldsymbol{p}$. The parity of the spin angular momentum $\boldsymbol{S}$ is not clear, but we can assume it has the same property as its orbital counterpart. The above examples show that there are two kinds of (three-dimensional space) vectors. Those that change their sign under the parity transformation are called polar vectors and the others, which do not, axial vectors. Similarly, a quantity that is rotationally invariant but changes its sign under the P operation is called a pseudoscalar. One example is the helicity operator $\boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|$, the spin component along the momentum direction.

Intrinsic Parity of a Particle When the Hamiltonian is a function of the distance $r = |\boldsymbol{x}|$

$$H = \frac{\boldsymbol{p}^2}{2m} + V(r) \tag{9.61}$$

it is invariant under parity operation, and if $\psi(\boldsymbol{x})$ is a solution then $\psi(-\boldsymbol{x})$ is also a solution:

$$\psi'(-\boldsymbol{x}) = U_P \psi(\boldsymbol{x}) = \pm\psi(\boldsymbol{x}) \tag{9.62}$$

Generally, a particle has its own intrinsic parity. When operated on a one particle state,

$$P|q\rangle = \pm\eta_P|Pq\rangle \tag{9.63}$$

Here, q is a quantum number that specifies the state, and η_P is a phase that accompanies the parity transformation. The phase degree of freedom is allowed because the expectation value of an observable is always sandwiched by a state and its conjugate. Usually, the phase is taken to be 1 for simplicity.

Parity of Fields The parity of the Dirac bilinears was already given in Sect. 4.3.3. The parity of the scalar, vector and tensor fields are the same as the corresponding Dirac bilinears. We reproduce in Table 9.2 the parity transformation properties of the fields.

Parity of the Photon Now let us consider the parity of the photon. Since the time component of a charged current ($\rho = j^0$) is a scalar with positive parity and the space component is a polar vector, the Maxwell equations

$$\nabla \cdot \boldsymbol{E} = q\rho, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \tag{9.64}$$

tell us that

$$\boldsymbol{E}(\boldsymbol{x}) \to \boldsymbol{E}'(-\boldsymbol{x}) = -\boldsymbol{E}(\boldsymbol{x}), \quad \boldsymbol{B}(\boldsymbol{x}) \to \boldsymbol{B}'(-\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x}) \tag{9.65}$$

Since the electromagnetic potential is connected to the field by

$$\boldsymbol{E} = \nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{9.66}$$

the transformation property of the potential is given by

$$\phi(\boldsymbol{x}) \to \phi'(-\boldsymbol{x}) = \phi(\boldsymbol{x}), \quad \boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}'(-\boldsymbol{x}) = -\boldsymbol{A}(\boldsymbol{x}) \tag{9.67}$$

This means the photon is a polar vector and has negative parity.

Table 9.2 Property of Dirac bilinears under parity transformation.

	$S(t, \boldsymbol{x})$	$P(t, \boldsymbol{x})$	$V^\mu(t, \boldsymbol{x})$	$A^\mu(t, \boldsymbol{x})$	$T^{\mu\nu}(t, \boldsymbol{x})$
P	$S(t, -\boldsymbol{x})$	$-P(t, -\boldsymbol{x})$	$V_\mu(t, -\boldsymbol{x})$	$-A_\mu(t, -\boldsymbol{x})$	$T_{\mu\nu}(t, -\boldsymbol{x})$

Note 1: $S = \overline{\psi}\psi$, $P = i\overline{\psi}\gamma^5\psi$, $V^\mu = \overline{\psi}\gamma^\mu\psi$, $A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$, $T^{\mu\nu} = \overline{\psi}\sigma^{\mu\nu}\psi$
Note 2: $V^\mu = (V^0, \boldsymbol{V})$, $V_\mu = (V^0, -\boldsymbol{V})$

Parity of Many-Particle Systems

The quantum number of a many-particle system is additive for those corresponding to continuous symmetry and multiplicative for those corresponding to discrete symmetry, provided they are independent. The wave function of a many-particle system when there are no interactions between the particles is described by

$$\Psi(x_1, x_2, \cdots, x_n) = \psi_1(x_1)\psi_2(x_2)\cdots\psi_n(x_n) \tag{9.68a}$$

$$\Psi' = U\Psi = \psi_1'\psi_2'\cdots\psi_n' = U\psi_1 U\psi_2\cdots U\psi_n(x_n) \tag{9.68b}$$

When $U = e^{iQ\alpha}$

$$\begin{aligned} &e^{iQ\alpha}\Psi = \Pi e^{iQ_i\alpha}\psi_i = e^{i(Q_1+Q_2+\cdots)\alpha}\Pi\psi_i \\ &\therefore \quad Q = Q_1 + Q_2 + \cdots + Q_n \end{aligned} \tag{9.69}$$

When the transformation is discrete, U itself is converted to its quantum number, the rhs becomes $\Pi\eta_i\Pi\psi_n$ and

$$\eta = \eta_1\eta_2\cdots\eta_n \tag{9.70}$$

Parity of a Two-Body System

Consider the case where the intrinsic parity of spinless particles 1, 2 is P_1, P_2 and the wave function of the relative motion is $\Phi(x) = f(r)Y_{LM}(\theta,\phi)$, where Y_{LM} is a spherical harmonic function. Under the parity operation $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$. Using the property of the spherical harmonic function $Y_{LM}(\pi-\theta,\phi+\pi) = (-1)^L Y_{LM}(\theta,\phi)$,

$$U\Psi(x_1, x_2) = U\left[\phi_1(x_1)\Phi(x_1 - x_2)\psi_2(x_2)\right] = P_1 P_2(-1)^L\Psi \tag{9.71}$$

The parity of a three-body system can be determined similarly as $P_1 P_2 P_3(-1)^{\ell+L}$, where ℓ is the relative angular momentum between the particles 1, 2 and L is that of the particle 3 relative to the CM of the 1 + 2 system. Now consider a reaction $a + b \to c + d$, which has relative angular momentum ℓ, ℓ' before and after the reaction, respectively:

$$\begin{aligned} &\langle cd|S|ab\rangle = \langle cd|P^{-1}PSP^{-1}P|ab\rangle = P_a P_b(-1)^\ell P_c P_d(-1)^{\ell'}\langle cd|S|ab\rangle \\ &\therefore \quad \{1 - P_a P_b(-1)^\ell P_c P_d(-1)^{\ell'}\}\langle cd|S|ab\rangle = 0 \end{aligned} \tag{9.72}$$

If $\langle cd|S|ab\rangle \neq 0$ and $[P, S] = 0$, the parity before and after the reaction is conserved:

$$P_a P_b(-1)^\ell = P_c P_d(-1)^{\ell'} \tag{9.73}$$

When the parity of the particles a, b, c is known, that of d can be determined using the above equation. We shall determine the parity of the pion in the next section this way. Equation (9.73) does not hold if $\langle cd|S|ab\rangle = 0$ by some other selection rules. Therefore, there are as many ambiguities in the determination of the parity assignment as the number of selection rules. For instance, to determine the parity of π^0, one can use a process $p \to p + \pi^0$ or $n \to n + \pi^0$, but to determine that of π^+ one cannot use $p \to p + \pi^+$, because it is forbidden by charge conservation. The process $p \to n + \pi^+$ is allowed, but one needs to know the relative parity of n to p. Similarly one cannot determine the parity of K^+ using the process $p \to n + K^+$ as it is forbidden by strangeness conservation. $p \to \Lambda + K^+$ is allowed, then again the parity of K^+ and Λ cannot be determined independently. The usual assumption is that all the quarks u, d, s, etc. have relatively positive parities. Since all the hadrons are made of quarks, the parities of the hadrons can be determined in principle once the relative angular momentum among the quarks is specified. The parity of the familiar baryons $(p, n, \Lambda, \cdots)$, therefore, is assumed to be positive.

Parity Transformation of Helicity States*

The parity transformation property of the helicity amplitude is a bit complicated, because it does not use the orbital angular momentum explicitly. Those who are not interested in the complication of the derivation may skip this part and use only the relevant formula Eq. (9.87). The following arguments follow closely those of Jacob and Wick [217].

To determine the parity of a partial wave state $|JM\lambda\rangle$ in the helicity formalism, we must go back to the beginning and evaluate the effect on the helicity states Eq. (9.31)

$$|p00\lambda_1\lambda_2\rangle = \psi_1(p\lambda_1)\psi_2(p\lambda_2) \tag{9.74}$$

where particles 1 and 2 are moving in $+z$ and $-z$ directions, respectively. ψ_1 is obtained by Lorentz boost from its rest frame. Since the eigenvalues of the parity operation are only phases, we need to fix the relative phases of various states beforehand. They can be determined from a requirement that $\psi(0\lambda)$ satisfies the usual angular momentum formula

$$J_\pm|s\lambda\rangle = (J_x \pm J_y)|s\lambda\rangle = [(s \mp \lambda)(s \pm \lambda + 1)]^{1/2}|s, \lambda \pm 1\rangle \tag{9.75}$$

The parity operation does not change the spin direction and at rest $J_z = \lambda$

$$P\psi(0\lambda) = \eta\psi(0\lambda) \tag{9.76}$$

where η is the particle's intrinsic parity. It is convenient to define a mirror operation with respect to the xz-plane:

$$Y \equiv e^{-i\pi J_y} P \tag{9.77}$$

Using the relations

$$e^{-i\theta J_y}|JM\lambda\rangle = \sum_{M'} d^J_{M'M}(\theta)|JM'\lambda\rangle \tag{9.78a}$$

$$d^J_{M'M}(\pi) = (-1)^{j+M'}\delta_{M',-M} \tag{9.78b}$$

we obtain

$$Y\psi_1(p\lambda_1) = \eta_1(-1)^{s_1-\lambda_1}\psi_1(p,-\lambda_1) \tag{9.79}$$

Similarly

$$Y\psi_2(p\lambda_2) = \eta_2(-1)^{s_2+\lambda_2}\psi_2(p,-\lambda_2) \tag{9.80}$$

The reason for $-\lambda_2$ in the exponent is that the particle's momentum is $-\boldsymbol{p}$, hence $M = J_z = -\lambda$. Combining the two equations

$$Y|p00\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}|p00,-\lambda_1,-\lambda_2\rangle \tag{9.81a}$$

$$\therefore \quad P|p00\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}e^{i\pi J_y}|p00,-\lambda_1,-\lambda_2\rangle \tag{9.81b}$$

Now, what we want is the phase that appears in

$$P|JM\lambda_1\lambda_2\rangle = e^{i\alpha}|JM,-\lambda_1,-\lambda_2\rangle \tag{9.82}$$

Since the parity and the angular momentum operators commute, the phase α does not depend on M. Therefore, without loss of generality, we can set $M = \lambda = \lambda_1 - \lambda_2$. For the same reason, we can confine our argument to the case $\theta = \phi = 0$. Expanding the rhs in Eq. (9.81b) in partial waves

$$\begin{aligned} e^{i\pi J_y}|p00,-\lambda_1,-\lambda_2\rangle &= \sum_{JM\mu_1\mu_2}\sum_{J'M'\nu_1\nu_2}|JM\mu_1\mu_2\rangle \\ &\times \langle JM\mu_1\mu_2|e^{i\pi J_y}|J'M'\nu_1\nu_2\rangle\langle J'M'\nu_1\nu_2|p00,-\lambda_1,-\lambda_2\rangle \\ &\overset{(9.39)}{=} \sum_{JM}|JM,-\lambda_1,-\lambda_2\rangle N_J d^J_{M,-\lambda}(-\pi) \\ &= \sum_J N_J(-1)^{\lambda-J}|J\lambda,-\lambda_1-\lambda_2\rangle \end{aligned} \tag{9.83}$$

Inserting this into Eq. (9.81b) gives

$$P|p00\lambda_1\lambda_2\rangle = \sum_J N_J\eta_1\eta_2(-1)^{s_1+s_2-J}|J\lambda,-\lambda_1,-\lambda_2\rangle \tag{9.84}$$

On the other hand, expanding the lhs of Eq. (9.81b) directly using Eq. (9.42) gives

$$\begin{aligned} P|p00\lambda_1\lambda_2\rangle &= \sum_{JM} P|JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2|p00\lambda_1\lambda_2\rangle \\ &= \sum_J N_J P|J\lambda\lambda_1\lambda_2\rangle \end{aligned} \tag{9.85}$$

