Further Investigations

Solve Problems 64–67 for the indicated variable. Assume that all letters represent positive numbers.

64. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for y

65. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for x

66. $s = \frac{1}{2}gt^2$ for t

67. $A = \pi r^2$ for r

Solve each of the following equations for x.

68. $x^2 + 8ax + 15a^2 = 0$

69. $x^2 - 5ax + 6a^2 = 0$

70. $10x^2 - 31ax - 14a^2 = 0$

71. $6x^2 + ax - 2a^2 = 0$

72. $4x^2 + 4bx + b^2 = 0$

73. $9x^2 - 12bx + 4b^2 = 0$

Answers to the Concept Quiz

1. False **2.** True **3.** False **4.** False **5.** True **6.** False **7.** True **8.** False **9.** True **10.** True

6.4 Quadratic Formula

OBJECTIVES

1. Use the quadratic formula to solve quadratic equations
2. Determine the nature of roots to quadratic equations

As we saw in the last section, the method of completing the square can be used to solve any quadratic equation. Thus if we apply the method of completing the square to the equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, we can produce a formula for solving quadratic equations. This formula can then be used to solve any quadratic equation. Let's solve $ax^2 + bx + c = 0$ by completing the square.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c \qquad \text{Isolate the } x^2 \text{ and } x \text{ terms}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Multiply both sides by } \frac{1}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \frac{1}{2}\left(\frac{b}{a}\right) = \frac{b}{2a} \text{ and } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Complete the square by adding $\frac{b^2}{4a^2}$ to both sides

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \qquad \text{Common denominator of } 4a^2 \text{ on right side}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \qquad \text{Commutative property}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{The right side is combined into a single fraction}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \sqrt{4a^2} = |2a| \text{ but } 2a \text{ can be used because of the use of } \pm$$

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula is usually stated as follows:

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad a \neq 0$$

We can use the quadratic formula to solve *any* quadratic equation by expressing the equation in the standard form $ax^2 + bx + c = 0$ and substituting the values for a, b, and c into the formula. Let's consider some examples.

Classroom Example
Solve $n^2 - 5n - 9 = 0$.

EXAMPLE 1 Solve $x^2 + 5x + 2 = 0$.

Solution

$$x^2 + 5x + 2 = 0$$

The given equation is in standard form with $a = 1$, $b = 5$, and $c = 2$. Let's substitute these values into the formula and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 8}}{2}$$

$$x = \frac{-5 \pm \sqrt{17}}{2}$$

The solution set is $\left\{\frac{-5 \pm \sqrt{17}}{2}\right\}$.

Classroom Example
Solve $a^2 + 8a + 5 = 0$.

EXAMPLE 2 Solve $x^2 - 2x - 4 = 0$.

Solution

$$x^2 - 2x - 4 = 0$$

We need to think of $x^2 - 2x - 4 = 0$ as $x^2 + (-2)x + (-4) = 0$ to determine the values $a = 1$, $b = -2$, and $c = -4$. Let's substitute these values into the quadratic formula and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 + 16}}{2}$$

$$x = \frac{2 \pm \sqrt{20}}{2}$$

$$x = \frac{2 \pm 2\sqrt{5}}{2}$$

$$x = \frac{2(1 \pm \sqrt{5})}{2} \qquad \text{Factor out a 2 in the numerator}$$

$$x = \frac{\cancel{2}(1 \pm \sqrt{5})}{\cancel{2}} = 1 \pm \sqrt{5}$$

The solution set is $\{1 \pm \sqrt{5}\}$.

Classroom Example
Solve $f^2 - 8f + 18 = 0$.

EXAMPLE 3

Solve $x^2 - 2x + 19 = 0$.

Solution

$$x^2 - 2x + 19 = 0$$

We can substitute $a = 1$, $b = -2$, and $c = 19$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(19)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 - 76}}{2}$$

$$x = \frac{2 \pm \sqrt{-72}}{2}$$

$$x = \frac{2 \pm 6i\sqrt{2}}{2} \qquad \sqrt{-72} = i\sqrt{72} = i\sqrt{36}\sqrt{2} = 6i\sqrt{2}$$

$$x = \frac{2(1 \pm 3i)}{2} \qquad \text{Factor out a 2 in the numerator}$$

$$x = \frac{\cancel{2}(1 \pm 3i\sqrt{2})}{\cancel{2}} = 1 \pm 3i\sqrt{2}$$

The solution set is $\{1 \pm 3i\sqrt{2}\}$.

Classroom Example
Solve $2b^2 + 6b - 5 = 0$.

EXAMPLE 4

Solve $2x^2 + 4x - 3 = 0$.

Solution

$$2x^2 + 4x - 3 = 0$$

Here $a = 2$, $b = 4$, and $c = -3$. Solving by using the quadratic formula is unlike solving by completing the square in that there is no need to make the coefficient of x^2 equal to 1.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{-4 \pm \sqrt{40}}{4}$$

$$x = \frac{-4 \pm 2\sqrt{10}}{4}$$

$$x = \frac{2(-2 \pm \sqrt{10})}{4} \quad \text{Factor out a 2 in the numerator}$$

$$x = \frac{\cancel{2}(-2 \pm \sqrt{10})}{\cancelto{2}{4}} = \frac{-2 \pm \sqrt{10}}{2}$$

The solution set is $\left\{\frac{-2 \pm \sqrt{10}}{2}\right\}$.

Classroom Example
Solve $x(5x - 7) = 6$.

EXAMPLE 5 Solve $n(3n - 10) = 25$.

Solution

$$n(3n - 10) = 25$$

First, we need to change the equation to the standard form $an^2 + bn + c = 0$.

$$n(3n - 10) = 25$$
$$3n^2 - 10n = 25$$
$$3n^2 - 10n - 25 = 0$$

Now we can substitute $a = 3$, $b = -10$, and $c = -25$ into the quadratic formula.

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$n = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(-25)}}{2(3)}$$

$$n = \frac{10 \pm \sqrt{100 + 300}}{2(3)}$$

$$n = \frac{10 \pm \sqrt{400}}{6}$$

$$n = \frac{10 \pm 20}{6}$$

$$n = \frac{10 + 20}{6} \quad \text{or} \quad n = \frac{10 - 20}{6}$$

$$n = 5 \quad \text{or} \quad n = -\frac{5}{3}$$

The solution set is $\left\{-\frac{5}{3}, 5\right\}$.