coordinate transformation, as illustrated below, is required.

$$\int_0^{1/2}\left(\frac{d\phi_1}{dt}\phi_1 + k\phi_1\phi_1\right)dt$$

$$= \int_{-1}^{1}\left(\frac{d\phi_1}{d\xi}\frac{d\xi}{dt}\phi_1 + k\phi_1\phi_1\right)\frac{dt}{d\xi}d\xi = \int_{-1}^{1}\left(4\frac{d\phi_1}{d\xi}\phi_1 + k\phi_1\phi_1\right)\tfrac{1}{4}d\xi$$

$$= \frac{d\phi_1}{d\xi}\phi_1\bigg|_{\xi=0}(2) + \frac{k\phi_1\phi_1}{4}\bigg|_{\xi=0}(2) = 2\left(-\tfrac{1}{2}\tfrac{1}{2} + \frac{k}{16}\right) = \tfrac{2}{4}\left(-1 + \frac{k}{4}\right),$$

where the Gaussian point is located at $\xi = 0$* and the weighting function is 2. The use of single-point Gaussian integration for the remaining elements of (2.2.15) yields

$$(2.2.24)\quad \tfrac{1}{2}\begin{bmatrix} -1+\frac{k}{4} & 1+\frac{k}{4} & 0 \\ -1+\frac{k}{4} & \left(1+\frac{k}{4}\right)+\left(-1+\frac{k}{4}\right) & 1+\frac{k}{4} \\ 0 & -1+\frac{k}{4} & 1+\frac{k}{4} \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} \frac{kT_e}{2} \\ kT_e \\ \frac{kT_e}{2} \end{bmatrix}.$$

Equation (2.2.24) is similar to (2.2.16), which was obtained through an exact integration of the Galerkin equations. The more interesting comparison, however, involves the collocation matrix of (2.2.23). Indeed, the second and third equations, the only ones that must be satisfied in view of the boundary condition, are completely equivalent. This is easily verified by multiplication of (2.2.23) by a factor of $\frac{1}{4}$ and addition of rows two and three to obtain row two of (2.2.24). Thus the orthogonal collocation method can be interpreted in this example as being analogous to Galerkin's method when the Galerkin integrations are performed using Gaussian quadrature. This interesting observation can, in fact, be readily extended to more complex differential operators and other basis functions (see Prenter, 1975).

2.2.3 The Choice of Basis Functions

The accuracy and efficiency of a MWR scheme is largely dependent upon the choice of basis functions. In the preceding section we introduced the piecewise linear chapeau basis function. This is perhaps the simplest of the many

*Gaussian quadrature is most conveniently performed over the interval $-1 \le \xi \le 1$, with the basis functions written in terms of ξ. The derivation of $\phi_1(\xi)$ will be outlined shortly, whereupon the relationship $dt/d\xi = \frac{1}{4}$ should become apparent.

piecewise continuous polynomials that can be used effectively as bases. Figure 2.5 reveals that the chapeau function spans two elements with two nodes located on each element. In this section we introduce higher-degree polynomials as bases, with the expectation of enhanced accuracy.

Linear Polynomial Bases. The chapeau functions serve to interpolate nodal information linearly over each element. In other words, one can obtain a solution anywhere within an element, given only the solution at the nodes. Consider, for example, the most accurate solution we were able to obtain using the Galerkin approach in the preceding section. We have at the nodes

$$[T_1,\ T_2,\ T_3]=[1,\ 0.678,\ 0.571].$$

Suppose that a solution is required at the point $t=0.25$, which does not correspond to the location of a node. From (2.2.11) we have

$$T\approx\hat{T}=\sum_{j=1}^{3}T_j\phi_j(t),$$

which for the case of $t=0.25$ becomes

$$\begin{aligned}T|_{0.25}\approx\hat{T}|_{0.25}&=T_1\phi_1|_{0.25}+T_2\phi_2|_{0.25}+T_3\phi_3|_{0.25}\\&=(1)(0.5)+(0.678)(0.5)+(0.571)(0.0)\\&=0.839.\end{aligned}$$

The analytical solution yields $T(0.25)=0.803$.

Quadratic Polynomial Bases. Let us now consider the possibility of introducing a quadratic interpolation over each element. The interpolating function, which is also a basis function, is of the form

$$\phi=a+bt+ct^2 \tag{2.2.25}$$

over each element. Because this is a quadratic function, three point values are required to uniquely define the coefficients a, b, c. We therefore require three nodes in each quadratic element, as illustrated in Figure 2.6. Three conditions must be imposed on (2.2.25) to evaluate the three constants a, b, and c. We require the basis function for node $i-1$ to be unity at node $i-1$ and zero at the remaining two nodes. In matrix notation, this constraint becomes

$$\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1 & t_{i-1} & t_{i-1}^2\\1 & t_i & t_i^2\\1 & t_{i+1} & t_{i+1}^2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}. \tag{2.2.26}$$