Comparing Eq. (9.84) and Eq. (9.85)

$$P|J\lambda\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{s_1+s_2-J}|J\lambda, -\lambda_1, -\lambda_2\rangle \tag{9.86}$$

Noting $s_1 + s_2 - J$ is an integer, we finally obtain

$$P|J\lambda\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{J-s_1-s_2}|J\lambda, -\lambda_1, -\lambda_2\rangle \tag{9.87}$$

When the parity is conserved, applying Eq. (9.87) to the scattering matrix,

$$\begin{aligned} \langle -\lambda_3, -\lambda_4|S^J| - \lambda_1, -\lambda_2\rangle &= \eta_S\langle \lambda_3\lambda_4|S^J|\lambda_1\lambda_2\rangle \\ \eta_S &= \eta_1\eta_2\eta_3\eta_4(-1)^{s_3+s_4-s_1-s_2} \end{aligned} \tag{9.88}$$

Using Eq. (9.47) and $d^J_{nm}(\pi - \theta) = (-1)^{J+n} d^J_{n,-m}(\theta)$, a similar formula for the scattering amplitude can be obtained:

$$\langle -\lambda_3, -\lambda_4|f(\theta, \phi)| - \lambda_1, -\lambda_2\rangle = \eta_S\langle \lambda_3\lambda_4|f(\theta, \pi - \phi)|\lambda_1\lambda_2\rangle \tag{9.89}$$

Parity Violation in the Weak Interaction

We will describe the detailed dynamics of the weak interaction in Chap. 15. Here, we describe only the essence of parity violation. The momentum $\boldsymbol{p}$ is an observable with negative parity, but it does not mean that parity is violated, since particles with $-\boldsymbol{p}$ exist equally. However, if the S-matrix contains an observable that has negative parity, it means parity is violated in the scattering. For instance,

$$\boldsymbol{J} \cdot \boldsymbol{p} = J p \cos\theta \tag{9.90}$$

is such an observable. In the strong magnetic field at ultra-low temperatures, a nucleus can be polarized along the magnetic field. An asymmetry in the angular distribution of the decay particles from the nuclei means that parity is violated in the decay. Here, the angle of the particle is defined relative to the magnetic field, i.e. the spin orientation of the parent nuclei and the asymmetry is relative to a plane perpendicular to the polarization axis, i.e. whether θ is $\gtrless \pi/2$. The experiment of Wu et al. [392] that proved parity violation in the weak interaction was determined this way.

In π-p scattering where the initial proton is unpolarized, the final proton can be polarized perpendicular to the plane of scattering, namely $\boldsymbol{\sigma} \cdot \boldsymbol{k}_1 \times \boldsymbol{k}_2 \neq 0$; where $\boldsymbol{k}_1, \boldsymbol{k}_2$ are the pion momenta before and after scattering, respectively. This is because the parity of $\boldsymbol{\sigma} \cdot \boldsymbol{k}_1 \times \boldsymbol{k}_2$ is positive. However, if the recoiled proton is longitudinally polarized, i.e. polarized along its momentum, which means $\boldsymbol{\sigma} \cdot \boldsymbol{p} \neq 0$, conservation of parity is violated.

The origin of the parity-violating transition can be traced back to the Hamiltonian. If parity is conserved in an interaction, the parity operator commutes with the Hamiltonian. Since the scattering matrix is made from the Hamiltonian, it commutes with the S-matrix, too. In order for the S-matrix to contain the parity-violating term, the Hamiltonian must contain a parity-violating component, too:

$$H = H_0 + H_{\mathrm{PV}} \tag{9.91a}$$

$$P H_0 P^{-1} = H_0, \quad P H_{\mathrm{PV}} P^{-1} = -H_{\mathrm{PV}} \tag{9.91b}$$

where H_0 and H_{PV} are a scalar and a pseudoscalar, respectively. H_{PV} is the origin of the parity-violating transition. We shall see how the parity-violating observable is related to H_{PV}.

Let us pause to consider the meaning of the extra parity-violating term in the Lagrangian. There are observables that change sign by the parity transformation P. The momentum $\boldsymbol{p}$ and helicity $\boldsymbol{\sigma} \cdot \boldsymbol{p}$ are examples. Their existence alone does not mean parity violation. We claim that parity is violated if a phenomenon exhibits a different behavior in the mirror world than in ours. This means the dynamical motion in the mirror world is different for a given initial state. Namely, it happens if the transition amplitude includes a term that behaves differently in the mirror world, in other words, if it includes odd-parity terms such as $\boldsymbol{\sigma} \cdot \boldsymbol{p}$. As the transition amplitude is a matrix element of the Hamiltonian for given initial and final states, the Hamiltonian itself has to include an odd-parity term to induce the parity-violating phenomenon. In mathematical language, it simply means $[P, H] \neq 0$. Calculations of the transition amplitude will show that if C' is the coefficient of the odd-parity term in the Hamiltonian, the parity-violating observables always appear multiplied by C'.

The weak interaction is known to violate parity. Beta decay, $(A, Z) \to (A, Z + 1) + e^- + \overline{\nu}_e$ or in the quark model $d \to u + e^- + \overline{\nu}_e$, is mediated by a charged force carrier, the $W^{\mp}$ vector boson, but in the low-energy limit, the interaction Hamiltonian is well described by the four-Fermi interaction, which can be written as

$$\begin{aligned} H_{\mathrm{INT}} = {} & (\overline{u}\gamma_\mu d)\left\{\overline{e}\left(C_{\mathrm{V}}\gamma^\mu + C'_{\mathrm{V}}\gamma^5\right)\nu_e\right\} \\ & + (\overline{u}\gamma_\mu\gamma^5 d)\left\{\overline{e}\left(C_{\mathrm{A}}\gamma^\mu\gamma^5 + C'_{\mathrm{A}}\gamma^\mu\right)\nu_e\right\} + \text{h.c.} \end{aligned} \tag{9.92}$$

where u, d, ν_e, e^- stand for the Dirac fields and h.c. means the hermitian conjugate of the preceding term. Referring to Table 9.2, we see that terms with C_{V}, C_{A}, which contain $V_\mu V^\mu$, $A_\mu A^\mu$, are parity conserving, while those with C'_{V}, C'_{A}, which contain $V_\mu A^\mu$, $A_\mu V^\mu$, are parity violating. Namely, C_{V}, C_{A} and C'_{V}, C'_{A} are the strengths of the parity-conserving and parity-violating terms, respectively. Since the observed rate is proportional to the square of the amplitude, the effect of the parity violation appears as the interference term. The total rate Γ is proportional to

$|H_0|^2 + |H_{\mathrm{PV}}|^2$. By preparing suitable initial and final states of the nucleus, it can be arranged to pick up only the C_{A}, C'_{A} part.[3] Then the angular distribution of the electron is shown to have the form

$$\frac{d\Gamma}{d\Omega} \sim 1 + \alpha \frac{(\boldsymbol{J} \cdot \boldsymbol{p}_e)}{m_e} = 1 + \alpha P v \cos\theta \tag{9.93a}$$

$$\alpha = \frac{2\,\mathrm{Re}\,(C_{\mathrm{A}}^* C'_{\mathrm{A}})}{|C_{\mathrm{A}}|^2 + |C'_{\mathrm{A}}|^2} \tag{9.93b}$$

P is the polarization of the parent nucleus. This shows clearly that the origin of the parity-violating observable $\boldsymbol{J} \cdot \boldsymbol{p}_e$ is the parity-violating Hamiltonian H_{PV}. The effect is maximal when $|C_{\mathrm{A}}| = |C'_{\mathrm{A}}|$, and experimental data have shown this is the case ($C'_{\mathrm{A}} \simeq -C_{\mathrm{A}}$). Experimental data have fixed the strengths as well as the phases of the coupling constants ($C'_{\mathrm{V}} = -C_{\mathrm{V}}$, $C'_{\mathrm{A}} = -C_{\mathrm{A}}$, $C_{\mathrm{A}} = C_{\mathrm{V}}$) and excluded other types (i.e. S, P, T) of quadrilinear forms. The Hamiltonian for beta decay has been shown to be

$$H_{\mathrm{INT}} = \frac{G_{\mathrm{F}}}{\sqrt{2}} \{\overline{u}\gamma_\mu (1-\gamma^5) d\}\{\overline{e}\gamma^\mu (1-\gamma^5)\nu_e\} \tag{9.94}$$

Here, C_{V} was replaced by the universal Fermi coupling constant $G_{\mathrm{F}}/\sqrt{2}$, $G_{\mathrm{F}} = 10^{-5} \times m_{\mathrm{p}}^2$, where m_{p} is the mass of the proton. This is called the V–A interaction and was the standard phenomenological Hamiltonian before the advent of the Standard Model.

9.2.2 Time Reversal

Time Reversal in Quantum Mechanics

Time reversal of a process means to make it proceed backward in time. In a macro-world, when a glass is dropped on the floor, the reverse process is that the broken pieces fly back to where the glass was broken and amalgamate to shape the original glass. Everybody knows this does not happen. But in the micro-world, Newton's equation of motion under the influence of a force $F(\boldsymbol{x})$ that does not depend on time

$$m\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F}(\boldsymbol{x}) \tag{9.95}$$

is invariant under the time reversal operation $t \to -t$, and this is indeed realized. Physical objects that are dependent on time, for example the velocity $\boldsymbol{v} = d\boldsymbol{x}/dt$, the momentum $\boldsymbol{p} = m\boldsymbol{v}$, the angular momentum $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ change their sign under time reversal and are suitable observables to test the hypothesis. Take the typical example of π–p scattering:

$$\pi(\boldsymbol{p}_1) + p(\boldsymbol{p}_2) \to \pi(\boldsymbol{p}_3) + p(\boldsymbol{p}_4) \tag{9.96}$$

3) If the so-called Gamow–Teller nuclear transitions with $|\Delta J| = 1$ and no parity change are selected, the vector interaction does not contribute ($C_{\mathrm{V}} = C'_{\mathrm{V}} = 0$).

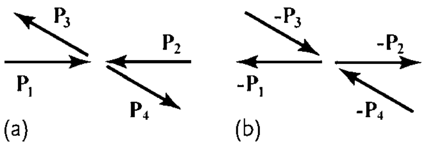


Figure 9.1 Time reversal. (a) Particles having momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ enter, and exit as two particles with momenta $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$. (b) Time-reversed reaction of (a). Particles with momenta $-\boldsymbol{p}_3$ and $-\boldsymbol{p}_4$ enter, and exit as particles with momenta $-\boldsymbol{p}_1$ and $-\boldsymbol{p}_2$.

The time-reversed process is a reaction where the outgoing particles reverse their momenta, go back to where they were scattered, undergo the reverse interaction and recover the initial states with their momenta reversed (Fig. 9.1):

$$\pi(-\boldsymbol{p}_3) + p(-\boldsymbol{p}_4) \to \pi(-\boldsymbol{p}_1) + p(-\boldsymbol{p}_2) \tag{9.97}$$

Namely, their positions are unchanged, but their momenta reversed and the initial and final states are interchanged. Expressing this in terms of the transition amplitude

$$\langle T\phi(\boldsymbol{p}_f)|T\psi(\boldsymbol{p}_i)\rangle = \langle \psi'(\boldsymbol{p}'_i)|\phi'(\boldsymbol{p}'_f)\rangle = \langle \phi(-\boldsymbol{p}_f)|\psi(-\boldsymbol{p}_i)\rangle^* \tag{9.98}$$

The time-reversal operation seems to include the complex conjugate operation in addition to ordinary unitary transformation. Let us see if the notion is valid in quantum mechanics. Consider a Schrödinger equation

$$i\frac{\partial\psi(t)}{\partial t} = H\psi(t) \tag{9.99}$$

and reverse the time direction of the wave function (H is independent of time)

$$t \to t' = -t\,, \quad \psi(t) \to \psi'(t') = T\psi(t) \tag{9.100}$$

Then

$$i\frac{\partial\psi'(t')}{\partial t'} = H'\psi'(t')\,, \quad H' = THT^{-1} \tag{9.101}$$

If $H' = H$,

$$-i\frac{\partial(T\psi)}{\partial t} = H(T\psi) \tag{9.102}$$

This shows that in order for the transformed wave function $T\psi$ to satisfy the same equation as Eq. (9.99), requiring $THT^{-1} = H$ is not enough; it is also necessary to take the complex conjugate of both sides. The process is known as Wigner's time reversal. In other words, Eq. (9.98) is a necessary condition for the time reversal operation in quantum mechanics. Under time reversal, the wave function is transformed to

$$\psi(t) \xrightarrow{\mathrm{T}} \psi'(t') = T\psi(t) = \psi^*(-t) \tag{9.103}$$

When ψ represents a plane wave with momentum $\boldsymbol{p}$

$$\psi(t, \boldsymbol{x}\,; \boldsymbol{p}) \sim e^{i\boldsymbol{p}\cdot\boldsymbol{x} - iEt} \xrightarrow{\mathrm{T}} e^{i\boldsymbol{p}\cdot\boldsymbol{x} - iEt'}|^{*}_{t'=-t} = e^{-i\boldsymbol{p}\cdot\boldsymbol{x} - iEt} \tag{9.104}$$

time reversal reverses the momentum of the state to $-\boldsymbol{p}$ as expected. In general, a transformation of the form

$$\begin{aligned} \psi(t) &= a\phi_1(t) + b\phi_2(t) \\ &\quad \to \psi' = a^*\phi_1'(t') + b^*\phi_2'(t') = a^*\phi_1^*(-t) + b^*\phi_2^*(-t) \end{aligned} \tag{9.105}$$

is called an antiunitary transformation. This means the T operation is expressed as a product of a unitary transformation U and complex conjugate operation K. Time-reversal invariance means the observables do not change under the operation Eq. (9.98). As the observables are expressed as the square of the transition amplitude, their antiunitarity does not produce any contradiction.

