TABLE 2.5. Comparison of Solutions Obtained with Several Weighted Residual Schemes[a]

Method	Solution $t=0.0$	$t=0.5$	$t=1.0$
Galerkin linear bases	1.00	0.678	0.571
Collocation linear bases	1.00	0.667	0.555
Subdomain linear bases	1.00	0.684	0.579
Galerkin quadratic bases	1.00	0.696	0.571
Galerkin Hermite cubic bases	1.00	0.684	0.570
Collocation Hermite cubic bases	1.00	0.683	0.566
Analytical	1.00	0.684	0.568

[a]Problem definition is given in (2.2.9).

Calculation of Derivatives. There are occasions when it is necessary to obtain not only the unknown function but also its derivative. The Hermite formulation provided this information directly, but it is also available using earlier formulations. It is only necessary to differentiate the approximating expression

$$\hat{T}=\sum_{j=1}^{3} T_j\phi_j(t)$$

to obtain

$$\frac{d\hat{T}}{dt}=\sum_{j=1}^{3} T_j\frac{d\phi(t)}{dt}=\sum_{j=1}^{3} T_j\frac{d\phi(\xi)}{d\xi}\frac{d\xi}{dt}. \tag{2.2.61}$$

Because the coefficients T_j are known and the basis function derivatives readily obtained, (2.2.61) is easily evaluated. When linear bases for the derivative at $t=0$ (using the Galerkin solution for T_j) are used, we have

$$\left.\frac{d\hat{T}}{dt}\right|_{t=0}=\sum_{j=1}^{3} T_j\left.\frac{d\phi_j(\xi)}{d\xi}\frac{d\xi}{dt}\right|_{t=0}=T_1\left(-\tfrac{1}{2}\right)(4)$$

$$+T_2\left(\tfrac{1}{2}\right)(4)=-2.00+1.36=-0.640.$$

Similarly, we obtain using quadratic bases

$$\left.\frac{d\hat{T}}{dt}\right|_{t=0}=\sum_{j=1}^{3} T_j\left.\frac{d\phi_j(\xi)}{d\xi}\frac{d\xi}{dt}\right|_{t=0}=T_1\left(-\tfrac{3}{2}\right)(2)$$

$$+T_2(2)(2)+T_3\left(-\tfrac{1}{2}\right)(2)=-0.787.$$

The analytical value of the derivative at $t=0$ is -1.00. In summary, the linear, quadratic, and Hermite bases, when used with Galerkin's formulation, yield derivative values of -0.640, -0.787, and -0.937, respectively, compared to the correct value, -1.00.

Let us now consider calculation of the derivative at the location $t=0.5$. This is important because it represents an interelement boundary for linear basis functions. Because the linear bases are C^0 continuous, the derivative $d\phi_j/dt|_{t=0.5}$ is not formally defined. The most common approximation of the derivative at this point is the average of $d\phi_j/dt|_{t=0.5+\epsilon}$ and $d\phi_j/dt|_{t=0.5-\epsilon}$, where $0<\epsilon<\Delta t$. Derivatives calculated using the three bases are summarized in Table 2.6. The Hermite bases provide a significantly more accurate solution for the derivative, whereas the quadratic scheme is only marginally better, overall, than the linear case. Note that the problem encountered in defining the derivative at the edges of the linear element also arise at the edges of quadratic and C^0 continuous cubic elements.

2.2.4 Two-Dimensional Basis Functions

The extension of the weighted residual method to higher dimensions is relatively straightforward, provided that regular rectangular subspaces are employed. We now focus on this special case and examine the more conceptually difficult irregular subspace problem in Chapter 3. As in the earlier discussion of one-dimensional schemes, we illustrate the two-dimensional case using an elementary example. The discretized domain, which we return to shortly, is illustrated in Figure 2.10.

Lagrangian Basis Functions. The simplest way to generate a two-dimensional basis function is to form the product of two one-dimensional bases, one written for each coordinate direction. This approach can be applied whether linear, quadratic, cubic, or Hermite cubic bases are used. When C^0 continuous Lagrange interpolating polynomials are used, we refer to these as Lagrangian bases and the associated finite elements as Lagrangian elements.

