5 Equations of motion: integral approach

Integral principles and, in particular, Hamilton's principle, have long occupied a prominent position in analytical mechanics. Hamilton's principle, first announced in 1834, presents a *variational principle* as the basis for the dynamical description of a holonomic system. This approach tends to view the motion as a whole and involves a search for the path in configuration space which yields a stationary value for a certain integral. As a result, one obtains the differential equations of motion.

The requirement of stationarity does not apply to nonholonomic systems. Nevertheless, one can use integral methods to obtain the equations of motion for nonholonomic systems. Here we use the integral of the variation rather than the variation of the integral. In this chapter, we shall discuss the derivation and application of these methods, particularly with respect to nonholonomic systems.

5.1 Hamilton's principle

Holonomic system

Consider a dynamical system whose motion satisfies *Lagrange's principle*, namely,

$$\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}-Q_i\right]\delta q_i=0 \tag{5.1}$$

There are n generalized coordinates and the δqs satisfy the instantaneous constraints. The kinetic energy $T(q, \dot{q}, t)$ is written for the unconstrained system, and is assumed to have at least two continuous derivatives in each of its arguments. Q_i is the generalized applied force associated with q_i.

Now integrate (5.1) with respect to time over the fixed interval t_1 to t_2. Using integration by parts, we find that

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i dt=-\int_{t_1}^{t_2}\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)dt+\left[\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} \tag{5.2}$$

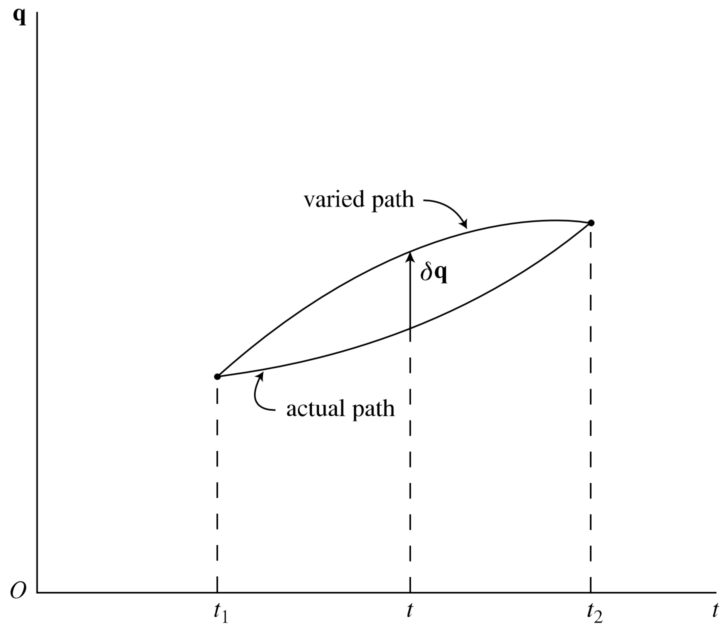

Figure 5.1.

Hence, we obtain

$$\int_{t_1}^{t_2}\left[\sum_{i=1}^{n}\frac{\partial T}{\partial q_i}\delta q_i+\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)+\sum_{i=1}^{n}Q_i\delta q_i\right]dt=\left[\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} \tag{5.3}$$

The δqs satisfy the m instantaneous constraint equations

$$\sum_{i=1}^{n}a_{ji}(q,t)\delta q_i=0 \qquad (j=1,\ldots,m) \tag{5.4}$$

and are assumed to equal zero at the fixed end points t_1 and t_2. Thus, the right-hand side of (5.3) vanishes.

The actual and varied paths in extended configuration space are shown in Fig. 5.1. The δqs are *contemporaneous* variations, that is, they take place with time held fixed. Note that, for a given actual path, the varied path is specified by the δqs which satisfy (5.4).

Let us assume that the qs and $\dot{q}$s are continuous functions of time along the actual and varied paths. Then we can write

$$\frac{d}{dt}(\delta q_i)=\delta\dot{q}_i \qquad (i=1,\ldots,n) \tag{5.5}$$

The transposition of d and δ operators will be discussed later in the chapter.