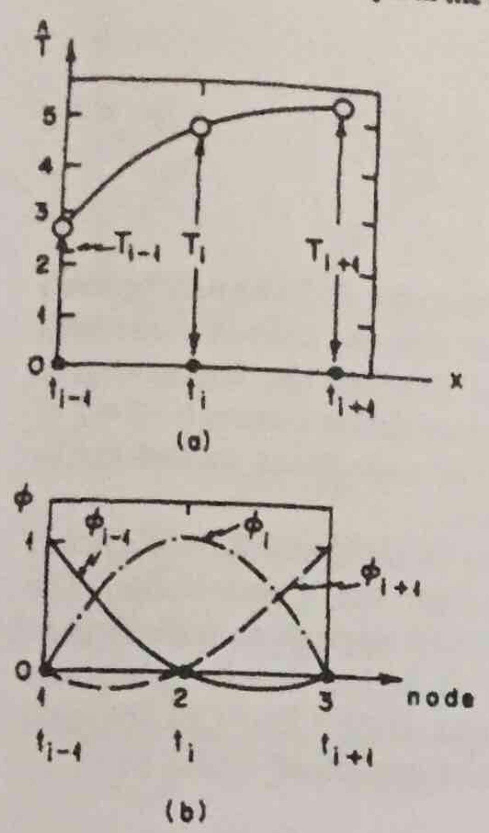

Figure 2.6. Quadratic basis functions ϕ_i appear in (b) and a function represented by the series $\hat{T} = T_{i-1}\phi_{i-1} + T_i\phi_i + T_{i+1}\phi_{i+1}$, in ($a$).

This equation can be solved to yield

$$a = \frac{t_i t_{i+1}}{(t_i - t_{i-1})(t_{i+1} - t_{i-1})}, \tag{2.2.27a}$$

$$b = \frac{-(t_i + t_{i+1})}{(t_i - t_{i-1})(t_{i+1} - t_{i-1})}, \tag{2.2.27b}$$

$$c = \frac{1}{(t_i - t_{i-1})(t_{i+1} - t_{i-1})}. \tag{2.2.27c}$$

The basis function is now obtained by substituting (2.2.27) into (2.2.25) to give

$$\phi_{i-1} = \left(\frac{t_i t_{i+1}}{\Delta_{i-1}}\right) - \left(\frac{t_i + t_{i+1}}{\Delta_{i-1}}\right)t + \frac{t^2}{\Delta_{i-1}}, \tag{2.2.28a}$$

where $\Delta_{i-1} \equiv (t_i - t_{i-1})(t_{i+1} - t_{i-1})$.

An expression analogous to (2.2.26) may be obtained for each of the remaining two nodes in the element by imposing the same type of conditions as indicated above. The resulting basis functions are

$$\phi_i = \left(\frac{t_{i-1} t_{i+1}}{\Delta_i}\right) - \left(\frac{t_{i+1} + t_{i-1}}{\Delta_i}\right)t + \frac{t^2}{\Delta_i}, \tag{2.2.28b}$$

where $\Delta_i \equiv (t_i - t_{i-1})(t_{i+1} - t_i)$ and

(2.2.28c) $$\phi_{i+1} = \left(\frac{t_i t_{i-1}}{\Delta_{i+1}}\right) - \left(\frac{t_{i-1} + t_i}{\Delta_{i+1}}\right)t + \frac{t^2}{\Delta_{i+1}},$$

where $\Delta_{i+1} \equiv (t_{i+1} - t_i)(t_{i+1} - t_{i-1})$.

Whereas the quadratic basis functions defined through (2.2.28) can be used directly in the MWR, the integrations, particularly for the Galerkin method, become somewhat tedious. It is advantageous to develop the higher-degree polynomial basis functions in the dimensionless ξ coordinate system ($-1 \leq \xi \leq 1$) described in the preceding section. This choice of coordinate system facilitates numerical integration by Gaussian quadrature.

The quadratic basis functions can be developed in the ξ coordinate system in a manner analogous to that employed above for the t coordinate. The resulting functions are defined in Table 2.4 and will appear as indicated in Figure 2.7*a*. To illustrate their development, let us derive the dimensionless equivalent of (2.2.28a) using the approach outlined in (2.2.25), (2.2.26), and (2.2.27). The quadratic polynomial used as a point of departure would be

(2.2.29) $$\phi(\xi) = a + b\xi + c\xi^2.$$

By introducing the requirement that $\phi_{-1}|_{-1} = 1$ and $\phi_{-1}|_0 = \phi_{-1}|_1 = 0$, we obtain the matrix equation

(2.2.30) $$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

TABLE 2.4. Linear, Quadratic, and Cubic Basis Functions Defined in the Dimensionless ξ Coordinate System

