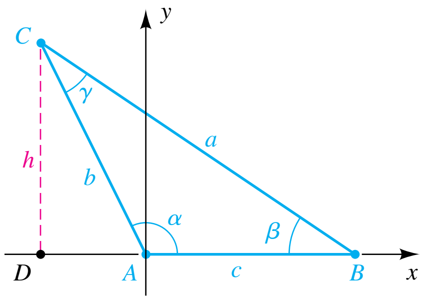
8.1

The Law of Sines

Figure 1



An **oblique triangle** is a triangle that does not contain a right angle. We shall use the letters A, B, C, a, b, c, α, β, and γ for parts of triangles, as we did in Chapter 6. Given triangle ABC, let us place angle α in standard position so that B is on the positive x-axis. The case for α obtuse is illustrated in Figure 1; however, the following discussion is also valid if α is acute.

Consider the line through C parallel to the y-axis and intersecting the x-axis at point D. If we let $d(C, D) = h$, then the y-coordinate of C is h. From the definition of the trigonometric functions of any angle,

$$\sin\alpha = \frac{h}{b}, \quad \text{so} \quad h = b\sin\alpha.$$

Referring to right triangle BDC, we see that

$$\sin\beta = \frac{h}{a}, \quad \text{so} \quad h = a\sin\beta.$$

Equating the two expressions for h gives us

$$b\sin\alpha = a\sin\beta,$$

which we may write as $$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}.$$

If we place α in standard position with C on the positive x-axis, then by the same reasoning,

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}.$$

The last two equalities give us the following result.

The Law of Sines

If ABC is an oblique triangle labeled in the usual manner (as in Figure 1), then

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

Note that the law of sines consists of the following three formulas:

$$\textbf{(1)}\ \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \qquad \textbf{(2)}\ \frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \qquad \textbf{(3)}\ \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

To apply any one of these formulas to a specific triangle, we must know the values of three of the four variables. If we substitute these three values into the appropriate formula, we can then solve for the value of the fourth variable. It follows that the law of sines can be used to find the remaining parts of an oblique triangle whenever we know either of the following (the three letters in parentheses are used to denote the known parts, with S representing a side and A an angle):

(1) two sides and an angle *opposite* one of them (SSA)

(2) two angles and any side (AAS or ASA)

In the next section we will discuss the law of cosines and show how it can be used to find the remaining parts of an oblique triangle when given the following:

(1) two sides and the angle *between* them (SAS)

(2) three sides (SSS)

The law of sines cannot be applied directly to the last two cases.

The law of sines can also be written in the form

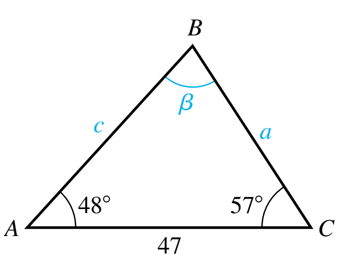
$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

Instead of memorizing the three formulas associated with the law of sines, it may be more convenient to remember the following statement, which takes all of them into account.

The Law of Sines (General Form)	In any triangle, the ratio of the sine of an angle to the side opposite that angle is equal to the ratio of the sine of another angle to the side opposite that angle.

In examples and exercises involving triangles, we shall assume that known lengths of sides and angles have been obtained by measurement and hence are approximations to exact values. Unless directed otherwise, when finding parts of triangles we will round off answers according to the following rule: *If known sides or angles are stated to a certain accuracy, then unknown sides or angles should be calculated to the same accuracy*. To illustrate, if known sides are stated to the nearest 0.1, then unknown sides should be calculated to the nearest 0.1. If known angles are stated to the nearest 10′, then unknown angles should be calculated to the nearest 10′. Similar remarks hold for accuracy to the nearest 0.01, 0.1°, and so on.

Figure 2



EXAMPLE 1 **Using the law of sines (ASA)**

Solve $\triangle ABC$, given $\alpha = 48°$, $\gamma = 57°$, and $b = 47$.

SOLUTION The triangle is sketched in Figure 2. Since the sum of the angles of a triangle is 180°,

$$\beta = 180° - 57° - 48° = 75°.$$

(continued)