Referring again to (5.3), note that

$$\delta T=\sum_{i=1}^{n}\frac{\partial T}{\partial q_i}\delta q_i+\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\delta\dot{q}_i \tag{5.6}$$

and the virtual work is

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i \tag{5.7}$$

Thus, we obtain

$$\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0 \tag{5.8}$$

This important result applies to the same wide variety of dynamical systems as does Lagrange's principle as given by (5.1). We shall return to this equation when we consider nonholonomic systems.

Now let us assume that all the applied forces are associated with a potential energy function $V(q, t)$. Then $\delta W = -\delta V$ and we can write (5.8) in the form

$$\int_{t_1}^{t_2} \delta L dt = 0 \tag{5.9}$$

where the Lagrangian function $L(q, \dot{q}, t) = T - V$. Assuming a *holonomic system*, the operations of integration and variation can be interchanged. Thus, we obtain

$$\delta \int_{t_1}^{t_2} L dt = 0 \tag{5.10}$$

which is the usual form of *Hamilton's principle*.

The variation of the integral in (5.10) implies that the actual and varied paths satisfy the m constraint equations of the form

$$\sum_{i=1}^{n} q_{ji}(q, t)\dot{q}_i + a_{jt}(q, t) = 0 \qquad (j = 1, \ldots, m) \tag{5.11}$$

where these expressions are integrable in this holonomic case. On the other hand, the integral of the variation, as in (5.9), implies that the δqs in the expression for δL must satisfy the instantaneous constraints of (5.4). For holonomic systems, the varied paths satisfy both the actual and instantaneous constraint equations. The solutions of (5.10) have the property of *stationarity*; whereas the solutions of (5.9) may or may not have this property, depending on the nature of the constraints.

Now let us restate Hamilton's principle, as it applies to holonomic systems, as follows: *The actual path in configuration space followed by a holonomic dynamical system between the fixed times t_1 and t_2 is such that the integral*

$$I = \int_{t_1}^{t_2} L dt \tag{5.12}$$

is stationary with respect to path variations which vanish at the end-points.

To reiterate, the primary assumptions in the derivation of Hamilton's principle are that: (1) the variations δq_i satisfy the instantaneous constraint equations; (2) the end-points are fixed in configuration space and time; and (3) all the applied forces are derivable from a potential energy function $V(q, t)$. Note that the system need not be conservative.

An alternate approach to obtaining the equations of motion for a holonomic system is to begin with Hamilton's principle as a stationarity principle. With this as a starting point, and using the same assumptions as before, we can derive Lagrange's principle. To see how this develops, let us begin with

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt = 0 \tag{5.13}$$

Then, using (5.5) and integrating by parts, we obtain

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right] \delta q_i dt = \sum_{i=1}^{n} \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} = 0 \tag{5.14}$$

If the δqs are unconstrained, and therefore arbitrary, each coefficient must equal zero, yielding

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{5.15}$$

which is Lagrange's equation. This is also the Euler–Lagrange equation of the calculus of variations.

On the other hand, if the δqs are constrained by (5.4), then the integrand must equal zero at each instant of time since the limits t_1 and t_2 are arbitrary. Thus, we obtain Lagrange's principle, namely,

$$\sum_{i=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right] \delta q_i = 0 \tag{5.16}$$

for this case where all the generalized applied forces are obtained from the potential energy $V(q, t)$.

Nonholonomic system

Although stationarity, as expressed in Hamilton's principle, is central to the dynamical theory of holonomic systems, it does not apply to nonholonomic systems. To see how this comes about, let us consider a nonholonomic system and require that each varied path must satisfy the actual constraint equations of the general form

$$f_j(q, \dot{q}, t) = 0 \qquad (j = 1, \ldots, m) \tag{5.17}$$

These constraints are enforced by invoking the *multiplier rule*. The multiplier rule states that the constrained stationary values of the integral of (5.12) are found by considering the *free variations* of

$$I = \int_{t_1}^{t_2} \Lambda dt \tag{5.18}$$

where $\Lambda(q, \dot{q}, \mu, t)$ is the *augmented Lagrangian function* which is formed by adjoining the constraint functions to the Lagrangian function by using Lagrange multipliers. Thus,

$$\Lambda = L(q, \dot{q}, t) + \sum_{j=1}^{m} \mu_j f_j(q, \dot{q}, t) \tag{5.19}$$

where the Lagrange multipliers $\mu_j(t)$ are treated as additional variables to be determined.

