Generalized momentum

The kinetic energy of a system of particles, written as a function of $(q, \dot{q}, t)$, has the form of (1.258) which, in detail, is

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(q, t)\,\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} a_i(q, t)\,\dot{q}_i + T_0(q, t) \tag{1.290}$$

The *generalized momentum* associated with the generalized coordinate q_i is

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \tag{1.291}$$

or

$$p_i = \sum_{j=1}^{n} m_{ij}\dot{q}_j + a_i \tag{1.292}$$

For a simple system such as a single particle whose position is given by the Cartesian coordinates (x, y), the x-component of generalized momentum is just the x-component of linear momentum $m\dot{x}$. On the other hand, if the position of the particle is given by the polar coordinates (r, θ), then p_θ is equal to $mr^2\dot{\theta}$, which is the angular momentum about the origin. For more general choices of coordinates, the generalized momentum may not have an easily discerned physical meaning. Because the generalized coordinates associated with a given system do not necessarily have the same units or dimensions, neither will the corresponding generalized momenta. However, the product $p_i\dot{q}_i$ will always have the units of energy.

Example 1.12 Two particles, each of mass m are connected by a rigid massless rod of length l to form a dumbbell that can move in the xy-plane. The position of the first particle is (x, y) and the direction of the second particle relative to the first is given by the angle θ (Fig. 1.22). We wish to find the kinetic energy and the generalized momenta.

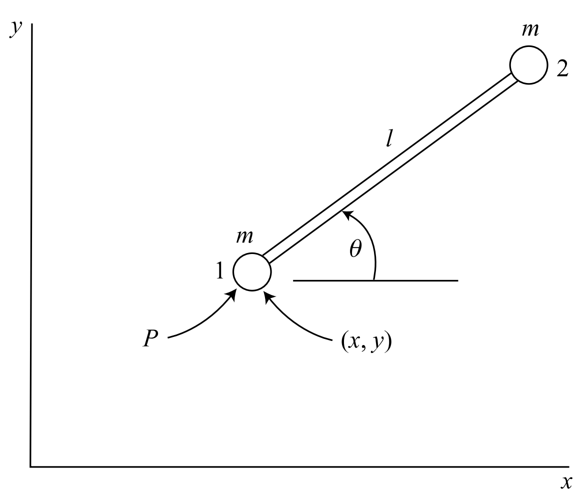

Figure 1.22.

Choose the first particle as the reference point P and use the general kinetic energy expression

$$T = \frac{1}{2}m\dot{\mathbf{r}}_p^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\boldsymbol{\rho}}_i^2 + \dot{\mathbf{r}}_p \cdot m\dot{\boldsymbol{\rho}}_c \tag{1.293}$$

which is of the form

$$T = T_p + T_r + T_c \tag{1.294}$$

Here T_p, the kinetic energy due to the reference point motion, is

$$T_p = \frac{1}{2}m\dot{\mathbf{r}}_p^2 \tag{1.295}$$

where m is the total mass. The kinetic energy due to motion relative to the reference point is

$$T_r = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\boldsymbol{\rho}}_i^2 \tag{1.296}$$

Finally, the kinetic energy due to coupling between the two previous motions is

$$T_c = \dot{\mathbf{r}}_p \cdot m\dot{\boldsymbol{\rho}}_c \tag{1.297}$$

If these general equations are applied to the present system, we obtain

$$T = m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta) \tag{1.298}$$

Then, using (1.291), we find that

$$p_x = \frac{\partial T}{\partial \dot{x}} = 2m\dot{x} - ml\dot{\theta}\sin\theta \tag{1.299}$$

$$p_y = \frac{\partial T}{\partial \dot{y}} = 2m\dot{y} + ml\dot{\theta}\cos\theta \tag{1.300}$$

These are the x and y components of the total linear momentum.

The generalized momentum associated with θ is

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml(\dot{y}\cos\theta - \dot{x}\sin\theta) \tag{1.301}$$

This is equal to absolute angular momentum about P, that is, ml times the velocity component of particle 2 which is perpendicular to the rod. Note that p_θ is not equal to H_p as it is ordinarily defined. To obtain H_p we use the relative kinetic energy T_r.

$$H_p = \frac{\partial T_r}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}ml^2\dot{\theta}^2\right) = ml^2\dot{\theta} \tag{1.302}$$

This is the angular momentum relative to the reference point P.

On the other hand, if we use (x_c, y_c, θ) as generalized coordinates, where (x_c, y_c) are the Cartesian coordinates of the center of mass, the total energy of the system is, in accordance with Koenig's theorem,

$$T = m\left(\dot{x}_c^2 + \dot{y}_c^2\right) + \frac{1}{4}ml^2\dot{\theta}^2 \tag{1.303}$$

The generalized momenta

$$p_x = \frac{\partial T}{\partial \dot{x}_c} = 2m\dot{x}_c, \qquad p_y = \frac{\partial T}{\partial \dot{y}_c} = 2m\dot{y}_c \tag{1.304}$$

are once again the x and y components of the total linear momentum. The third generalized momentum is

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2}ml^2\dot{\theta} \tag{1.305}$$

This is the angular momentum about the center of mass.

1.5 Impulse response

Linear impulse and momentum

Consider a system of N particles whose position vectors $\mathbf{r}_i$ are measured relative to an inertial frame. Newton's law of motion for the system is

$$\dot{\mathbf{p}} = \mathbf{F} \tag{1.306}$$

where $\mathbf{F}$ is the total external force and $\mathbf{p}$ is the total linear momentum.

$$\mathbf{p} = \sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i = m\dot{\mathbf{r}}_c \tag{1.307}$$

where m is the total mass and $\mathbf{r}_c$ is the position vector of the center of mass.

Now integrate (1.307) with respect to time over an interval t_1 to t_2. We obtain

$$\Delta\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1 = \hat{\mathbf{F}} \tag{1.308}$$

where the *linear impulse* $\hat{\mathbf{F}}$ is given by

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F}dt \tag{1.309}$$

Equation (1.308) is a statement of the *principle of linear impulse and momentum: The change in the total linear momentum of a system of particles over a given time interval is equal to the total impulse of the external forces acting on the system*. Notice that the interval is arbitrary and not necessarily small.

Since a vector equation is involved in this principle, a similar scalar equation must apply to each component, provided that the component represents a fixed direction in inertial space.