27. A tank contains 20 gallons of water. One-half of the water is removed and replaced with antifreeze. Then one-half of this mixture is removed and replaced with antifreeze. This process is continued eight times. How much water remains in the tank after the eighth replacement process?

28. The radiator of a truck contains 10 gallons of water. Suppose we remove 1 gallon of water and replace it with antifreeze. Then we remove 1 gallon of this mixture and replace it with antifreeze. This process is carried out seven times. To the nearest tenth of a gallon, how much antifreeze is in the final mixture?

Thoughts Into Words

29. Your friend solves Problem 6 as follows: If the car depreciates 20% per year, then at the end of 5 years it will have depreciated 100% and be worth zero dollars. How would you convince him that his reasoning is incorrect?

30. A contractor wants you to clear some land for a housing project. He anticipates that it will take 20 working days to do the job. He offers to pay you one of two ways: (1) a fixed amount of $3000 or (2) a penny the first day, 2 cents the second day, 4 cents the third day, and so on, doubling your daily wages each day for the 20 days. Which offer should you take and why?

14.4 Mathematical Induction

OBJECTIVE 1 Use mathematical induction to prove mathematical statements

Is $2^n > n$ for all positive integer values of n? In an attempt to answer this question, we might proceed as follows:

If $n = 1$, then $2^n > n$ becomes $2^1 > 1$, a true statement.

If $n = 2$, then $2^n > n$ becomes $2^2 > 2$, a true statement.

If $n = 3$, then $2^n > n$ becomes $2^3 > 3$, a true statement.

We can continue in this way as long as we want, but obviously we can never show in this manner that $2^n > n$ for *every* positive integer n. However, we do have a form of proof, called **proof by mathematical induction**, that can be used to verify the truth of many mathematical statements involving positive integers. This form of proof is based on the following principle.

Principle of Mathematical Induction

Let P_n be a statement in terms of n, where n is a positive integer. If

1. P_1 is true, and

2. the truth of P_k implies the truth of P_{k+1} for every positive integer k,

then P_n is true for every positive integer n.

The principle of mathematical induction, a proof that some statement is true for all positive integers, consists of two parts. First, we must show that the statement is true for the positive integer 1. Second, we must show that if the statement is true for some positive integer, then it follows that it is also true for the next positive integer. Let's illustrate what this means.

Classroom Example
Prove that $3^n > n$ for all positive values of n.

EXAMPLE 1 Prove that $2^n > n$ for all positive integer values of n.

Proof

Part 1 If $n = 1$, then $2^n > n$ becomes $2^1 > 1$, which is a true statement.

Part 2 We must prove that if $2^k > k$, then $2^{k+1} > k + 1$ for all positive integer values of k. In other words, we should be able to start with $2^k > k$ and from that deduce $2^{k+1} > k + 1$. This can be done as follows:

$$2^k > k$$

$$2(2^k) > 2(k) \quad \text{Multiply both sides by 2}$$

$$2^{k+1} > 2k$$

We know that $k \geq 1$ because we are working with positive integers. Therefore

$$k + k \geq k + 1 \quad \text{Add } k \text{ to both sides}$$

$$2k \geq k + 1$$

Because $2^{k+1} > 2k$ and $2k \geq k + 1$, by the transitive property we conclude that

$$2^{k+1} > k + 1$$

Therefore, using parts 1 and 2, we proved that $2^n > n$ for *all* positive integers.

It will be helpful for you to look back over the proof in Example 1. Note that in part 1, we established that $2^n > n$ is true for $n = 1$. Then, in part 2, we established that if $2^n > n$ is true for any positive integer, then it must be true for the next consecutive positive integer. Therefore, because $2^n > n$ is true for $n = 1$, it must be true for $n = 2$. Likewise, if $2^n > n$ is true for $n = 2$, then it must be true for $n = 3$, and so on, for *all* positive integers.

We can depict proof by mathematical induction with dominoes. Suppose that in Figure 14.1, we have infinitely many dominoes lined up. If we can push the first domino over (part 1 of a mathematical induction proof), and if the dominoes are spaced so that each time one falls over, it causes the next one to fall over (part 2 of a mathematical induction proof), then by pushing the first one over we will cause a chain reaction that will topple all of the dominoes (Figure 14.2).

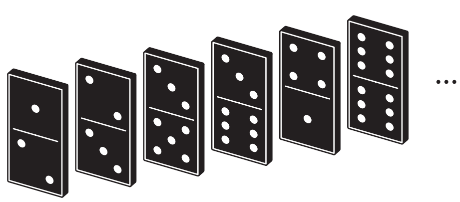

Figure 14.1

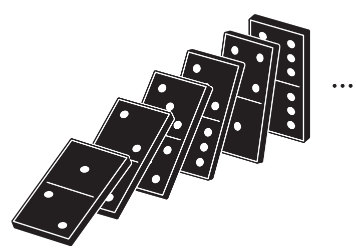

Figure 14.2

Recall that in the first three sections of this chapter, we used a_n to represent the nth term of a sequence and S_n to represent the sum of the first n terms of a sequence. For example, if $a_n = 2n$, then the first three terms of the sequence are $a_1 = 2(1) = 2$, $a_2 = 2(2) = 4$, and $a_3 = 2(3) = 6$. Furthermore, the kth term is $a_k = 2(k) = 2k$, and the $(k + 1)$ term is $a_{k+1} = 2(k + 1) = 2k + 2$. Relative to this same sequence, we can state that $S_1 = 2$, $S_2 = 2 + 4 = 6$, and $S_3 = 2 + 4 + 6 = 12$.

There are numerous sum formulas for sequences that can be verified by mathematical induction. For such proofs, the following property of sequences is used:

$$S_{k+1} = S_k + a_{k+1}$$