The stationarity of the free variations of the integral of (5.18) results in the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}_i}\right) - \frac{\partial \Lambda}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{5.20}$$

$$\frac{\partial \Lambda}{\partial \mu_j} = f_j(q, \dot{q}, t) = 0 \qquad (j = 1, \ldots, m) \tag{5.21}$$

Note that (5.21) merely restates the constraint equations.

Now let us apply (5.20) to a nonholonomic system in which the constraint functions are linear in the $\dot{q}$s. Thus, the constraint equations have the familiar form

$$f_j(q, \dot{q}, t) = \sum_{i=1}^{n} a_{ji}(q, t)\dot{q}_i + a_{jt}(q, t) = 0 \qquad (j = 1, \ldots, m) \tag{5.22}$$

Then, using (5.19) and (5.20), we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= -\sum_{j=1}^{m} \frac{d}{dt}(\mu_j a_{ji}) + \sum_{j=1}^{m}\sum_{k=1}^{n} \mu_j \frac{\partial a_{jk}}{\partial q_i}\dot{q}_k + \sum_{j=1}^{m} \mu_j \frac{\partial a_{jt}}{\partial q_i} \\
&= -\sum_{j=1}^{m} \dot{\mu}_j a_{ji} + \sum_{j=1}^{m}\sum_{k=1}^{n} \mu_j \left(\frac{\partial a_{jk}}{\partial q_i} - \frac{\partial a_{ji}}{\partial q_k}\right)\dot{q}_k \\
&\quad + \sum_{j=1}^{m} \mu_j \left(\frac{\partial a_{jt}}{\partial q_i} - \frac{\partial a_{ji}}{\partial t}\right) \qquad (i = 1, \ldots, n)
\end{aligned} \tag{5.23}$$

These are the Euler–Lagrange equations for finding the solution path leading to a stationary value of the integral I of (5.12), where both the actual and varied paths satisfy the constraints given by (5.22). Comparing (5.23) with the known form of Lagrange's equation for this nonholonomic system, namely,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{5.24}$$

we see that, in general, the equations are different. We conclude that the requirement of stationarity leads to *incorrect dynamical equations* for the general case of *nonholonomic* constraints. Conversely, the solution path of a nonholonomic system will not, in general, result in a stationary value of the integral in (5.12).

On the other hand, if we equate $-\dot{\mu}_j$ with λ_j and set

$$\frac{\partial a_{jk}}{\partial q_i} - \frac{\partial a_{ji}}{\partial q_k} = 0 \quad \text{and} \quad \frac{\partial a_{jt}}{\partial q_i} - \frac{\partial a_{ji}}{\partial t} = 0 \quad \begin{pmatrix} i, k = 1, \ldots, n \\ j = 1, \ldots, m \end{pmatrix} \tag{5.25}$$

which are the exactness conditions, then the system is *holonomic* and (5.23) reduces to the correct equation (5.24).

The correct equations of motion of a nonholonomic system can be obtained from

$$\int_{t_1}^{t_2} \delta L dt = 0 \tag{5.26}$$

which may be considered to be the *nonholonomic form of Hamilton's principle*. It is not a stationarity principle, however, and thereby differs fundamentally from its usual holonomic form given in (5.10). Equation (5.26) assumes that: (1) the actual and varied paths are continuous functions of time and their difference $\delta\mathbf{q}$ satisfies the instantaneous constraint equations; (2) the δqs equal zero at the fixed end-points t_1 and t_2; and (3) all the applied forces arise from a potential energy function $V(q, t)$.