In fact, the necessity as well as consistency of complex conjugation can be shown in many examples. For instance, in classical mechanics $\boldsymbol{p} = m d\boldsymbol{x}/dt \xrightarrow{\mathrm{T}} -\boldsymbol{p}$, but in quantum mechanics $\boldsymbol{p}$ is a space differential operator and has no t dependence. By complex conjugation it obtains the right sign. Besides, the quantum condition

$$[x_i, p_j] = i\delta_{ij} \tag{9.106}$$

and the commutator of the angular momentum

$$[L_i,\ L_j] = i\varepsilon_{ijk}L_k \tag{9.107}$$

are not invariant for the change $\boldsymbol{p} \to -\boldsymbol{p},\ \ \boldsymbol{L} \to -\boldsymbol{L}$ but are invariant after an additional complex conjugate operation. In conclusion, the time-reversal operation requires, in addition to flipping the time, the complex conjugate to be taken of all c-numbers in the equation.

Problem 9.4

Show the transformation properties under the time-reversal operation.

$$\begin{array}{llll} \text{Electric field} & \boldsymbol{E}(t) & \to \boldsymbol{E}'(-t) & = \ \ \boldsymbol{E}(t) \\ \text{Magnetic field} & \boldsymbol{B}(t) & \to \boldsymbol{B}'(-t) & = -\boldsymbol{B}(t) \\ \text{Potential} & A^{\mu} = (\phi, \boldsymbol{A}) & \to A^{\mu\prime}(t') & = (\phi'(-t), \boldsymbol{A}'(-t)) \\ & & & = (\phi(t), -\boldsymbol{A}(t)) = A_{\mu}(t) \end{array} \tag{9.108}$$

Time Reversal of the S-Matrix

The S-matrix is defined as $\lim_{\substack{t\to\infty \\ t'\to-\infty}} U(t, t')$ [Eqs. (6.11), (6.12)]. Since $T\,U(t)\,T^{-1} = U(-t)$, it is obvious that $T\,S\,T^{-1} = S^{\dagger}$. Another way of seeing this is to use

$$\langle \text{out} : q_f | q_i : \text{in} \rangle = \langle \text{out} : q_f | S | q_i : \text{out} \rangle = \langle \text{in} : q_f | S | q_i : \text{in} \rangle \tag{9.109}$$

From the first expression, we obtain

$$\begin{aligned}\langle \text{out} : q_f | q_i; \text{in}\rangle &= \langle \text{out} : q_f | T^{-1} T | q_i : \text{in}\rangle = \langle \text{in} : T(q_f) | T(q_i) : \text{out}\rangle^* \\ &= \langle \text{out} : T(q_i) | T(q_f); \text{in}\rangle = \langle \text{out} : T(q_i) | S | T(q_f); \text{out}\rangle\end{aligned} \tag{9.110a}$$

where we have used $T|\text{in}\rangle = |\text{out}\rangle$, $T|\text{out}\rangle = |\text{in}\rangle$ to go to the third equality. From the last expression in Eq. (9.109), we obtain

$$\begin{aligned}\langle \text{in} : q_f | S | q_i : \text{in}\rangle &= \langle \text{in} : q_f | T^{-1} T S T^{-1} T | q_i : \text{in}\rangle \\ &= \langle \text{out} : T(q_f) | (T S T^{-1}) | T(q_i) : \text{out}\rangle^* \\ &= \langle \text{out} : T(q_i) | (T S T^{-1})^\dagger | T(q_f) : \text{out}\rangle\end{aligned} \tag{9.110b}$$

Comparing the last expressions of Eq. (9.110a) and Eq. (9.110b), we conclude

$$T S T^{-1} = S^\dagger \tag{9.111}$$

Time Reversal of Partial Waves

As the spin operator has no correspondence in classical mechanics, we are not sure how it transforms under the T operation. Let us assume its transformation property is the same as that of the orbital angular momentum, then $\boldsymbol{s} \sim \boldsymbol{x} \times \boldsymbol{p}$ changes its sign, too. Therefore

$$T J_i T^{-1} = -J_i\,, \quad T J_\pm T^{-1} = -J_\mp \tag{9.112a}$$

$$J_z T | J M\rangle = -T J_z | J M\rangle = -M T | J M\rangle$$

$$\therefore \quad T | J M\rangle = \eta(J, M) | J, -M\rangle \tag{9.112b}$$

The second equality of the first line follows from complex conjugation of i. To determine the phase η, we use the conventional relation among the angular momentum eigenstates:

$$\begin{aligned}T J_- | J M\rangle &= [(J+M)(J-M+1)]^{1/2} T | J M - 1\rangle \\ &= [(J+M)(J-M+1)]^{1/2} \eta(J, M-1) | J, -M+1\rangle \\ &= -J_+ T | J, M\rangle = -\eta(J, M) J_+ | J, -M\rangle \\ &= -\eta(J, M) [(J+M)(J-M+1)]^{1/2} | J, -M+1\rangle \\ \therefore \quad \eta(J, M) &= -\eta(J, M-1)\end{aligned} \tag{9.113a}$$

This equation means

$$\eta(J, M) = \eta(J)(-1)^M \tag{9.114}$$

$\eta(J)$ is a factor independent of M, and can be chosen considering the rotational invariance of the S-matrix and the antiunitarity of the T reversal. By choosing

$$\eta(J) = (-1)^J \text{ [4]} \tag{9.115}$$

we can fix the phase as

$$T|JM\rangle = (-1)^{J-M}|J, -M\rangle \tag{9.116}$$

The phase of the helicity state can be determined similarly:

$$T|JM\lambda\rangle = (-1)^{J-M}|J, -M\lambda\rangle \tag{9.117}$$

To prove the equality, we use the fact that the helicity does not change its sign under T and that the T-transformed one-particle state $|\boldsymbol{p}\lambda\rangle$ is $\eta|-\boldsymbol{p}\lambda\rangle$ which is equivalent to 180° rotation on the y axis. Namely

$$T|p00\lambda_1\lambda_2\rangle = \varepsilon e^{-i\pi J_y}|p00\lambda_1\lambda_2\rangle \tag{9.118}$$

where ε is a phase factor. The rest of the argument uses the same logic as we derived the parity of the state in Eq. (9.87).

Problem 9.5

Prove that the T-operated helicity eigenfunctions for a spin 1/2 particle given by Eq. (4.14)

$$\chi+ = \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix}, \quad \chi- = \begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{9.119}$$

are given by

$$T\chi_\pm = -i\sigma_2\chi_\pm^* = (-1)^{S-M}\chi_\mp \tag{9.120}$$

Finally, as to the helicity scattering amplitude, using the antiunitarity of the T transformation and Eq. (9.117) we find

$$\langle\lambda_3\lambda_4|S^J|\lambda_1\lambda_2\rangle = \langle\lambda_1\lambda_2|S^J|\lambda_3\lambda_4\rangle \tag{9.121}$$

Time Reversal of Fields

Scalar and Vector Fields We define the T reversal of a scalar field that obeys the Klein–Gordon equation by

$$\varphi(t, \boldsymbol{x}) \xrightarrow{\text{T}} \varphi'(x') = T\varphi T^{-1} = \varphi'(-t, \boldsymbol{x}) \tag{9.122}$$

Since the free-field Lagrangian is bilinear in form in φ and $\varphi^\dagger$, both choices

$$\varphi' = \varphi \quad \text{and} \quad \varphi' = \varphi^\dagger \tag{9.123}$$

4) The conventional spherical harmonic function Y_{LM} is defined with $\eta(J) = 1$. To be consistent with our definition, it has to be changed to $Y_{LM} \to i^L Y_{LM}$, but this is not a problem in this book.

are possible. But here we impose a constraint by requiring that the wave function (denoted as ψ = c-number) should respect the rule we saw in quantum mechanics. As the field φ can be expanded in terms of plane waves as

$$\varphi = \sum_k \frac{1}{\sqrt{2\omega}}\left[a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}\right] \tag{9.124}$$

the wave function $\psi(t, \boldsymbol{x})$ describing a particle with momentum $\boldsymbol{p}$ can be extracted by

$$\psi(t, \boldsymbol{x}) = \langle 0|\varphi(t, \boldsymbol{x})|p\rangle \tag{9.125}$$

Similarly, the T-reversed state should be able to be extracted by

$$\psi'(-t, \boldsymbol{x}) = \langle T0|\varphi(t, \boldsymbol{x})|Tp\rangle \tag{9.126}$$

But for a c-number wave function, it obeys the rule of Eqs. (9.98) and (9.103)

$$\begin{aligned} \psi'(-t, \boldsymbol{x}) &= \psi(-t, \boldsymbol{x})^* \overset{\text{Eq. (9.98)}}{=} \langle 0|\varphi(-t, \boldsymbol{x})|p\rangle^* \\ &= \langle T0|T\{\varphi(-t, \boldsymbol{x})p\}\rangle = \langle T0|T\varphi(-t, \boldsymbol{x})T^{-1}|Tp\rangle \end{aligned} \tag{9.127}$$

Comparing Eqs. (9.126) and (9.127), we can choose the T-transformed field as

$$T\varphi(t, \boldsymbol{x})T^{-1} = \varphi(-t, \boldsymbol{x}) \tag{9.128}$$

subject to the additional constraint of taking the complex conjugate of all the c-numbers.[5] Referring to the expansion formula of φ, the transformation properties of the annihilation and creation operators are

$$a_k \to T a_k T^{-1} = a_{-k}\,, \quad b_k^\dagger \to T b^\dagger T^{-1} = b_{-k}^\dagger \tag{9.129}$$

The transformation is consistent with the expression

$$|k\rangle = \sqrt{2\omega}\, a_k^\dagger|0\rangle \overset{\mathrm{T}}{\to} \sqrt{2\omega}\, a_{-k}^\dagger|0\rangle = |-k\rangle \tag{9.130}$$

Under T reversal, the current operator of the scalar field

$$j^\mu = iq(\varphi^\dagger \partial^\mu \varphi - \partial^\mu \varphi^\dagger \varphi) \tag{9.131}$$

changes its argument $t \to -t$ and $i \to -i$. Therefore the transformation is

$$\begin{aligned} &T j^0(t, \boldsymbol{x})T^{-1} = j^0(-t, \boldsymbol{x})\,, \quad T\boldsymbol{j}(t, \boldsymbol{x})T^{-1} = -\boldsymbol{j}(-t, \boldsymbol{x}) \\ &\text{i.e.} \quad j^\mu(t, \boldsymbol{x}) \overset{\mathrm{T}}{\to} j_\mu(-t, \boldsymbol{x}) \end{aligned} \tag{9.132}$$

As the vector field obeys Proca's equation

$$\partial_\mu f^{\mu\nu} + m^2 V^\nu = j^\nu\,, \quad f^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu \tag{9.133}$$

the consistency argument tells us that

$$V^\mu(t, \boldsymbol{x}) \overset{\mathrm{T}}{\to} V_\mu(-t, \boldsymbol{x}) \tag{9.134}$$

5) As a matter of fact, an alternative definition of the reversal accompanied by the hermitian conjugation operation $\varphi(t, \boldsymbol{x}) \to \varphi^\dagger(-t, \boldsymbol{x})$ exists, but we will not elaborate on it in this book.