TABLE 2.6. Temperature and Temperature Derivatives Computed Using Linear, Quadratic, and Hermite Cubic-Basis Functions

Function:	$\hat{T}$		$d\hat{T}/dt$	
Location:	$t=0.5$	$t=1.0$	$t=0.0$	$t=0.5$
Analytic	0.684	0.569	−1.00	−0.368
Linear	0.678	0.571	−0.640	−0.429
Quadratic	0.696	0.571	−0.787	−0.429
Hermite	0.684	0.570	−0.937	−0.378

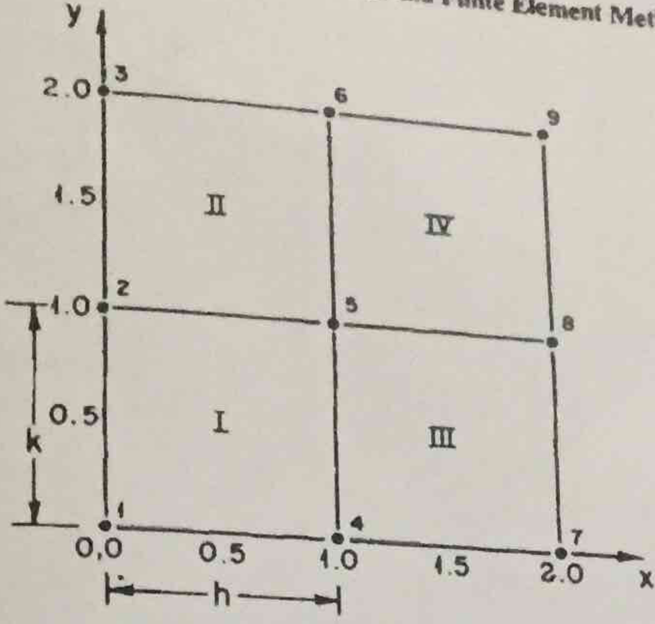


Figure 2.10. Discretized domain for two-dimensional heat flow. Finite element nodes are indicated by dots and the elements themselves by Roman numerals (from Pinder and Gray, 1977).

As an example, the two-dimensional Lagrangian basis functions arising out of the linear bases of Table 2.4 are

$$\phi_1(\xi,\eta)=[\tfrac{1}{2}(1-\xi)][\tfrac{1}{2}(1-\eta)], \tag{2.2.62a}$$

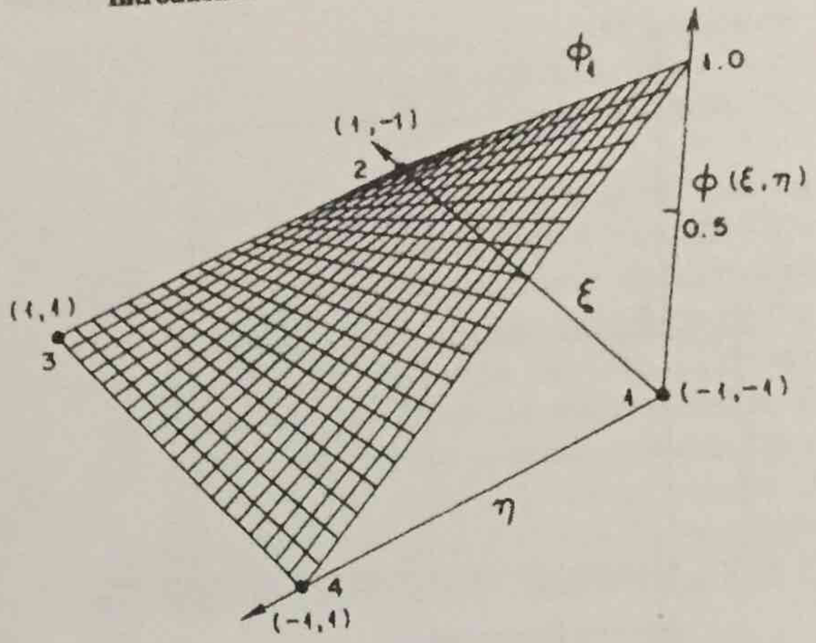
$$\phi_2(\xi,\eta)=[\tfrac{1}{2}(1+\xi)][\tfrac{1}{2}(1-\eta)], \tag{2.2.62b}$$