More generally, when the applied forces do not arise from a potential energy function, one can use

$$\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0 \tag{5.27}$$

where the virtual work is

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i \tag{5.28}$$

and the δqs satisfy (5.4). As we found in the derivation of (5.8), this result is essentially an integrated form of Lagrange's principle, (5.1).

Example 5.1 A flat rigid body of mass m moves in the horizontal xy-plane (Fig. 5.2). There is a knife-edge constraint at the reference point P, about which the moment of inertia

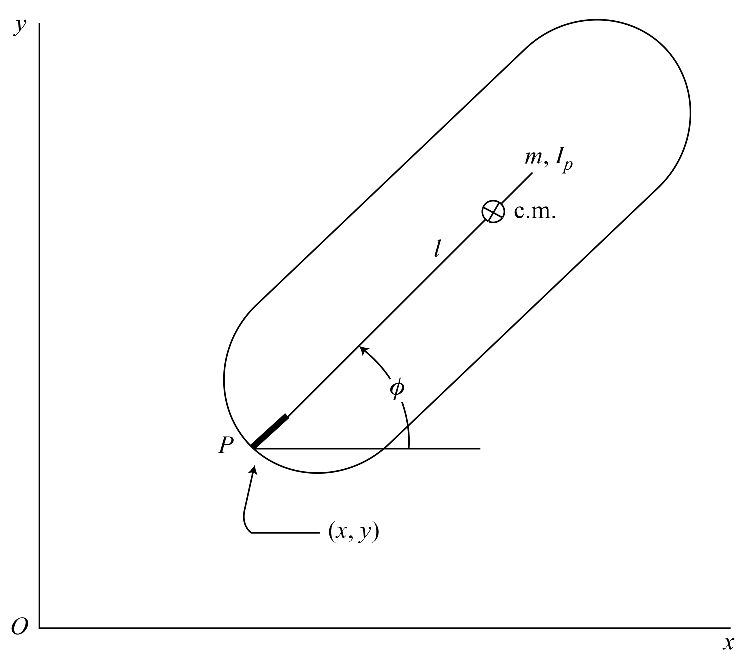

Figure 5.2.

is I_p. Assuming the center of mass is located at a distance l from P, and using (x, y, ϕ) as generalized coordinates, let us find the differential equations of motion.

We can take $V = 0$, so Hamilton's principle has the nonholonomic form

$$\int_{t_1}^{t_2} \delta T \, dt = 0 \tag{5.29}$$

The unconstrained kinetic energy is, from (3.146),

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_p \dot{\phi}^2 + ml\dot{\phi}(-\dot{x} \sin\phi + \dot{y} \cos\phi) \tag{5.30}$$

and we obtain

$$\begin{aligned} \delta T &= m\dot{x}\delta\dot{x} + m\dot{y}\delta\dot{y} + I_p\dot{\phi}\delta\dot{\phi} - ml\dot{\phi}\sin\phi\,\delta\dot{x} + ml\dot{\phi}\cos\phi\,\delta\dot{y} \\ &\quad + ml(-\dot{x}\sin\phi + \dot{y}\cos\phi)\delta\dot{\phi} + ml\dot{\phi}(-\dot{x}\cos\phi - \dot{y}\sin\phi)\delta\phi \end{aligned} \tag{5.31}$$

Noting that

$$\delta\dot{x} = \frac{d}{dt}(\delta x), \qquad \delta\dot{y} = \frac{d}{dt}(\delta y), \qquad \delta\dot{\phi} = \frac{d}{dt}(\delta\phi) \tag{5.32}$$

we find that

$$\begin{aligned} \delta T &= (m\dot{x} - ml\dot{\phi}\sin\phi)\frac{d}{dt}(\delta x) + (m\dot{y} + ml\dot{\phi}\cos\phi)\frac{d}{dt}(\delta y) \\ &\quad + (I_p\dot{\phi} - ml\dot{x}\sin\phi + ml\dot{y}\cos\phi)\frac{d}{dt}(\delta\phi) - ml\dot{\phi}(\dot{x}\cos\phi + \dot{y}\sin\phi)\delta\phi \end{aligned} \tag{5.33}$$