Dirac Field As the component of angular momentum changes its sign under time reversal, rearrangement of the spin component is necessary for the Dirac field. What we have to do is to find a transformation such that

$$\psi(x) \xrightarrow{\mathrm{T}} T\psi(x)T^{-1} = \psi'(t', \boldsymbol{x}) = B\psi(-t, \boldsymbol{x}) \tag{9.135}$$

satisfies the same Dirac equation. Here, B is a c-number 4×4 matrix that operates on spin indices. Consider the Dirac equation for the wave field

$$(\gamma^\mu i\partial_\mu - m)\psi(t, \boldsymbol{x}) = 0 \xrightarrow{\mathrm{T}} (\gamma^\mu i\partial'_\mu - m)^* \psi'(t', \boldsymbol{x}) = 0 \tag{9.136}$$

where $\partial'_\mu = (-\partial_0, \nabla)$. Substituting Eq. (9.135) in Eq. (9.136) and multiplying B^{-1} from the left, we obtain

$$\begin{aligned} &B^{-1}[\gamma^{0*}(-i)(-\partial_0) + \boldsymbol{\gamma}^* \cdot (-i\nabla) - m]B\psi(-t, \boldsymbol{x}) \\ &= [B^{-1}\gamma^{0*}B(i\partial_0) + B^{-1}\boldsymbol{\gamma}^* B \cdot (-i\nabla) - m]\psi(-t, \boldsymbol{x}) = 0 \end{aligned} \tag{9.137}$$

Therefore, if B which satisfies the following relations exists

$$B^{-1}\gamma^{0*}B = \gamma^0 , \quad B^{-1}\gamma^{k*}B = -\gamma^k \tag{9.138}$$

the time reversed field ψ satisfies the same equation and T transformation invariance holds. Considering $\gamma^{2*} = -\gamma^2$, $\gamma^{\mu *} = \gamma^\mu (\mu \neq 2)$, we can adopt as B

$$B = i\gamma^1\gamma^3 = -i\gamma^5 C = -\Sigma_2 = \begin{bmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} \tag{9.139}$$

Then

$$\psi'(t') = T\psi(t)T^{-1} = B\psi(-t) = i\gamma^1\gamma^3\psi(-t) \tag{9.140a}$$

$$\overline{\psi}'(t') = T\overline{\psi}(t)T^{-1} = \overline{\psi}(-t)B^{-1} = \overline{\psi}(-t)i\gamma^1\gamma^3 . \tag{9.140b}$$

The matrix B has the property that

$$\begin{aligned} B &= B^\dagger = B^{-1} = -B^* \\ B^{-1}\gamma^{\mu *}B &= \gamma_\mu \end{aligned} \tag{9.141}$$

Therefore, the transformation property of the Dirac bilinear vector is given by

$$\begin{aligned} \overline{\psi}_1(t)\gamma^\mu\psi_2(t) &\xrightarrow{\mathrm{T}} \overline{\psi}'_1(-t)\gamma^{\mu *}\psi'_2(-t) = \overline{\psi}_1 B^{-1}\gamma^{\mu *}B\psi_2 \\ &= \overline{\psi}_1(-t)\gamma_\mu\psi_2(-t) \end{aligned} \tag{9.142a}$$

$$\overline{\psi}_1(t)\gamma^\mu\gamma^5\psi_2(t) \xrightarrow{\mathrm{T}} \overline{\psi}_1(-t)\gamma_\mu\gamma^5\psi_2(-t) \tag{9.142b}$$

The transformation properties of other Dirac bilinears are given similarly and we list them in Table 9.3.

Table 9.3 Properties of Dirac bilinears under T transformation.

	$S(t, x)$	$P(t, x)$	$V^{\mu}(t, x)$	$A^{\mu}(t, x)$	$T^{\mu\nu}(t, x)$
T	$S(-t, x)$	$-P(-t, x)$	$V_{\mu}(-t, x)$	$A_{\mu}(-t, x)$	$-T_{\mu\nu}(-t, x)$

Note 1: $S = \overline{\psi}\psi$, $P = i\overline{\psi}\gamma^5\psi$, $V^{\mu} = \overline{\psi}\gamma^{\mu}\psi$, $A^{\mu} = \overline{\psi}\gamma^{\mu}\gamma^5\psi$, $T^{\mu\nu} = \overline{\psi}\sigma^{\mu\nu}\psi$
Note 2: $V^{\mu} = (V^0, \boldsymbol{V})$, $V_{\mu} = (V^0, -\boldsymbol{V})$

Experimental Tests

Principle of Detailed Balance: The amplitude of a scattering process from a initial state $|i\rangle$ to a final state $|f\rangle$ and its inverse $f \to i$ are related by Eq. (9.121). The cross section of a process A + B → C + D, assuming both A and B are unpolarized is expressed as

$$\frac{d\sigma}{d\Omega}(\mathrm{AB} \to \mathrm{CD}) = \frac{1}{(2S_{\mathrm{A}}+1)(2S_{\mathrm{B}}+1)} \sum_{\lambda} |f_{\lambda_{\mathrm{C}}\lambda_{\mathrm{D}},\lambda_{\mathrm{A}}\lambda_{\mathrm{B}}}|^2 \tag{9.143}$$

On the other hand the cross section of the inverse process is, again assuming both C and D are unpolarized,

$$\frac{d\sigma}{d\Omega}(\mathrm{CD} \to \mathrm{AB}) = \frac{1}{(2S_{\mathrm{C}}+1)(2S_{\mathrm{D}}+1)} \sum_{\lambda} |f_{\lambda_{\mathrm{A}}\lambda_{\mathrm{B}},\lambda_{\mathrm{C}}\lambda_{\mathrm{D}}}|^2 \tag{9.144}$$

As Eqs. (9.121) and (9.47) mean

$$p_{\mathrm{AB}}^2 \sum_{\lambda} |f_{\lambda_{\mathrm{C}}\lambda_{\mathrm{D}},\lambda_{\mathrm{A}}\lambda_{\mathrm{B}}}|^2 = p_{\mathrm{CD}}^2 \sum_{\lambda} |f_{\lambda_{\mathrm{A}}\lambda_{B},\lambda_{C}\lambda_{D}}|^2 \tag{9.145}$$

where p_{AB} and p_{CD} are the momenta of particles in CM of AB and CD, respectively. we obtain

$$\frac{d\sigma(\mathrm{AB} \to \mathrm{CD})}{d\sigma(\mathrm{CD} \to \mathrm{AB})} = \frac{p_{\mathrm{CD}}{}^2(2S_{\mathrm{C}}+1)(2S_{\mathrm{D}}+1)}{p_{\mathrm{AB}}{}^2(2S_{\mathrm{A}}+1)(2S_{\mathrm{B}}+1)} \tag{9.146}$$

The equation is referred to as the "principle of detailed balance".

Strong Interaction As an example of the application of the principle of detailed balance to test time-reversal invariance, we show both cross sections of the reactions $p + {}^{27}\mathrm{Al} \rightleftharpoons \alpha + {}^{24}\mathrm{Mg}$ in Fig. 9.2. The data show T-reversal invariance is preserved in the strong interaction to better than 10^{-3}.

Weak Interaction

There is evidence that small CP violation exists in the kaon and B-meson decays (see Chap. 16), hence the same amount of T violation is expected to exist as long as CPT invariance holds, which is discussed in Sect. 9.3.3. In fact, a T-violation effect consistent with CPT invariance has been observed in the neutral K meson decays [107], which will be described in detail in Chap. 16. With this one exception, there is no other evidence of T violation. As it offers some of the most sensitive tests for models of new physics, we will discuss some of the experimental work below.

Observables that violate T invariance can be constructed as a triple product of T-odd variables like $\boldsymbol{\sigma} \cdot \boldsymbol{p}_1 \times \boldsymbol{p}_2$, $\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 \times \boldsymbol{p}_3$. In the first example, $\boldsymbol{\sigma}$ can be prepared by a polarized beam, polarized targets or spin of scattered/decayed particles.

$\mathbf{K^+ \to \pi^0 + \mu^+ + \nu_\mu}$: In the decay $K^+ \to \pi^0 + \mu^+ + \nu_\mu$, the transverse component of muon polarization relative to the decay plane determined by $\boldsymbol{p}_\mu \times \boldsymbol{p}_\pi$ (which is $\boldsymbol{\sigma}_\mu \cdot \boldsymbol{p}_\mu \times \boldsymbol{p}_\pi$) was determined to be $1.7 \pm 2.5 \times 10^{-3}$ [225, 226]. The imaginary (i.e. T-violating) part of the decay amplitude was also determined to be $\mathrm{Im}\,(\xi) = -0.006 \pm 0.008$ (see Sect. 15.6.3 for the definition of ξ).

$\mathbf{n \to p + e^- + \bar{\nu}_e}$: Another example is the triple correlation D of the neutron polarization and the momenta of electron and antineutrino in the beta decay $n \to p\,e^-\overline{\nu}_e$. It is defined as

$$d\,W \propto 1 + D\,\boldsymbol{P}_n \cdot \boldsymbol{p}_e \times \boldsymbol{p}_{\overline{\nu}} \tag{9.147a}$$

where $\boldsymbol{P}_n$ is the polarization of the neutron. The experimental value is given as $D = -4 \pm 6 \times 10^{-4}$ [350]. The data can be used to define the imaginary part of the coupling constant. The relative phase ϕ_{AV} is related to D by

$$\lambda = \left|\frac{g_{\mathrm{A}}}{g_{\mathrm{V}}}\right|, \quad \phi_{\mathrm{AV}} = \mathrm{Arg}\left[\frac{g_{\mathrm{A}}}{g_{\mathrm{V}}}\right] \tag{9.147b}$$

$$\sin\phi_{\mathrm{AV}} = D\frac{(1+3\lambda^2)}{2\lambda} \tag{9.147c}$$

where $g_{\mathrm{A}}, g_{\mathrm{V}}$ are axial and vector coupling constants of the weak interaction [C_{V} and C_{A} of Eq. (9.92)]. ϕ_{AV} is given as $180.06° \pm 0.07°$.

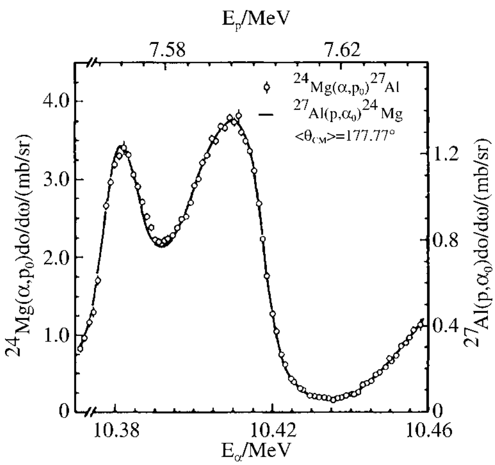


Figure 9.2 Test of detailed balance and time reversibility in the reaction $^{27}\mathrm{Al} + p \rightleftharpoons {}^{24}\mathrm{Mg} + \alpha$. The intensity of the T-violating effect ($\xi = |f_{\text{T-violating}}/f_{\text{T-inv}}|^2$) is smaller than 5×10^{-4} [66].

The Electric Dipole Moment of the Neutron

The definition of the electric dipole moment (EDM) is classically

$$\boldsymbol{d} = \int \rho(\boldsymbol{x})\boldsymbol{x}\, d^3x \tag{9.148}$$

which is nonzero if the charge distribution within an object is polarized, in other words, not distributed evenly. When the object is a particle like the neutron, regardless of whether of finite or point size, its only attribute that has directionality is the total spin $\boldsymbol{\sigma}$. If the particle has a finite $\boldsymbol{d}$, it has to be proportional to $\boldsymbol{\sigma}$. While the transformation property of $\boldsymbol{\sigma}$ under P, T is $+, -$, respectively, that of $\boldsymbol{d}$ is $-, +$, as can be seen from Eq. (9.148). Therefore the existence of the EDM of a particle violates both P and T. As the neutron is a neutral composite of quarks, its EDM can be a sensitive test of T-reversal invariance in the strong interaction sector as well as the weak interaction. The interaction of the magnetic and electric dipole moments with the electromagnetic field is given by

$$H_{\text{INT}} = -\boldsymbol{\mu}\cdot\boldsymbol{B} - \boldsymbol{d}\cdot\boldsymbol{E} \tag{9.149}$$

Its relativistic version can be expressed as

$$H_{\text{INT}} = -q\overline{\psi}\gamma^{\mu}\psi A_{\mu} + \frac{i}{2}d\overline{\psi}\gamma^5\sigma^{\mu\nu}\psi F_{\mu\nu} \tag{9.150}$$

Problem 9.6

Show that in the nonrelativistic limit, Eq. (9.150) reduces to Eq. (9.149).