$$\phi_3(\xi,\eta)=[\tfrac{1}{2}(1+\xi)][\tfrac{1}{2}(1+\eta)], \tag{2.2.62c}$$

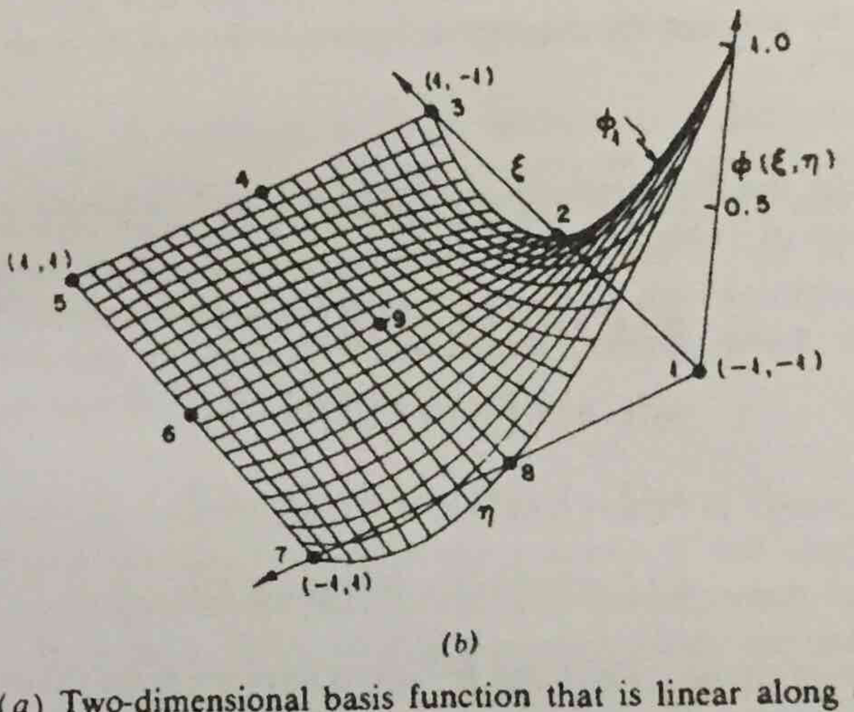
$$\phi_4(\xi,\eta)=[\tfrac{1}{2}(1-\xi)][\tfrac{1}{2}(1+\eta)]. \tag{2.2.62d}$$

Figure 2.11*a* illustrates ϕ_1. The other functions have the same shape. Lagrangian quadratic, cubic, and Hermitian cubic two-dimensional basis functions are presented in Table 2.7*a* and illustrated in Figures 2.11*b*, *c* and 2.13. Note that the Lagrangian element requires interior nodes when quadratic or cubic basis functions are generated.

Serendipity Basis Functions. Although Lagrangian bases are the logical extension of their one-dimensional counterparts, there is another choice which preserves many of the important properties of the Lagrangian bases while decreasing the degrees of freedom by eliminating interior nodes. Examination of Figure 2.11*b* and Table 2.7*a* reveals that the basis function at the central node spans only the rectangular domain illustrated. In light of this observation, let us now examine the possibility of defining a set of two-dimensional bases that are quadratic along each side of the rectangle of Figure 2.11, yet have no



(a)



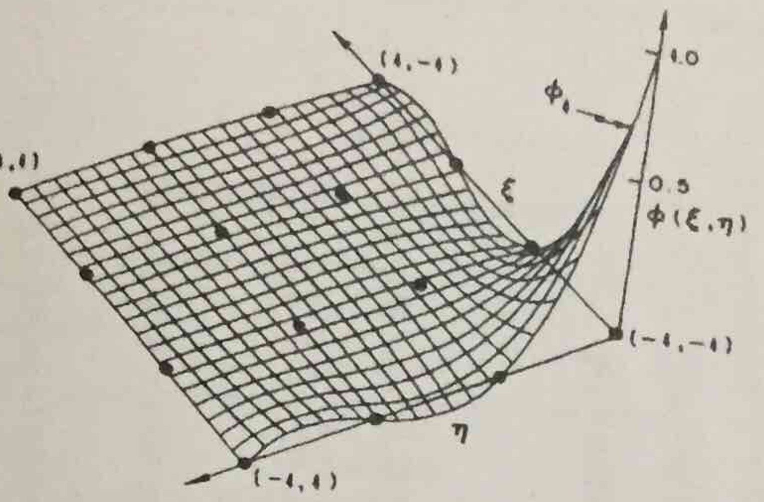
(b)