Now integrate by parts, noting that the δqs equal zero at the end-points. We obtain

$$\begin{aligned} \int_{t_1}^{t_2} \delta T \, dt = -\int_{t_1}^{t_2} \Bigg\{ &\frac{d}{dt}(m\dot{x} - ml\dot{\phi}\sin\phi)\delta x + \frac{d}{dt}(m\dot{y} + ml\dot{\phi}\cos\phi)\delta y \\ &+ \bigg[\frac{d}{dt}(I_p\dot{\phi} - ml\dot{x}\sin\phi + ml\dot{y}\cos\phi) \\ &+ ml\dot{\phi}(\dot{x}\cos\phi + \dot{y}\sin\phi) \bigg] \delta\phi \Bigg\} dt = 0 \end{aligned} \tag{5.34}$$

The end-points t_1 and t_2 are arbitrary, so the integrand must be zero continuously. Thus, we obtain

$$\begin{aligned} &(m\ddot{x} - ml\ddot{\phi}\sin\phi - ml\dot{\phi}^2\cos\phi)\delta x + (m\ddot{y} + ml\ddot{\phi}\cos\phi - ml\dot{\phi}^2\sin\phi)\delta y \\ &\quad + (I_p\ddot{\phi} - ml\ddot{x}\sin\phi + ml\ddot{y}\cos\phi)\delta\phi = 0 \end{aligned} \tag{5.35}$$

which is essentially Lagrange's principle of (5.1).

The nonholonomic constraint equation is

$$\dot{x}\sin\phi - \dot{y}\cos\phi = 0 \tag{5.36}$$

which states that the velocity of point P perpendicular to the knife edge is zero. The corresponding instantaneous constraint equation is

$$\sin\phi\ \delta x - \cos\phi\ \delta y = 0 \tag{5.37}$$

We can choose two independent sets of $(\delta x, \delta y, \delta\phi)$ which satisfy (5.37). Let us choose virtual displacements proportional to $(\cos\phi, \sin\phi, 0)$ and $(0, 0, 1)$. Then, from (5.35) we obtain the following two differential equations of motion:

$$m\ddot{x}\cos\phi + m\ddot{y}\sin\phi - ml\dot{\phi}^2 = 0 \tag{5.38}$$

$$I_p\ddot{\phi} - ml\ddot{x}\sin\phi + ml\ddot{y}\cos\phi = 0 \tag{5.39}$$

Alternatively, we could have noted that

$$\delta y = \tan\phi\ \delta x \tag{5.40}$$

and then considered δx and $\delta\phi$ to be independent.

We need a third differential equation which is obtained by differentiating (5.36) with respect to time.

$$\ddot{x}\sin\phi - \ddot{y}\cos\phi + \dot{x}\dot{\phi}\cos\phi + \dot{y}\dot{\phi}\sin\phi = 0 \tag{5.41}$$

Equations (5.38), (5.39), and (5.41) are linear in the $\ddot{q}$s and can be solved for $\ddot{x}$, $\ddot{y}$, and $\ddot{\phi}$, which are integrated to obtain the motion as a function of time.

The approach used in this example has yielded three second-order equations, namely, two dynamical equations and one kinematical equation. Notice that $\dot{q}$s have been used as velocity variables in the kinetic energy function; quasi-velocities should be avoided because equations similar to (5.32) will not apply.

5.2 Transpositional relations

Now let us examine the kinematical effects due to the nonintegrability of the quasi-velocity expressions and constraint equations often associated with nonholonomic systems. As before, we shall ultimately be concerned with time integrals of variational expressions, although the transpositional relations under consideration here are differential in nature. Hence, we will actually study the kinematics of differential paths in configuration space.

The d and δ operators

Let us begin with the differential form

$$d\theta_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)dq_i + \Psi_{jt}(q,t)dt \qquad (j = 1, \ldots, n) \tag{5.42}$$

In general, the differential form is not integrable, so θ_j is a quasi-coordinate. The operator d, as in dq_i, represents an infinitesimal change in the variable q_i which occurs during the