and the potential energy is

$$V = mg\left[-x\sin\alpha + \frac{1}{2}r\cos(\theta+\alpha)\right] \tag{2.126}$$

The generalized momenta are

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x} + \frac{1}{2}mr\dot{\theta}\cos\theta \tag{2.127}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2}mr\dot{x}\cos\theta + \frac{1}{4}mr^2\dot{\theta} \tag{2.128}$$

Equations (2.127) and (2.128) can be solved for $\dot{x}$ and $\dot{\theta}$. We obtain

$$\dot{x} = \frac{rp_x - 2p_\theta\cos\theta}{mr\sin^2\theta} \tag{2.129}$$

$$\dot{\theta} = \frac{4p_\theta - 2rp_x\cos\theta}{mr^2\sin^2\theta} \tag{2.130}$$

Substituting these expressions for $\dot{x}$ and $\dot{\theta}$ into the kinetic energy equation, we obtain, after some algebraic simplification, the Hamiltonian function

$$H = T + V = \frac{1}{2mr^2\sin^2\theta}\left(r^2p_x^2 + 4p_\theta^2 - 4rp_xp_\theta\cos\theta\right) + V \tag{2.131}$$

where V is given by (2.126).

We shall use Hamilton's equations in the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{m}\lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.132}$$

where, from (2.124), the constraint coefficients are

$$a_{11} = 1, \qquad a_{12} = -r \tag{2.133}$$

The $\dot{x}$ equation is

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{rp_x - 2p_\theta\cos\theta}{mr\sin^2\theta} \tag{2.134}$$

The $\dot{p}_x$ equation is

$$\dot{p}_x = -\frac{\partial H}{\partial x} + \lambda = -\frac{\partial V}{\partial x} + \lambda = mg\sin\alpha + \lambda \tag{2.135}$$

The $\dot{\theta}$ equation is

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{4p_\theta - 2rp_x\cos\theta}{mr^2\sin^2\theta} \tag{2.136}$$

Finally, the $\dot{p}_\theta$ equation is

$$\begin{aligned}\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} - r\lambda \\ &= \frac{1}{mr^2 \sin^3\theta}\left[\left(r^2 p_x^2 + 4p_\theta^2\right)\cos\theta - 2rp_x p_\theta(1+\cos^2\theta)\right] \\ &\quad + \frac{1}{2}mgr\sin(\theta+\alpha) - r\lambda \end{aligned} \tag{2.137}$$

These four first-order Hamiltonian equations plus the constraint equation (2.124) can be integrated numerically to solve for the 2 qs, 2ps, and λ as functions of time.

For a *scleronomic system* such as the one we have been considering, relatively simple matrix equations can be used to obtain the Hamiltonian function. The kinetic energy is quadratic in the $\dot{q}$s and of the form

$$T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{m}\dot{\mathbf{q}} \tag{2.138}$$

where $\mathbf{m}$ is the $n \times n$ matrix of generalized mass coefficients. The generalized momenta are given by the matrix equation

$$\mathbf{p} = \mathbf{m}\dot{\mathbf{q}} \tag{2.139}$$

and, conversely,

$$\dot{\mathbf{q}} = \mathbf{b}\mathbf{p} \tag{2.140}$$

where $\mathbf{b} = \mathbf{m}^{-1}$. For scleronomic systems, the Hamiltonian function is equal to the total energy, or

$$H = T + V = \frac{1}{2}\mathbf{p}^{\mathrm{T}}\mathbf{b}\mathbf{p} + V \tag{2.141}$$

For the system of this example, we have

$$\mathbf{m} = \begin{bmatrix} m & \frac{1}{2}mr\cos\theta \\ \frac{1}{2}mr\cos\theta & \frac{1}{4}mr^2 \end{bmatrix} \tag{2.142}$$

and

$$\mathbf{b} = \frac{4}{mr^2\sin^2\theta}\begin{bmatrix} \frac{1}{4}r^2 & -\frac{1}{2}r\cos\theta \\ -\frac{1}{2}r\cos\theta & 1 \end{bmatrix} \tag{2.143}$$

in agreement with (2.131) and (2.141).

2.3 Integrals of the motion

We have found that for a dynamical system whose configuration is given by n independent generalized coordinates, the Lagrangian method results in n second-order differential