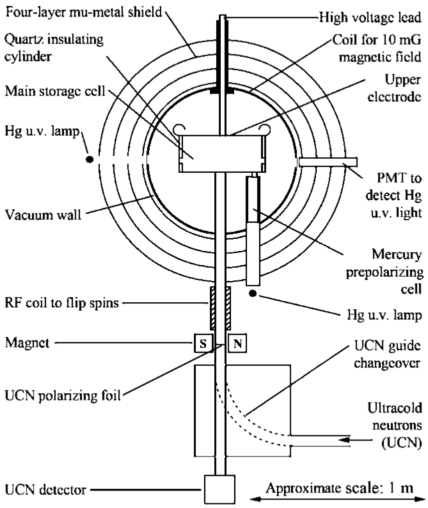
For an $s = 1/2$ particle, depending on the magnetic quantum number $m = \pm 1/2$, the energy level splits in the electromagnetic field. When an oscillating field is injected, a resonance occurs corresponding to the Larmor frequency

$$\Delta E = 2\mu B \pm 2dE = h(\nu \pm \Delta\nu) \tag{9.151}$$

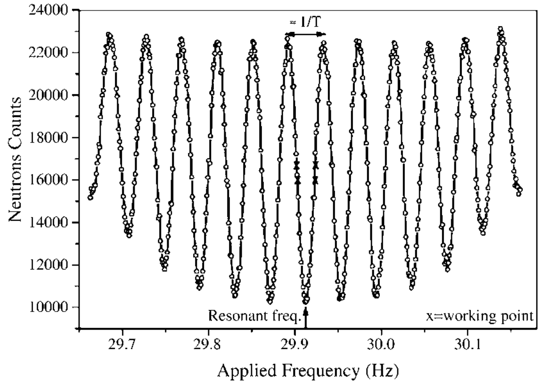
Here $\pm$ corresponds to the directions of the magnetic and electric fields being either parallel or antiparallel to each other. Therefore from the deviation $\Delta\nu$ of the resonance frequency when the electric field is applied, one can determine the strength of the EDM. What has to be carefully arranged experimentally is the parallel alignment of the electric field relative to the magnetic field, because if there exists a perpendicular component $\boldsymbol{E}_\perp$ a magnetic field of $\gamma\boldsymbol{v}\times\boldsymbol{E}_\perp$ for a moving system with velocity $\boldsymbol{v}$ is induced and gives a false signal.

An example material used for the experiment is the ultra cold neutron (UCN) of $T \sim 0.002$ K, which has energy $\sim 2\times 10^{-7}$ eV and velocity $v \sim 6\,\text{ms}^{-1}$. This is the energy the neutron obtains in the earth's gravitational field when it drops 2 m. The de Broglie wavelength is very long ($\lambda \sim 670$ Å), and consequently the neutron interferes coherently with material, allowing the index of refraction n to be defined:

$$n = \left[1 - \frac{\lambda^2 N a_{\text{coh}}}{\pi} \pm \frac{\mu B}{Mv^2/2}\right]^{1/2} \tag{9.152}$$



(a)



(b)

Figure 9.3 (a) The neutron EDM experimental apparatus of the RALySussex experiment at ILL [37, 198]. The bottle is a 20 liter storage cell composed of a hollow upright quartz cylinder closed at each end by aluminum electrodes that are coated with a thin layer of carbon. A highly uniform 1 μT magnetic field $\boldsymbol{B}_0$ parallel to the axis of the bottle is generated by a coil and the electric field ($\boldsymbol{E}_0$) is generated by applying high voltage between the electrodes. The storage volume is situated within four layers of mu metal, giving a shielding factor of about 10 000 against external magnetic fluctuations. (b) A magnetic resonance plot showing the UCNs with spin-up count after the spin precessing magnetic field has been applied. The peak (valley) corresponds to the maximum (minimum) transmission through the analyzing foil. The measured points to detect the EDM effect are marked as four crosses. The corresponding pattern for spin down is inverted but otherwise identical.

where N is the number of nuclei per unit volume, a_{coh} is the coherent forward scattering amplitude and $Mv^2/2$ is the kinetic energy of the neutron. The $\pm$ sign depends on whether the magnetic moment and the field are parallel or antiparallel. Inserting the actual values, we obtain a total reflection angle $\sim 5°$ for $v \sim 80\,\text{m}^{-1}$, but for a velocity $v < 6\,\text{m}^{-1}$ the angle exceeds 90° and total reflection is obtained at any angle. Namely, the neutron, if it is slow enough, can be transported through a bent tube just like light through an optical fiber and also can be confined in a bottle for a long duration of time (referred to as a magnetic bottle or a neutron bottle). That the refractive index differs depending on the polarization is used to separate neutrons of different polarization.

Thermal neutrons emerging from the deuterium moderator of a reactor are transported through a curved nickel pipe and further decelerated by a totally reflecting turbine before arriving at the entrance of the apparatus. Figure 9.3 shows the apparatus for the experiment [37, 198].

A magnetized iron–cobalt foil of 1 μm thickness is used to block neutrons of one spin orientation using the principle of total reflection stated above. The neutron bottle, a cylindrical 20-liter trap within a 1 μT uniform magnetic field $\boldsymbol{B}_0$, is able to store neutrons for more than 100 s. The electric field, of approximately 10 kV/cm, was generated by applying high voltage (HV) to the electrode. After application of a resonant oscillating magnetic field ($\sim$ 30 Hz) perpendicular to $\boldsymbol{B}_0$, which turns the neutron spin perpendicular to the magnetic field and makes it precess freely, the shutter is opened, and the neutrons leave the bottle, dropping and passing through the iron–nickel foil again, which this time acts as an analyzer of the polarization. Only those that remain in the initial spin state can reach the detector, which is located below the UCN polarizing foil. The ratio of the two spin states is determined by the exact value of the applied frequency, as shown in Eq. (9.151).

Four points, marked with crosses, slightly off resonance, are measured where the slope is steepest. The existence of the EDM should appear as a frequency shift when the electric field is applied. The measurement did not detect any shift, giving the upper limit of the dipole moment of the neutron as

$$d_n < 2.9 \times 10^{-26}\, e\,\text{cm} \tag{9.153}$$

Let us pause and think of the sensitivity that the measurement represents. Crudely speaking, the neutron is spatially spread to the size of the pion Compton wavelength ($\sim 10^{-13}$ cm). If the T violation effect is due to the weak interaction, it is probably reasonable to assume that the relative strength of emission and reabsorption of the W boson is $\sim (g_\text{w}^2/m_\text{w}^2)/(g_\text{s}^2/m_\pi^2)$. Here, g_w, g_s are the strength of the weak and strong interactions. If we assume $g_\text{w} \sim g_\text{s}$, following the spirit of the unified theory, then the strength of the EDM induced by the weak interaction is expected to be roughly of the order $10^{-13} \times (m_\pi/m_W)^2 \sim 10^{-19} e$ cm. Since the experimental value is 6 orders of magnitude smaller than the expectation, we can consider the EDM of the neutron a good test bench for T-reversal invariance. At present, the expected value of the Standard Model using the Kobayashi–Maskawa model is 10^{-33}–10^{-34} e cm. The probability of detecting the finite value of the EDM is small as long as the Standard Model is correct. However, there are models that predict a value just below the present experimental upper limit [2, 129], and an improved experimental measurement is desired.

9.3 Internal Symmetries

9.3.1 *U*(1) Gauge Symmetry

Conserved Charge

We have already introduced the most fundamental result of the internal symmetry, the conserved charge current Eq. (5.38), in Chap. 5:

$$J^\mu = iq\left[\varphi^\dagger \partial^\mu \varphi - \partial^\mu \varphi^\dagger \varphi\right] \tag{9.154}$$

This was the result of the Lagrangian's symmetry, namely invariance under the phase transformation

$$\varphi \to e^{-iq\alpha}\varphi\,, \quad \varphi^\dagger \to \varphi^\dagger e^{iq\alpha} \tag{9.155}$$

which generally holds when the Lagrangian is bilinear in complex fields, including Dirac fields. The transformation Eq. (9.155) has nothing to do with the space-time coordinates and is an example of internal symmetry, referred to as $U(1)$ gauge symmetry of the first kind.[6] The space integral of the time component of Eq. (9.154) is a conserved quantity and generically called a charge operator:

$$Q = -iq\int d^3x \sum_r \left[\pi_r(x)\varphi_r(x) - \pi_r^\dagger(x)\varphi_r^\dagger(x)\right] \tag{9.156a}$$

$$= iq\int d^3x \left[\varphi^\dagger\dot{\varphi} - \dot{\varphi}^\dagger\varphi\right] \tag{9.156b}$$

The first expression is a more general form, while the second one is specific to the complex Klein–Gordon field. It is easy to show that Q satisfies the commutation relations

$$[Q, \varphi] = -q\varphi\,, \quad [Q, \varphi^\dagger] = q\varphi^\dagger \tag{9.157}$$

This means that φ and $\varphi^\dagger$ are operators to decrease and increase the charge of the state by q. Then using the BCH formulae Eq. (9.15),

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots + \frac{1}{n!}[A, [A, [A \cdots [A, B]]]] + \cdots \tag{9.158}$$

we can show that

$$e^{i\alpha Q}\varphi e^{-i\alpha Q} = e^{-iq\alpha}\varphi\,, \quad e^{i\alpha Q}\varphi^\dagger e^{-i\alpha Q} = e^{+iq\alpha}\varphi^\dagger \tag{9.159}$$

Namely, the charge operator is the generator of the gauge transformation. In summary, we rephrase Noether's theorem in quantum field theory. If there is a continuous symmetry, i.e. a unitary transformation that keeps the Lagrangian invariant, a corresponding conserved (Noether) current exists. The space integral of its 0th component is a constant of time and is also the generator of the symmetry transformation.

9.3.2
Charge Conjugation

Charge conjugation (CC for short or C transformation) is an operation to exchange a particle and its antiparticle, or equivalently an operation to change the sign of

6) It is also called a global symmetry. When the phase is a function of space-time, i.e. $\alpha = \alpha(x)$, it is called a gauge transformation of the second kind or a local gauge transformation, which plays a key role in generating the known fundamental forces (See Chapter 18).

7) Here we use the word "charge" in a general sense. It includes not just electric charge, but also strangeness, hypercharge, etc.

the charge Q.[7] Examples of the C transformation are exchanges of an electron and positron $e^- \leftrightarrow e^+$, proton and antiproton $p \leftrightarrow \overline{p}$, and $\pi^+ \leftrightarrow \pi^-$. The photon and the neutral pion are not changed since they are charge neutral and the particles are their own antiparticles. The charge conjugation operation deals with variables in internal space, but because of the CPT theorem, it is inseparably connected with PT, space-time symmetry. This is because the antiparticle can be considered mathematically as a particle with negative energy-momentum traveling backward in time.

Charge Conjugation of the Field Operators

We learned in Chap. 4 that the charge conjugation operation generally involves taking the complex (hermitian for the operator) conjugate of the wave function or the field. This resulted from the fact that the interaction with the electromagnetic field is obtained by the gauge principle, which means replacement of the derivative by its covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu \tag{9.160}$$

Charge conjugation means essentially changing the sign of the charge, and the above form suggests it involves complex conjugation. Let us see if this is true.

