11.1 Systems of Two Linear Equations in Two Variables

OBJECTIVES

1. Solve systems of two linear equations by graphing
2. Solve systems of two linear equations by using the substitution method
3. Solve systems of two linear equations by using the elimination-by-addition method
4. Solve application problems using a system of equations

In Chapter 7 we stated that any equation of the form $Ax + By = C$, when A, B, and C are real numbers (A and B not both zero), is a **linear equation** in the two variables x and y, and its graph is a straight line. Two linear equations in two variables considered together form a **system of two linear equations in two variables**, as illustrated by the following examples:

$$\begin{pmatrix} x + y = 6 \\ x - y = 2 \end{pmatrix} \qquad \begin{pmatrix} 3x + 2y = 1 \\ 5x - 2y = 23 \end{pmatrix} \qquad \begin{pmatrix} 4x - 5y = 21 \\ -3x + y = -7 \end{pmatrix}$$

To *solve* such a system means to find all of the ordered pairs that simultaneously satisfy both equations in the system. For example, if we graph the two equations $x + y = 6$ and $x - y = 2$ on the same set of axes, as in Figure 11.1, then the ordered pair associated with the point of intersection of the two lines is the **solution of the system**. Thus we say that $\{(4, 2)\}$ is the solution set of the system

$$\begin{pmatrix} x + y = 6 \\ x - y = 2 \end{pmatrix}$$

To check the solution, we substitute 4 for x and 2 for y in the two equations.

$x + y = 6$ becomes $4 + 2 = 6$, a true statement

$x - y = 2$ becomes $4 - 2 = 2$, a true statement

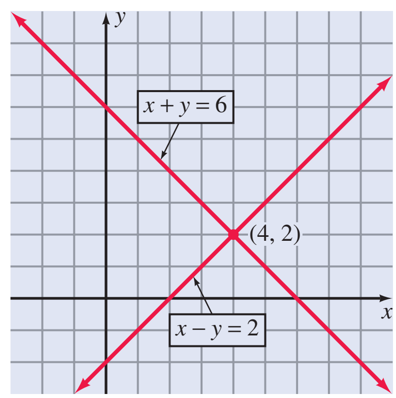

Figure 11.1

Because the graph of a linear equation in two variables is a straight line, three possible situations can occur when we are solving a system of two linear equations in two variables. These situations are shown in Figure 11.2.

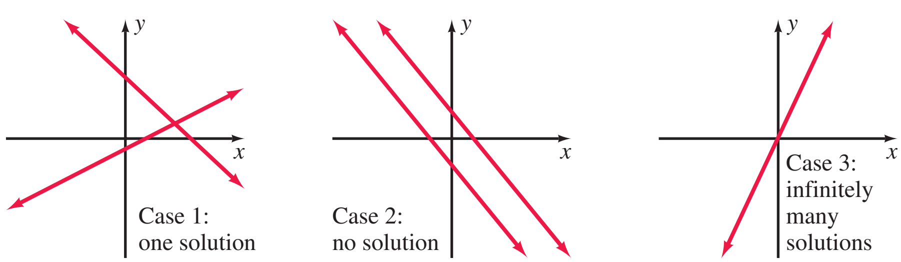

Figure 11.2

Case 1 The graphs of the two equations are two lines intersecting in one point. There is exactly one solution, and the system is called a **consistent system**.

Case 2 The graphs of the two equations are parallel lines. There is *no solution*, and the system is called an **inconsistent system**.

Case 3 The graphs of the two equations are the same line, and there are *infinitely many solutions* of the system. Any pair of real numbers that satisfies one of the equations also satisfies the other equation, and we say that the equations are dependent.

Thus, as we solve a system of two linear equations in two variables, we can expect one of three outcomes: The system will have *no* solutions, *one* ordered pair as a solution, or *infinitely many* ordered pairs as solutions.

The Substitution Method

Solving specific systems of equations by graphing requires accurate graphs. However, unless the solutions are integers, it is difficult to obtain exact solutions from a graph. Therefore we will consider some other techniques for solving systems of equations.

The **substitution method**, which works especially well with systems of two equations in two unknowns, can be described as follows.

Step 1 Solve one of the equations for one variable in terms of the other. (If possible, make a choice that will avoid fractions.)

Step 2 Substitute the expression obtained in step 1 into the other equation, producing an equation in one variable.

Step 3 Solve the equation obtained in step 2.

Step 4 Use the solution obtained in step 3, along with the expression obtained in step 1, to determine the solution of the system.

Classroom Example
Solve the system $\begin{pmatrix} 5x + y = 5 \\ 3x - 2y = 16 \end{pmatrix}$.

EXAMPLE 1

Solve the system $\begin{pmatrix} x - 3y = -25 \\ 4x + 5y = 19 \end{pmatrix}$.

Solution

Solve the first equation for x in terms of y to produce

$$x = 3y - 25$$

Substitute $3y - 25$ for x in the second equation and solve for y.

$$\begin{aligned} 4x + 5y &= 19 \\ 4(3y - 25) + 5y &= 19 \\ 12y - 100 + 5y &= 19 \\ 17y &= 119 \\ y &= 7 \end{aligned}$$

Next, substitute 7 for y in the equation $x = 3y - 25$ to obtain

$$x = 3(7) - 25 = -4$$

The solution set of the given system is $\{(-4, 7)\}$. (You should check this solution in both of the original equations.)

Classroom Example
Solve the system:
$\begin{pmatrix} 12x - 3y = -7 \\ 8x + 9y = -1 \end{pmatrix}$

EXAMPLE 2

Solve the system $\begin{pmatrix} 5x + 9y = -2 \\ 2x + 4y = -1 \end{pmatrix}$.

Solution

A glance at the system should tell you that solving either equation for either variable will produce a fractional form, so let's just use the first equation and solve for x in terms of y.

$$\begin{aligned} 5x + 9y &= -2 \\ 5x &= -9y - 2 \\ x &= \frac{-9y - 2}{5} \end{aligned}$$