Figure 2.11. (*a*) Two-dimensional basis function that is linear along each side. (*b*) Two-dimensional Lagrangian basis function that is quadratic along each side. Note the occurrence of a central node where the basis function must be zero.

center node. In a manner analogous to that used to obtain the one-dimensional basis functions, we first write a polynomial expression quadratic in ξ and in η.

$$\phi = a + b\xi + c\eta + d\xi^2 + e\eta^2 + f\xi\eta + g\xi^2\eta + h\xi\eta^2 + i\xi^2\eta^2. \tag{2.2.63}$$

There are nine coefficients to be evaluated in this expression, which requires the specification of nine constraints. The Lagrangian scheme involves nine nodes, and consequently the nine conditions are easily formulated. In fact, the bases we derived by taking the product of two one-dimensional bases will arise



(c)

Figure 2.11. (*Continued*)

Figure 2.11. (*c*) Two-dimensional Lagrangian basis function that is cubic along each side. Note the occurrence of four interior nodes where the basis function is defined to be zero.

out of (2.2.63) when each basis function is required to be unity at the node for which it is defined and zero at all other nodes.

Because we wish to obtain a basis function with no interior node, there can be only eight constraints imposed. Thus the general polynomial of (2.2.63) requires simplification. Let us achieve this objective by dropping the last term, leaving the expression

$$\phi = a + b\xi + c\eta + d\xi^2 + e\eta^2 + f\xi\eta + g\xi^2\eta + h\xi\eta^2. \tag{2.2.64}$$

By requiring each basis function $\phi_i(\xi, \eta)$ to be unity at node i and zero at all other nodes, we arrive at the matrix equation for the particular case of ϕ_1:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}. \tag{2.2.65}$$

TABLE 2.7a. Basis Functions Formulated Using Quadratic, Cubic, and Hermitian Cubic Polynomials[a]

	Lagrangian	Serendipity
Linear	$\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$	$\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$
Quadratic		
Corner node	$\frac{1}{4}\xi\xi_i(1+\xi\xi_i)\eta\eta_i(1+\eta\eta_i)$	$\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)$
Side node, $\xi_i=0$	$\frac{1}{2}\eta\eta_i(1+\eta\eta_i)(1-\xi^2)$	$\frac{1}{2}(1-\xi^2)(1+\eta\eta_i)$
Side node, $\eta_i=0$	$\frac{1}{2}\xi\xi_i(1+\xi\xi_i)(1-\eta^2)$	$\frac{1}{2}(1+\xi\xi_i)(1-\eta^2)$
Interior node	$(1-\xi^2)(1-\eta^2)$	—
Cubic		
Corner node	$\frac{1}{256}(9\xi^2-1)(\xi_i\xi+1)(9\eta^2-1)(\eta_i\eta+1)$	$\frac{1}{32}(1+\xi\xi_i)(1+\eta\eta_i)[9(\xi^2+\eta^2)-10]$
Side node $\begin{cases}\eta=\pm 1, \xi=\pm\frac{1}{3}\end{cases}$	$\frac{9}{256}(9\eta^2-1)(\eta_i\eta+1)(1-\xi^2)(1+9\xi\xi_i)$	$\frac{9}{32}(1-\xi^2)(1+9\xi\xi_i)(1+\eta\eta_i)$
Side node $\begin{cases}\eta=\pm\frac{1}{3}, \xi=\pm 1\end{cases}$	$\frac{9}{256}(9\xi^2-1)(\xi_i\xi+1)(1-\eta^2)(1+9\eta\eta_i)$	
Interior node, $\xi=\pm\frac{1}{3}, \eta=\pm\frac{1}{3}$	$\frac{81}{256}(1-\xi^2)(1-\eta^2)(1+9\xi\xi_i)(1+9\eta\eta_i)$	$\frac{9}{32}(1-\eta^2)(1+9\eta\eta_i)(1+\xi\xi_i)$
Hermite		
ϕ^1_{00i}	$\frac{1}{16}(\xi+\xi_i)^2(\xi\xi_i-2)(\eta+\eta_i)^2(\eta\eta_i-2)$	$\frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(2+\xi\xi_i+\eta\eta_i-\xi^2-\eta^2)$
ϕ^1_{10i}	$-\frac{1}{16}\xi_i(\xi+\xi_i)^2(\xi\xi_i-1)(\eta+\eta_i)^2(\eta\eta_i-2)$	$-\frac{\xi_i}{8}(1-\xi^2)(1+\xi\xi_i)(1+\eta\eta_i)$
ϕ^1_{01i}	$-\frac{1}{16}(\xi+\xi_i)^2(\xi\xi_i-2)\eta_i(\eta+\eta_i)^2(\eta\eta_i-1)$	$-\frac{\eta_i}{8}(1-\eta^2)(1+\xi\xi_i)(1+\eta\eta_i)$
ϕ^1_{11i}	$\frac{1}{16}\xi_i(\xi+\xi_i)^2(\xi\xi_i-1)\eta_i(\eta+\eta_i)^2(\eta\eta_i-1)$	—