We start by considering the charge eigenstate

$$Q|q, \boldsymbol{p}, s_z\rangle = q|q, \boldsymbol{p}, s_z\rangle \tag{9.161}$$

By definition, charge conjugation means

$$C|q, \boldsymbol{p}, s_z\rangle = \eta_C| - q, \boldsymbol{p}, s_z\rangle \tag{9.162}$$

η_C is a phase factor. Then

$$\begin{aligned} QC|q, \boldsymbol{p}, s_z\rangle &= \eta_C Q| - q, \boldsymbol{p}, s_z\rangle = -\eta_C q| - q, \boldsymbol{p}, s_z\rangle \\ CQ|q, \boldsymbol{p}, s_z\rangle &= qC|q, \boldsymbol{p}, s_z\rangle = q\eta_C| - q, \boldsymbol{p}, s_z\rangle \end{aligned}$$

Therefore

$$CQ = -QC \quad \text{or} \quad CQC^{-1} = -Q \tag{9.163}$$

As a particle state is constructed from the creation operator $a^\dagger$ and its antiparticle from $b^\dagger$, we define the charge conjugation operator C with the properties

$$Ca_q C^{-1} = \eta_C b_q\,, \quad Cb_q^\dagger C^{-1} = \eta_C a_q^\dagger \tag{9.164a}$$

$$Cb_q C^{-1} = \eta_C^\dagger a_q\,, \quad Ca_q^\dagger C^{-1} = \eta_C^\dagger b_q^\dagger \tag{9.164b}$$

$$CC^\dagger = C^\dagger C = 1, \quad \eta_C \eta_C^\dagger = 1 \tag{9.164c}$$

Applying the charge conjugation twice brings the field back to its original form, hence its eigenvalue is ± 1 and is referred to as C parity. The definition Eq. (9.164) of charge conjugation means the exchange $\varphi \leftrightarrow \varphi^\dagger$ for the Klein–Gordon field, as we guessed from a simple argument. Namely

$$C\varphi C^{-1} = \eta_C \varphi^\dagger , \quad C\varphi^\dagger C^{-1} = \eta_C^\dagger \varphi \tag{9.165}$$

The expression for the charge operator Eq. (9.156) shows clearly that the interchange $\varphi \rightleftharpoons \varphi^\dagger$ changes the sign of the charge operator. With the above property, it can easily be shown that the charge operator defined by Eq. (9.156) satisfies Eq. (9.163).

Problem 9.7

Prove

$$C = \exp\left[i\frac{\pi}{2} \sum_k \left(b_k^\dagger - a_k^\dagger\right)(b_k - a_k) \right] \tag{9.166}$$

has the required properties for the charge conjugation operator.

Charge Conjugation of the Dirac Field

For a field with spin, an additional operation is required to flip the spin component. For instance, for the Dirac field,

$$\begin{aligned} C\psi(x)C^{-1} &= \eta_C C' \overline{\psi}^{\mathrm{T}}(x) = \eta_C C' \gamma^0 \psi^*(x) \\ C\overline{\psi}(x)C^{-1} &= -\eta_C^* \psi^{\mathrm{T}}(x) C'^{-1} \end{aligned} \tag{9.167}$$

where C' on the rhs is a 4×4 matrix that acts on the spin components as given in Eq. (4.101). Using Eq. (9.167) and the property of the C' matrix [see Eq. (4.98)],

$$C'^{-1}\gamma^\mu C' = -(\gamma^\mu)^{\mathrm{T}} \tag{9.168}$$

the bilinear form of the Dirac operators is changed to

$$\begin{aligned} \overline{\psi}_2 \gamma^\mu \psi_1 &\xrightarrow{C} \left\{-\eta_C^* \psi_2^{\mathrm{T}} C'^{-1}\right\} \gamma^\mu \left\{\eta_C C' \overline{\psi}_1^{\mathrm{T}}\right\} \\ &= \psi_2^{\mathrm{T}} (\gamma^\mu)^{\mathrm{T}} \overline{\psi}_1^{\mathrm{T}} = -\left(\overline{\psi}_1 \gamma^\mu \psi_2\right)^{\mathrm{T}} \\ &= -\overline{\psi}_1 \gamma^\mu \psi_2 = -\left(\overline{\psi}_2 \gamma^\mu \psi_1\right)^\dagger \end{aligned} \tag{9.169}$$

The minus sign in the last equality of the second line comes from the anticommutativity of the field and the removal of the transpose in the last equality is allowed because it is a 1×1 matrix. By setting $\psi_2 = \psi_1 = \psi$, we have a C-transformed current

$$j_C^\mu = C(q\overline{\psi}\gamma^\mu \psi)C^{-1} = -q\overline{\psi}\gamma^\mu \psi = -j^\mu \tag{9.170}$$

In quantum mechanics, the change of the sign is a part of the definition and had to be put in by hand, but it is automatic in field theory. Transformation properties of other types of bilinear Dirac fields can also be obtained by using Eq. (9.168), see Table 9.4.

Table 9.4 Property of Dirac bilinears under C transformation.

	$S(t, x)$	$P(t, x)$	$V^\mu(t, x)$	$A^\mu(t, x)$	$T^{\mu\nu}(t, x)$
C	$S^\dagger(t, x)$	$P^\dagger(t, x)$	$-V^{\mu\dagger}(t, x)$	$A^{\mu\dagger}(t, x)$	$-T^{\mu\nu\dagger}(t, x)$

Note 1: $S = \overline{\psi}\psi$, $P = i\overline{\psi}\gamma^5\psi$, $V^\mu = \overline{\psi}\gamma^\mu\psi$, $A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$, $T^{\mu\nu} = \overline{\psi}\sigma^{\mu\nu}\psi$
Note 2: $V^\mu = (V^0, \boldsymbol{V})$, $V_\mu = (V^0, -\boldsymbol{V})$

Charge Conjugation of Vector Fields

As the vector and the scalar field satisfy the same Klein–Gordon equation, the charge conjugation operation transforms them as follows:

$$\varphi \to C\varphi C^{-1} = \eta_C \varphi^\dagger\,, \quad V^\mu \to C V^\mu C^{-1} = \eta_C V^{\mu\dagger} \tag{9.171}$$

The phase factor η_C cannot be determined for the free field. But if the field interacts, for instance, with the Dirac field, a Lorentz-invariant interaction Lagrangian has the form

$$\mathcal{L}_{\rm INT} = f\overline{\psi}_1\psi_2\varphi + g\overline{\psi}_1\gamma^\mu\psi_2 V_\mu + (\text{h.c.}) \tag{9.172}$$

The second term (h.c.) is the hermitian conjugate of the first term, and by its presence the Lagrangian becomes hermitian. The requirement of C invariance for the Lagrangian fixes the phase. Using Table 9.4 for the transformation of the Dirac bilinears, we obtain

$$\varphi \stackrel{C}{\to} \varphi^\dagger\,, \quad V^\mu \stackrel{C}{\to} -V^{\mu\dagger} \tag{9.173a}$$

$$f = f^*\,, \quad g = g^* \tag{9.173b}$$

As we noted earlier, by C operation, in addition to the electric charge, the strangeness, baryon number, lepton number and all other additive quantum numbers change their sign, and therefore the eigenstate of the charge conjugation has to be truly neutral in the sense that all the quantum numbers have to be zero.

C Parity of the Photon The photon is a massless vector boson and is described by the Maxwell equation.

$$\partial_\mu\partial^\mu A^\nu = q j^\nu \tag{9.174}$$

The interaction Lagrangian of the electromagnetic interaction is given by $\mathcal{L}_{\rm INT} = -q j^\mu A_\mu$. Since the electric current changes its sign by C transformation, A_μ must also change sign to keep the Lagrangian invariant. Therefore, the C parity of the photon is -1. This can also be derived from Eq. (9.173) when applied to a real vector field. The C parity of an n-photon system is $(-1)^n$. This is independent of both the orbital and the spin angular momentum.

Experimental Tests

C in the Strong Interaction: Let us consider a process

$$\overline{p} + p \to \pi^+(\pi^0) + X \quad \stackrel{C}{\longleftrightarrow} \quad p + \overline{p} \to \pi^-(\pi^0) + \overline{X} \tag{9.175}$$

If the strong interaction conserves the charge conjugation symmetry, the angular distribution measured relative to the incoming antiprotons should exhibit a symmetry between θ and $\pi - \theta$, as shown in Fig. 9.4 and in the following equations. The energy spectrum should be identical in both cases, too:

$$\frac{d\sigma}{d\Omega}(\pi^+; \theta) = \frac{d\sigma}{d\Omega}(\pi^-; \pi - \theta) \tag{9.176a}$$

$$\frac{d\sigma}{d\Omega}(\pi^0; \theta) = \frac{d\sigma}{d\Omega}(\pi^0; \pi - \theta) \tag{9.176b}$$

$$\frac{d\sigma}{dE}(\pi^+) = \frac{d\sigma}{dE}(\pi^-) \tag{9.176c}$$

These relations have been confirmed experimentally within the errors.

C Parity of the Neutral Two-Particle System: A two-particle system consisting of a particle and its antiparticle is neutral and can be a C eigenstate. We consider them as identical particles in different states. If the spin and orbital angular momentum of the system are L, S, the C parity of the system is expressed as

$$C = (-1)^{L+S} \tag{9.177}$$

Proof: If they are fermions ($f\overline{f} = e^-e^+$, $p\overline{p}$, etc.) they change sign by a particle exchange because of Fermi statistics. The exchange consists of that of the charge (C), space coordinates and spin coordinates. Therefore

$$-1 = C(-1)^L(-1)^{S+1} \tag{9.178}$$

Here, we have used the fact that the spin wave function is symmetric when $S = 1$ and antisymmetric when $S = 0$. The case for the boson system can be proved similarly. □

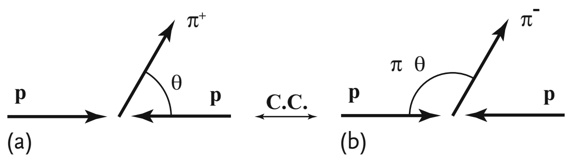


Figure 9.4 (a) $\overline{p} + p \to \pi^+ + X$, (b) $p + \overline{p} \to \pi^- + \overline{X}$. If C is conserved, $d\sigma(\overline{p} + p \to \pi^+) = d\sigma(p + \overline{p} \to \pi^-)$, where $d\sigma$ stands for either $d\sigma/d\Omega$ or $d\sigma/dE$. If the pion angle is measured relative to the incident antiprotons, $d\sigma/d\Omega\,(\theta) = d\sigma/d\Omega\,(\pi - \theta)$.

Problem 9.8

The ρ^0 meson is a vector particle with $J^P = 1^-$ and decays into $\pi^+ + \pi^-$. Prove it cannot decay into $2\pi^0$.

C in Electromagnetic Interactions: The most stringent test of C symmetry in electromagnetic interactions comes from the nonexistence of the $\pi^0 \to 3\gamma$ decay [311]:

$$\frac{\Gamma(\pi^0 \to 3\gamma)}{\Gamma_{\text{total}}} < 3.1 \times 10^{-8} \quad 90\%\text{CL} \quad (\text{CL} = \text{confidence level}) \tag{9.179}$$

The following decays are known to occur via the electromagnetic interaction through their strength (the decay rate). Asymmetry tests are not as stringent as that of decay branching ratios, but we list them here as another test. As $\eta(548)$ is a neutral scalar meson having 0^- and mass of 548 MeV, it changes to itself by C operation and so do π^0 and γ. Since π^+ and π^- are interchanged, the energy spectrum of the decays

$$\eta \to \pi^+ + \pi^- + \pi^0 \tag{9.180a}$$

$$\eta \to \pi^+ + \pi^- + \gamma \tag{9.180b}$$

should be identical. Experimentally, the asymmetry is less than 0.1%.

As π^0, η decay into 2γ, their C parity is positive and they cannot decay into 3γ. Conversely, if a particle decays into 3γ or $\pi^0\gamma$, $\eta\gamma$ its C parity is negative.

Problem 9.9

Positronium is a bound state of an electron and a positron connected by the Coulomb force. For the $L = 0$ ground state, there are two states, with $S = 0, 1$. Show that the $S = 0$ state decays into 2γ and the $S = 1$ state into 3γ, but not vice versa.

