7.1 Verifying Trigonometric Identities

A **trigonometric expression** contains symbols involving trigonometric functions.

ILLUSTRATION Trigonometric Expressions

- $x + \sin x$
- $\dfrac{\sqrt{\theta} + 2^{\sin\theta}}{\cot\theta}$
- $\dfrac{\cos(3t+1)}{t^2 + \tan^2(2 - t^2)}$

We assume that the domain of each variable in a trigonometric expression is the set of real numbers or angles for which the expression is meaningful. To provide manipulative practice in simplifying complicated trigonometric expressions, we shall use the fundamental identities (see page 364) and algebraic manipulations, as we did in Examples 5 and 6 of Section 6.2. In the first three examples our method consists of transforming the left-hand side of a given identity into the right-hand side, or vice versa.

EXAMPLE 1 Verifying an identity

Verify the identity $\sec\alpha - \cos\alpha = \sin\alpha\tan\alpha$.

SOLUTION We transform the left-hand side into the right-hand side:

$$
\begin{aligned}
\sec\alpha - \cos\alpha &= \frac{1}{\cos\alpha} - \cos\alpha && \text{reciprocal identity}\\
&= \frac{1 - \cos^2\alpha}{\cos\alpha} && \text{add expressions}\\
&= \frac{\sin^2\alpha}{\cos\alpha} && \sin^2\alpha + \cos^2\alpha = 1\\
&= \sin\alpha\left(\frac{\sin\alpha}{\cos\alpha}\right) && \text{equivalent expression}\\
&= \sin\alpha\tan\alpha && \text{tangent identity}
\end{aligned}
$$

EXAMPLE 2 Verifying an identity

Verify the identity $\sec\theta = \sin\theta(\tan\theta + \cot\theta)$.

SOLUTION Since the expression on the right-hand side is more complicated than that on the left-hand side, we transform the right-hand side into the left-hand side:

$$\begin{aligned}
\sin\theta\,(\tan\theta+\cot\theta) &= \sin\theta\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right) && \text{tangent and cotangent identities}\\
&= \sin\theta\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right) && \text{add fractions}\\
&= \sin\theta\left(\frac{1}{\cos\theta\sin\theta}\right) && \text{Pythagorean identity}\\
&= \frac{1}{\cos\theta} && \text{cancel } \sin\theta\\
&= \sec\theta && \text{reciprocal identity}
\end{aligned}$$

EXAMPLE 3 Verifying an identity

Verify the identity $\dfrac{\cos x}{1-\sin x}=\dfrac{1+\sin x}{\cos x}$.

SOLUTION Since the denominator of the left-hand side is a binomial and the denominator of the right-hand side is a monomial, we change the form of the fraction on the left-hand side by multiplying the numerator and denominator by the conjugate of the denominator and then use one of the Pythagorean identities:

$$\begin{aligned}
\frac{\cos x}{1-\sin x} &= \frac{\cos x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x} && \text{multiply numerator and denominator by } 1+\sin x\\
&= \frac{\cos x\,(1+\sin x)}{1-\sin^2 x} && \text{property of quotients}\\
&= \frac{\cos x\,(1+\sin x)}{\cos^2 x} && \sin^2 x+\cos^2 x=1\\
&= \frac{1+\sin x}{\cos x} && \text{cancel } \cos x
\end{aligned}$$

Another technique for showing that an equation $p = q$ is an identity is to begin by transforming the left-hand side p into another expression s, making sure that each step is *reversible*—that is, making sure it is possible to transform s back into p by reversing the procedure used in each step. In this case, the equation $p = s$ is an identity. Next, as a *separate* exercise, we show that the right-hand side q can also be transformed into the expression s by means of reversible steps and, therefore, that $q = s$ is an identity. It then follows that $p = q$ is an identity. This method is illustrated in the next example.

EXAMPLE 4 Verifying an identity

Verify the identity $(\tan\theta-\sec\theta)^2=\dfrac{1-\sin\theta}{1+\sin\theta}$.