[a] These functions are presented graphically in Figures 2.11–2.14.

TABLE 2.7*b*. **Basis Functions Formulated Using a Mixed Formulation**

Corner node

$\phi_i = \alpha_i \beta_i$

$\alpha_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$

$\beta_i = \beta_\xi + \beta_\eta$

where β_ξ and β_η are defined as

Side	β_ξ	β_η
Linear	$\frac{1}{2}$	$\frac{1}{2}$
Quadratic	$\xi\xi_i - \frac{1}{2}$	$\eta\eta_i - \frac{1}{2}$
Cubic	$\frac{9}{8}\xi^2 - \frac{5}{8}$	$\frac{9}{8}\eta^2 - \frac{5}{8}$

Side node

Quadratic

$\phi_i = \frac{1}{2}(1-\xi^2)(1+\eta\eta_i) \qquad \xi_i = 0, \quad \eta_i = \pm 1$

$\phi_i = \frac{1}{2}(1+\xi\xi_i)(1-\eta^2) \qquad \xi_i = \pm 1, \quad \eta_i = 0$

Cubic

$\phi_i = \frac{9}{32}(1-\xi^2)(1+9\xi\xi_i)(1+\eta\eta_i) \qquad \xi_i = \pm\frac{1}{3}, \quad \eta_i = \pm 1$

$\phi_i = \frac{9}{32}(1+\xi\xi_i)(1-\eta^2)(1+9\eta\eta_i) \qquad \xi_i = \pm 1, \quad \eta_i = \pm\frac{1}{3}$

Equation (2.2.65) yields the vector of coefficients

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ 0 \\ 0 \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix},$$

which, when substituted into (2.2.64) gives

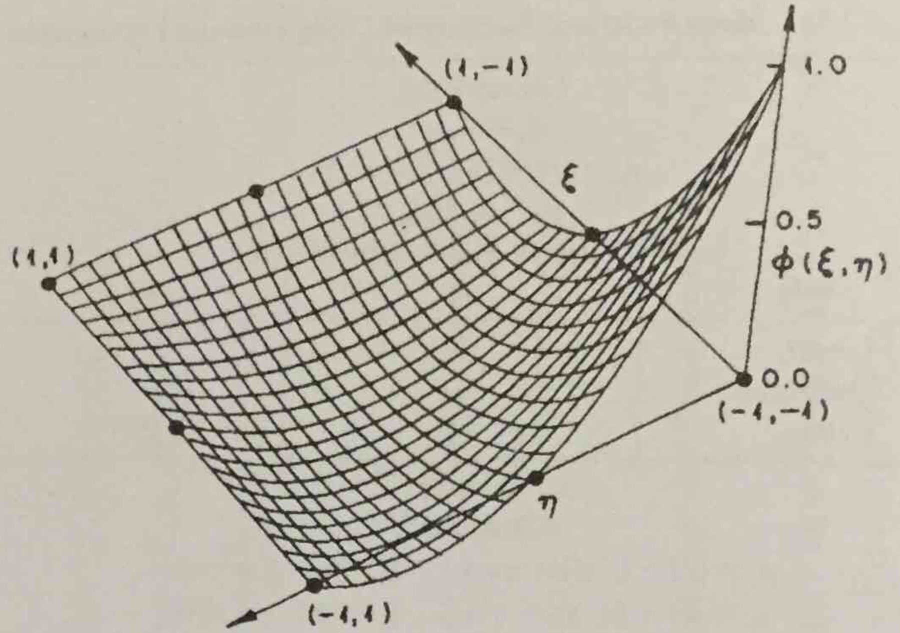
$$\phi_1 = \frac{1}{4}(-1+\xi^2+\eta^2+\xi\eta-\xi^2\eta-\eta^2\xi). \tag{2.2.66}$$