C Violation in Weak Interactions: Next we consider weak interactions, for example

$$\mu^{\mp} \to e^{\mp} + \nu + \overline{\nu} \tag{9.181}$$

If C symmetry is respected, the helicity of the electron and that of the positron should be the same provided other conditions are equal. The helicity can be measured first by making the electron emit photons by bremsstrahlung and then letting the photons pass through magnetized iron. The transmissivity of the photon depends on its polarization, which in turn depends on the electron helicity. The result has shown that $h(e^-) \simeq -1$, $h(e^+) \simeq +1$ [111, 267]. Therefore charge conjugation symmetry is almost 100% broken in weak interactions. The origin lies in the Hamiltonian. The parity-violating weak interaction Hamiltonian was given in Eq. (9.92). If we take into account the C transformation property of the axial vector (see Table 9.4) we have

$$\overline{\psi}_2 \gamma^\mu \gamma^5 \psi_1 \xrightarrow{\text{C}} \overline{\psi}_1 \gamma^\mu \gamma^5 \psi_2 \tag{9.182}$$

This and Eq. (9.173) mean under C transformation

$$\begin{aligned} C_V &\to C_V^*\,, \quad C_V{}' \to -C_V{}'^* \\ C_A &\to C_A^*\,, \quad C_A' \to -C_A'^* \end{aligned} \tag{9.183}$$

Namely, if C invariance holds, C_V, C_A have to be real and C_V', C_A' have to be purely imaginary. But the decay asymmetry of a polarized nucleus that was discussed following Eq. (9.93) has shown $C_A' = -C_A =$ real. Therefore, C is also maximally violated. But the combined CP symmetry does not change the V–A Hamiltonian of Eq. (9.94) and is conserved.

The chirality operator $(1 - \gamma^5)$ in front of the lepton field operator in Eq. (9.94) is the origin of $h(e^-) \simeq h(\nu_e) = -1$, $h(e^+) \simeq h(\overline{\nu}_e) = +1$ as we discussed in Sect. 4.3.5. The latter is typical of the weak interaction where the weak boson W^μ couples only to the left-handed field (i.e. $(1-\gamma^5)\psi$). This will be discussed in detail in Chap. 15.

The V–A interaction Eq. (9.94) respects T invariance as well as CP invariance. In principle, CP invariance and T invariance are independent. However, the combined CPT invariance is rooted deeply in the structure of quantum field theory and is the subject of the next section.

As long as CPT invariance holds, CP violation means T violation. There is evidence that small CP violation exists in the kaon and B-meson decays, hence the same amount of T violation is expected to exist. In fact, a T-violation effect consistent with CPT invariance has been observed in the neutral K meson decays [107], which will be described in detail in Chap. 16.

9.3.3 CPT Theorem

CPT is a combined transformation of C, P and T. All the coordinates are inverted $(t, \boldsymbol{x}) \to (-t, -\boldsymbol{x})$ and the operation is antiunitary because of the T transformation. All c-numbers are changed to their complex conjugates. Writing the combined operator as Θ, its action is summarized as

$$\begin{array}{ccc} (t, \boldsymbol{x}) & \xrightarrow{\text{CPT}} & (-t, -\boldsymbol{x}) \\ \text{c-numbers} & \to & (\text{c-numbers})^* \\ \langle \text{out}|H|\text{in}\rangle & \to & \langle \Theta\,\text{in}|H|\Theta\,\text{out}\rangle \\ \Theta|q, \boldsymbol{p}, \sigma\rangle & \to & |\overline{q}, \boldsymbol{p}, -\sigma\rangle \end{array} \tag{9.184}$$

The CPT combined transformation property of the Dirac bilinears is given by

$$\overline{\psi}_1 \Gamma \psi_2 \xrightarrow{\text{CPT}} \begin{cases} \overline{\psi}_2 \Gamma \psi_1 & \Gamma = \text{S, P, T} \\ -\overline{\psi}_2 \Gamma \psi_1 & \Gamma = \text{V, A} \end{cases} \tag{9.185}$$

independent of the order of the C, P, T operations. If we take the hermitian conjugate of the bilinears, the original form is recovered except for the sign of the coordinates. Transformation properties of the scalar, pseudoscalar, vector fields, etc.

are shown to be the same as the corresponding Dirac bilinears. It follows that any Lorentz scalar or second-rank tensor made of any number of fields are brought back to their original form by a combined operation of (CPT + hermitian conjugate). Therefore the Lagrangian density (a scalar) and the Hamiltonian density (a second rank tensor) transform under CPT

$$\mathscr{L}(t, \boldsymbol{x}) \xrightarrow{\text{CPT}} \mathscr{L}^\dagger(-t, -\boldsymbol{x}) \tag{9.186a}$$

$$\mathscr{H}(t, \boldsymbol{x}) \xrightarrow{\text{CPT}} \mathscr{H}^\dagger(-t, -\boldsymbol{x}) \tag{9.186b}$$

Since $\mathscr{L}$ and $\mathscr{H}$ are hermitian operators and the action is their integral over all space-time, we conclude that the equations of motion are invariant under CPT transformation. The Hamiltonian $H = \int d^3x \mathscr{H}$ changes the sign of time t, but as H is a conserved quantity and does not depend on time, the transformed Hamiltonian is the same as before it is transformed. As the scattering matrix is composed of the Hamiltonian, it is invariant, too. Unlike individual symmetry of C, P or T, the combined CPT invariance theorem can be derived with a few fundamental assumptions, such as the validity of Lorentz invariance, the local quantum field theory and the hermitian Hamiltonian. The violation of CPT means violation of either Lorentz invariance or quantum mechanics.[8] Theoretically it has firm foundations. Experimental evidence of CPT invariance is very strong, too.

We list a few examples of the CPT predictions:

(1) The mass of a particle and its antiparticle is the same.

Proof: Let H be the CPT-invariant Hamiltonian. As the mass is an eigenstate of the Hamiltonian in the particles rest frame

$$\begin{aligned} m &= \langle q, \sigma | H | q, \sigma \rangle = \langle q, \sigma | \Theta^{-1} \Theta \, H \Theta^{-1} \Theta | q, \sigma \rangle = \langle \Theta \, q, \sigma | H | \Theta \, q, \sigma \rangle^* \\ &= \langle \overline{q}, -\sigma | H | \overline{q}, -\sigma \rangle \end{aligned} \tag{9.187}$$

The spin orientation of $|\overline{q}, -\sigma\rangle$ is different from $|q, \sigma\rangle$, but the mass does not depend on the spin orientation (see Poincaré group specification for a particle state in Sect. 3.6). Therefore the masses of the particle and antiparticle are the same.

□

(2) The magnetic moment of a particle and its antiparticle is the same but its sign is reversed.

(3) The lifetime of a particle and its antiparticle is the same.

Proof: Writing the S-matrix as

$$S_{\beta\alpha} = 1 + i2\pi\delta(M_\alpha - E_f)\langle \beta; \text{out} | T | \alpha; \text{out} \rangle \tag{9.188}$$

8) It is pointed out that CPT might be violated in quantum gravity [130, 210].

Table 9.5 Summary of transformation properties under C, P and T.

	$S(t,x)$	$P(t,x)$	$V^\mu(t,x)$	$A^\mu(t,x)$	$T^{\mu\nu}(t,x)$
P	$S(t,-x)$	$-P(t,-x)$	$V_\mu(t,-x)$	$-A_\mu(t,-x)$	$T_{\mu\nu}(t,-x)$
C	$S^\dagger(t,x)$	$P^\dagger(t,x)$	$-V^{\mu\dagger}(t,x)$	$A^{\mu\dagger}(t,x)$	$-T^{\mu\nu\dagger}(t,x)$
T	$S(-t,x)$	$-P(-t,x)$	$V_\mu(-t,x)$	$A_\mu(-t,x)$	$-T_{\mu\nu}(-t,x)$
CP	$S^\dagger(t,-x)$	$-P^\dagger(t,-x)$	$-V_\mu{}^\dagger(t,-x)$	$-A_\mu{}^\dagger(t,-x)$	$-T_{\mu\nu}{}^\dagger(t,-x)$
CPT	$S^\dagger(-t,-x)$	$P^\dagger(-t,-x)$	$-V^{\mu\dagger}(-t,-x)$	$-A^{\mu\dagger}(-t,-x)$	$T^{\mu\nu\dagger}(-t,-x)$

Note 1: $S = \overline{\psi}\psi$, $P = \overline{\psi}\gamma^5\psi$, $V^\mu = \overline{\psi}\gamma^\mu\psi$, $A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$, $T^{\mu\nu} = \overline{\psi}\sigma^{\mu\nu}\psi$
Note 2: The differential operator ∂_μ transforms exactly the same as the vector field V_μ except it does not change sign under C-operation.

the total decay rate of α is given by

$$\begin{aligned}
\Gamma(\alpha, q_\alpha) &= 2\pi \sum_{\beta, q_\beta} \delta(M_\alpha - E_f)|\langle\beta, q_\beta; \text{out}|T|\alpha, q_\alpha; \text{out}\rangle|^2 \\
&= 2\pi \sum_{\beta, q_\beta} \delta(M_\alpha - E_f)|\langle\beta, q_\beta; \text{out}|\Theta^{-1}\Theta\, T\Theta^{-1}\Theta|\alpha, q_\alpha; \text{out}\rangle^*|^2 \\
&= 2\pi \sum_{\bar{\beta}, \bar{q}_\beta} \delta(M_{\bar{\alpha}} - E_f)|\langle\bar{\beta}, \bar{q}_\beta; \text{in}|T|\bar{\alpha}, \bar{q}_\alpha; \text{in}\rangle^*|^2 \\
&= \Gamma(\bar{\alpha}, \bar{q}_\alpha)
\end{aligned} \tag{9.189}$$

where, q_α, q_β denote quantum numbers other than particle species. We have used the fact that for one particle state, $|\alpha, q_\alpha; \text{in}\rangle = |\alpha, q_\alpha; \text{out}\rangle$, $M_\alpha = M_{\bar{\alpha}}$ and that both "in" and "out" states form complete sets of states, i.e.

$$\sum_{\beta, q_\beta} |\beta, q_\beta; \text{in}\rangle\langle\beta, q_\beta; in| = \sum_{\beta, q_\beta} |\beta, q_\beta; \text{out}\rangle\langle\beta, q_\beta; \text{out}| = 1 \tag{9.190}$$

□

Finally, in Table 9.5 we give a summary list of the transformation properties of C, P and T.

9.3.4
SU(2) (Isospin) Symmetry

Isospin Multiplets

Among hadrons there are many small groups in which members having different electric charge share some common properties. For instance, in the following example members have almost the same mass values (given in units of MeV) within groups.

$$\begin{array}{lllllllll} & \begin{bmatrix} p \\ \\ n \end{bmatrix} & \begin{array}{l} 938.272 \\ \\ 939.565 \end{array} & \begin{bmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{bmatrix} & \begin{array}{l} 139.570 \\ 134.977 \\ 139.570 \end{array} & \begin{bmatrix} K^+ \\ \\ K^0 \end{bmatrix} & \begin{array}{l} 493.68 \\ \\ 497.65 \end{array} & \begin{bmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{bmatrix} & \begin{array}{l} 1189.37 \\ 1192.64 \\ 1197.45 \end{array} \\ \delta m/m & & 0.14\,\% & & 3.3\,\% & & 0.8\,\% & & 0.33\,\% \end{array} \tag{9.191}$$

In addition, it is known that the strength of the interaction is almost the same within the accuracy of a few percent. Because they have different electric charges, the difference can be ascribed to the electromagnetic interaction, which is weaker by $\sim O(\alpha) = 1/137$. Therefore, if we neglect the small mass difference, there is no distinction among the members of the groups as far as the strong interaction is concerned. They constitute multiplets but are degenerate. In analogy to spin multiplets, which are degenerate under a central force, we call them "isospin multiplets". Just as the degeneracy of the spin multiplets is resolved by a magnetic field (the Zeeman effect), that of the isospin multiplets is resolved by turning on the electromagnetic force. We conceive of an abstract space (referred to as internal space in contrast to external or real space) and consider the strong interaction as a central force with the electromagnetic force violating the rotational invariance just like the magnetic field in external space. If we identify the isospin as the equivalent of spin in real space, the mathematical structure of isospin is exactly the same as that of spin, which is the origin of the name.