By employing a scheme analogous to the one used for ϕ_1, one can obtain bases for the remaining corner nodes. All corner node bases are described by

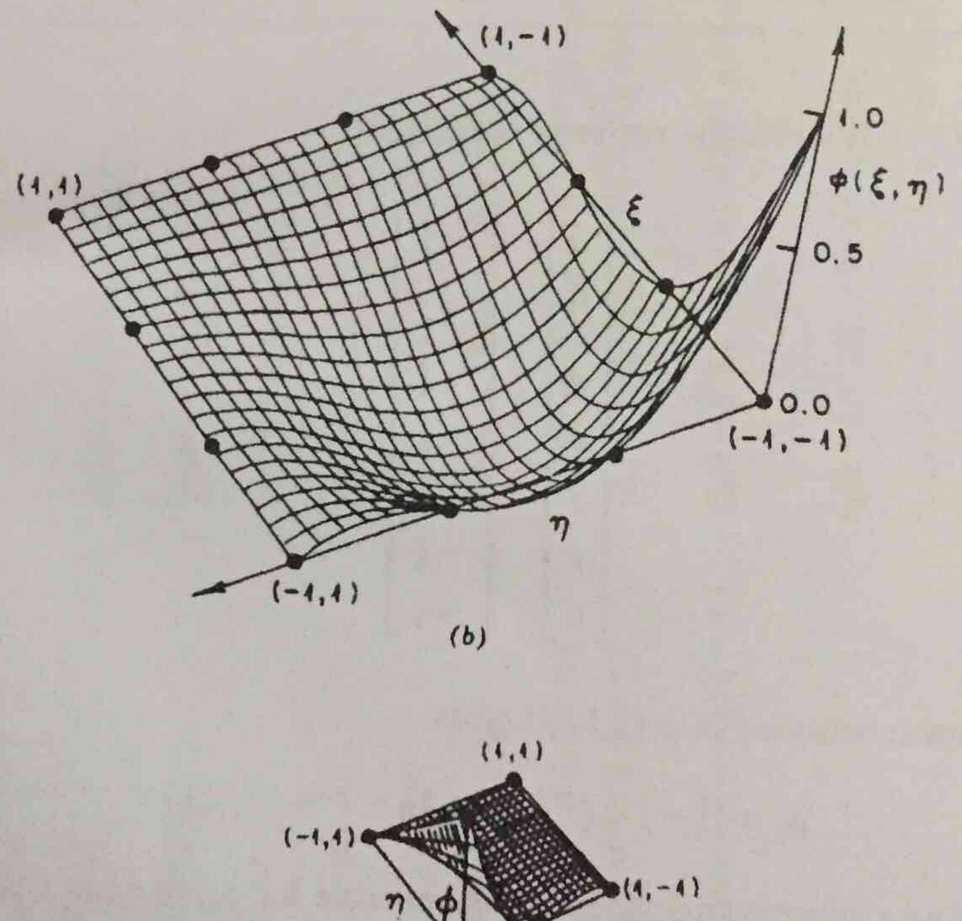
$$\phi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1), \tag{2.2.67}$$

where $\xi_i = \pm 1$, $\eta_i = \pm 1$, depending on the location of the corner node.

$$\phi_i = \frac{1}{2}(1-\xi^2)(1+\eta\eta_i) \qquad \text{for} \qquad \xi_i = 0$$



(a)



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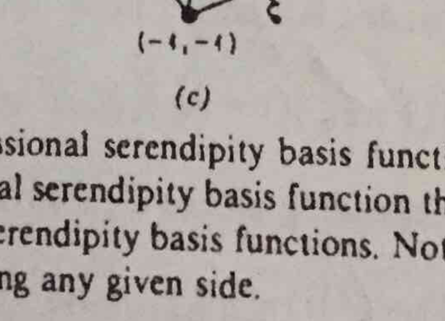

(c)

Figure 2.12. (*a*) Two-dimensional serendipity basis function that is quadratic along each side. (*b*) Two-dimensional serendipity basis function that is cubic along each side. (*c*) Two-dimensional mixed serendipity basis functions. Note that the function may be linear, quadratic, or cubic along any given side.