Charge Independence

Historically, the concept of isospin was first proposed by Heisenberg in 1932 to consider the proton and the neutron as two different states of the same particle (nucleon) when he recognized the fact that the energy levels of mirror nuclei are very similar. Consider the potentials between the nucleons V. When

$$V_{pp} = V_{nn} \qquad \text{charge symmetry} \tag{9.192}$$

we call it charge symmetry and when

$$V_{pp} = V_{nn} = V_{pn} \quad \text{charge independence} \tag{9.193}$$

we call it charge independence. To formulate the mathematics of the charge independence, we consider transformation of the proton wave function ψ_p and that of the neutron ψ_n and define the doublet function ψ by

$$\psi \equiv \begin{bmatrix} \psi_p \\ \psi_n \end{bmatrix} \to \psi' = \begin{bmatrix} \psi_p{}' \\ \psi_n{}' \end{bmatrix} = \begin{bmatrix} \alpha\psi_p + \beta\psi_n \\ \gamma\psi_p + \delta\psi_n \end{bmatrix} = U\psi \tag{9.194}$$

Charge independence means the expectation value of the potential does not change under the transformation U. Conservation of probability requires U to be a unitary matrix

$$UU^\dagger = U^\dagger U = 1 \tag{9.195}$$

Under this restriction, U should have the form

$$U = e^{i\phi}\begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1 \tag{9.196}$$

If we further require U to be unimodular ($\det U = 1$), we have $\phi = 0$. Generally, any transformation that keeps the norm of N complex numbers

$$\psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + \cdots + |\psi_N|^2 \tag{9.197}$$

invariant makes a group that is called a unitary group $U(N)$, and when $\det U = 1$ it is called a special unitary group $SU(N)$. An $SU(N)$ matrix can be expressed as

$$U = \exp\left[-\frac{i}{2}\sum_{i=1}^{N^2-1}\lambda_i\theta_i\right] \tag{9.198}$$

where λ_i are traceless hermitian matrices (Appendix G) called generators of the $SU(N)$ transformation. For $N = 2$, there are three traceless hermitian matrices and we can adopt the Pauli matrices for them:

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{9.199}$$

Since $\boldsymbol{I} = \boldsymbol{\tau}/2$ satisfies the same commutation relation

$$[I_i, I_j] = i\varepsilon_{ijk}I_k \tag{9.200}$$

as the angular momentum, the $SU(2)$ transformation is mathematically equivalent (holomorphic) to the rotational operation. The difference is that the operand is not a spin state in real space, but a multiplet consisting of particles of roughly the same mass as those in Eq. (9.191). Namely, the act of rotation is performed not in a real space but in a kind of abstract space (referred to as isospin space) and the state vector ψ denotes p when its direction is upward and n when downward. Such symmetry in the internal space is called an internal symmetry. External symmetries refer to those in a real space such as spin and parity. If we denote the difference of the states like p or n by a subscript r as $\phi_r(x)$, the internal symmetry operation acts on r, while the external symmetry acts on the space-time coordinate x and legs (components) of Lorentz tensors. They are generally independent operations, but sometimes connected like CPT.

As the mathematics of $SU(2)$ is the same as that of angular momentum, it is convenient to express various physical quantities using the same terminology as used in space rotation. For instance, p and n make a doublet; we say the nucleon has isospin $I = 1/2$. The three kinds of pions $\pi^\pm, \pi^0$ constitute a triplet, or a vector in isospace, and have $I = 1$. The term "charge independence" of the nuclear force means the interaction between the nucleon and the pion is rotationally invariant in isospin space, namely the interaction Hamiltonian is an isoscalar.

Conserved Isospin Current

We consider the proton and the neutron as the same particle, but they have different projections in the abstract isospin space and constitute a doublet, called an isospinor:

$$\psi = \begin{bmatrix} p \\ n \end{bmatrix} \tag{9.201}$$

Here we have adopted the notation p, n to denote ψ_p, ψ_n. The Lagrangian density is written as

$$\begin{aligned} \mathcal{L} &= \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\ &= \overline{p}(i\gamma^\mu \partial_\mu - m)p + \overline{n}(i\gamma^\mu \partial_\mu - m)n \end{aligned} \tag{9.202}$$

They should have the same mass because of the isospin symmetry. Note, p is a spinor with four components in real space, and ψ a two-component spinor in isospin space. Therefore, ψ has 8 independent components. In exactly the same manner as a rotation in real space, a rotation in isospin space can be carried out by using a unitary operator

$$U_I = e^{-i\sum_i \alpha_j I_j} = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{I}} \tag{9.203}$$

where $\boldsymbol{I} = (I_1, I_2, I_3)$ satisfies Eq. (9.200). Let us consider transformations defined as

$$\psi \to \psi' = e^{-\frac{i}{2}\tau_j \alpha_j}\psi \equiv e^{-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}}\psi, \tag{9.204a}$$

and

$$\overline{\psi} \to \overline{\psi}' = \overline{\psi} e^{\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}} \quad \text{where} \quad \overline{\psi} = (\overline{p},\ \overline{n}) \tag{9.204b}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is a set of three independent but constant variables. Since τ_i, α_i are independent of space-time, the transformation does not change any space-time structure of the spinors p, n, and the Lagrangian density is kept invariant under the continuous isospin rotation. Equations (9.204) have exactly the same form as the $U(1)$ phase transformation of Eq. (9.155) and is called the (global) $SU(2)$ gauge transformation. Now consider an infinitesimal rotation

$$\begin{aligned} &\psi \to \psi + \delta\psi\,, && \overline{\psi} \to \overline{\psi} + \delta\overline{\psi} \\ &\delta\psi = -\frac{i}{2}\tau^i \psi \varepsilon_i = -\frac{i}{2}[\tau^i]_{ab}\psi_b \varepsilon_i\,, && \delta\overline{\psi} = \frac{i}{2}\overline{\psi}\tau^i \varepsilon_i = \frac{i}{2}\overline{\psi}_a[\tau^i]_{ab}\varepsilon_i \end{aligned} \tag{9.205}$$

Applying Noether's formula Eq. (5.32) to the transformation, we obtain

$$\begin{aligned} J^\mu &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu \phi_r)}\delta\phi_r \overset{\phi_r = \psi,\overline{\psi}}{=} \frac{\delta\mathcal{L}}{\delta(\partial_\mu \psi)}\delta\psi + \delta\overline{\psi}\frac{\delta\mathcal{L}}{\delta(\partial_\mu \overline{\psi})} \\ &= \left[-\frac{\delta\mathcal{L}}{\delta(\partial_\mu \psi_a)}\frac{i}{2}[\tau^i]_{ab}\psi_b + \frac{i}{2}\overline{\psi}_a[\tau^i]_{ab}\frac{\delta\mathcal{L}}{\delta(\partial_\mu \overline{\psi}_b)}\right]\varepsilon_i \\ &= \frac{1}{2}\overline{\psi}\gamma^\mu \tau^i \psi \varepsilon_i \end{aligned} \tag{9.206}$$

The second term in the second line vanishes because there is no $\partial_\mu \overline{\psi}$ in the Lagrangian. Since ε_i is an arbitrary constant, Eq. (9.206) defines a conserved current referred to as isospin current:

$$(J^\mu)_j = \overline{\psi}\gamma^\mu \frac{\tau_j}{2}\psi \tag{9.207}$$

It is easy to check that an isospin operator defined by

$$\hat{\boldsymbol{I}}_j = \int d^3x\, \overline{\psi}\gamma^0 \frac{\tau_j}{2}\psi \tag{9.208}$$

satisfies the relations

$$[I_j, \psi] = -I_j\psi\,, \quad [I_j, \psi^\dagger] = I_j\psi\,, \quad e^{i\boldsymbol{\alpha}\cdot\hat{\boldsymbol{I}}}\psi e^{-i\alpha\hat{I}} = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{I}}\psi \tag{9.209}$$

Therefore the operator $\hat{\boldsymbol{I}}$ is the generator of the isospin rotation which appeared in Eq. (9.203). The isospin current differs from the Dirac current only in the presence of the isospin operator $\boldsymbol{\tau}/2$ sandwiched between the isospinors $\overline{\psi}$ and ψ.

In a similar vein, when we have three identical real Klein–Gordon fields $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, we can also establish the rotational invariance of $\boldsymbol{\phi}\cdot\boldsymbol{\phi} = {\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2$ and that of the Lagrangian, which can be expressed as

$$\mathcal{L} = \partial_\mu \boldsymbol{\phi}^\dagger \cdot \partial^\mu \boldsymbol{\phi} - m^2 \boldsymbol{\phi}^\dagger \cdot \boldsymbol{\phi} \tag{9.210}$$

leading to the existence of conserved isospin angular momentum, for which $\boldsymbol{\phi}$ is a $I = 1$ triplet. Since the pions have spin 0 and constitute a triplet of almost the same mass, we can identify them with the field $\boldsymbol{\phi}$.[9] In analogy to Eq. (9.208), it can be expressed as

$$\boldsymbol{I}_\pi = -i \int d^3x\, \pi_a [\boldsymbol{t}]_{ab} \phi_b \tag{9.211a}$$

$$t^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad t^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad t^3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{9.211b}$$

However, to identify $(|\pi^+\rangle, |\pi^0\rangle, |\pi^-\rangle)$ as the $T_3 = (+1, 0, -1)$ state, it is more convenient to use a slightly different representation:

$$\begin{aligned} {t_1}' &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad {t_2}' = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \\ {t_3}' &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \end{aligned} \tag{9.212}$$

9) Note, the pion field is a $I = 1$ triplet representation of the $SU(2)$ group. If the fundamental representation with $I = 1/2$ does not exist, we may have to consider the pion as the fundamental representation of the $SO(3)$ group, as π_j's are real fields.

Problem 9.10

Prove that $\boldsymbol{I}, \boldsymbol{I}'$, satisfy

$$[I_i,\ I_j] = i\varepsilon_{ijk} I_k\,, \quad [I_{\pi i},\ I_{\pi j}] = i\varepsilon_{ijk} I_{\pi k} \tag{9.213a}$$

$$[I_i, \psi_j] = i\varepsilon_{ijk}\psi_k\,, \quad [I_{\pi i},\ \phi_j] = i\varepsilon_{ijk}\phi_k \tag{9.213b}$$

Problem 9.11

Prove

$$\left[I_{\pi 3}, \frac{\phi_1 \pm i\phi_2}{\sqrt{2}}\right] = \pm\frac{\phi_1 \pm i\phi_2}{\sqrt{2}}\,, \quad [I_{\pi 3}, \phi_3] = 0 \tag{9.214}$$

This equation can be used to define

$$\frac{\phi_1 \pm i\phi_2}{\sqrt{2}}|0\rangle = \alpha_p|\pi^{\pm}\rangle = \mp|\pi^{\pm}\rangle\,, \quad \phi_3 = |\pi^0\rangle \tag{9.215}$$

The phase α_p was chosen to produce

$$I_{\pi\pm}|I = 1, I_3 = 0\rangle = \sqrt{2}|1, \pm 1\rangle \tag{9.216}$$

which is a special case of the usual convention adopted from the angular momentum operator:

$$I_{\pi\pm}|I, I_3\rangle = [(I \mp I_{\pi 3})(I \pm I_3 + 1)]^{1/2}|I, I_3 \pm 1\rangle \tag{9.217}$$

The real nucleon and the pion do not have exactly the same mass within the multiplet. The isospin rotation is not an exact symmetry, but as long as the difference is small we can treat the isospin as a conserved quantity.

Isospin of the Antiparticle

In order to find the isospin of the antiparticle, we apply the charge conjugation operation to Eq. (9.194). Then the field represents the antiparticle. As the C operation contains complex conjugation, the transformation matrix becomes

$$\overline{p}' = \quad \alpha^*\overline{p} + \beta^*\overline{n} \tag{9.218a}$$

$$\overline{n}' = -\beta\ \ \overline{p} + \alpha\ \ \overline{n} \tag{9.218b}$$

here, $\overline{p}, \overline{n}$ denote the fields of the antiparticles p, n. In general, if the symmetry transformation matrix is U, that of the antiparticle is given by its complex conjugate U^*. Rearranging Eq. (9.218), we obtain

$$(-\overline{n}') = \quad \alpha \ (-\overline{n}) + \beta \ \overline{p} \tag{9.219a}$$

$$\overline{p}' = -\beta^*(-\overline{n}) + \alpha^*\overline{p} \tag{9.219b}$$

which means

$$\begin{bmatrix} -\overline{n}' \\ \overline{p}' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix} \begin{bmatrix} -\overline{n} \\ \overline{p} \end{bmatrix} = U \begin{bmatrix} -\overline{n} \\ \overline{p} \end{bmatrix} \tag{9.220}$$

The equation means that if (p, n) is an isodoublet, $(-\overline{n}, \overline{p})$ is also an isodoublet that transforms in the same way.

Charge Symmetry We mentioned charge symmetry in Eq. (9.192). It can be defined more explicitly as the rotation in isospin space by π on the y-axis. For a doublet its operation is

$$e^{-i\pi I_2} = e^{-i(\pi/2)\tau_2} = -i\tau_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{9.221}$$