Degree	Function	Form ($-1 \leq \xi \leq 1$)
Linear	$\phi_{-1}(\xi)$	$\frac{1}{2}(1-\xi)$
	$\phi_1(\xi)$	$\frac{1}{2}(1+\xi)$
Quadratic	$\phi_{-1}(\xi)$	$-\frac{1}{2}\xi(1-\xi)$
	$\phi_0(\xi)$	$1-\xi^2$
	$\phi_1(\xi)$	$\frac{1}{2}\xi(1+\xi)$
Cubic	$\phi_{-1}(\xi)$	$\frac{1}{16}(-9\xi^3+9\xi^2+\xi-1)$ or $\frac{1}{16}(1-\xi)(9\xi^2-1)$
	$\phi_{-1/3}(\xi)$	$\frac{9}{16}(3\xi^3-\xi^2-3\xi+1)$ or $\frac{9}{16}(3\xi-1)(\xi^2-1)$
	$\phi_{1/3}(\xi)$	$\frac{9}{16}(-3\xi^3-\xi^2+3\xi+1)$ or $-\frac{9}{16}(3\xi+1)(\xi^2-1)$
	$\phi_1(\xi)$	$\frac{1}{16}(9\xi^3+9\xi^2-\xi-1)$ or $\frac{1}{16}(1+\xi)(9\xi^2-1)$

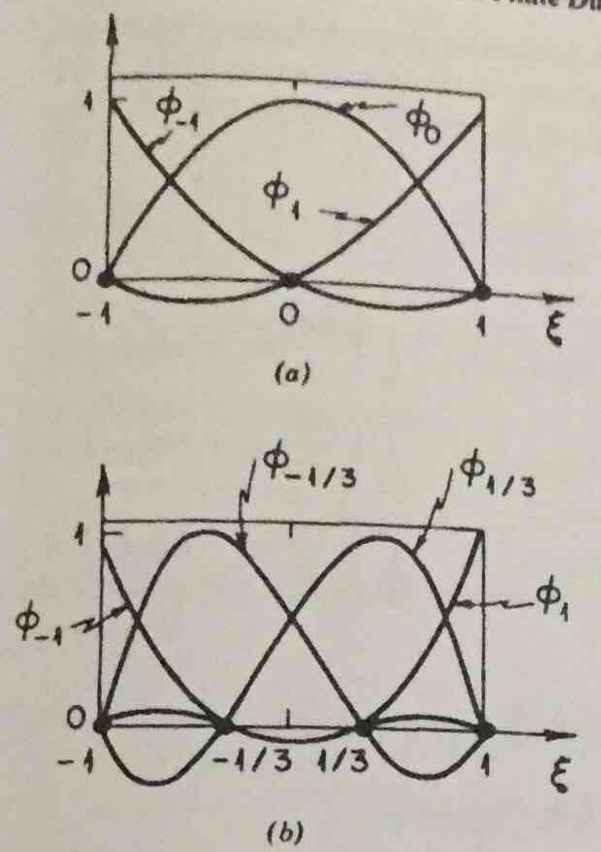

Figure 2.7. Quadratic (a) and cubic (b) basis functions defined in local ξ coordinate system.

which provides the following solution for a, b, and c:

$$[a,\ b,\ c]=[0,\ -\tfrac{1}{2},\ \tfrac{1}{2}]. \tag{2.2.31}$$

Substitution of (2.2.31) into (2.2.29) yields the required basis function,

$$\phi_{-1}(\xi)=-\tfrac{1}{2}\xi(1-\xi). \tag{2.2.32a}$$

The remaining two bases can be formulated in an analogous manner and the resulting equations are

$$\phi_0(\xi)=1-\xi^2 \tag{2.2.32b}$$

and

$$\phi_1(\xi)=\tfrac{1}{2}\xi(1+\xi). \tag{2.2.32c}$$

To demonstrate the use of higher-degree polynomial basis functions defined in the dimensionless ξ coordinate system, let us reexamine, using quadratic bases, the problem solved earlier using the Galerkin method with chapeau functions. The governing equations are

$$\frac{dT}{dt}+k(T-T_e)=0 \tag{2.2.9}$$

subject to $T(0)=1$, $T_e=0.5$, $k=2$, and $0 \leq t \leq 1$. The approximating equations generated using the Galerkin method are similar to those expressed by (2.2.15). However, because a single element constitutes the entire domain $0 \leq t \leq 1$, the coefficient matrix is full.

(2.2.33)

$$\begin{bmatrix} \int_0^1 \left(\frac{d\phi_1}{dt}\phi_1 + k\phi_1\phi_1 \right) dt & \int_0^1 \left(\frac{d\phi_2}{dt}\phi_1 + k\phi_2\phi_1 \right) dt & \int_0^1 \left(\frac{d\phi_3}{dt}\phi_1 + k\phi_3\phi_1 \right) dt \\ \int_0^1 \left(\frac{d\phi_1}{dt}\phi_2 + k\phi_1\phi_2 \right) dt & \int_0^1 \left(\frac{d\phi_2}{dt}\phi_2 + k\phi_2\phi_2 \right) dt & \int_0^1 \left(\frac{d\phi_3}{dt}\phi_2 + k\phi_3\phi_2 \right) dt \\ \int_0^1 \left(\frac{d\phi_1}{dt}\phi_3 + k\phi_1\phi_3 \right) dt & \int_0^1 \left(\frac{d\phi_2}{dt}\phi_3 + k\phi_2\phi_3 \right) dt & \int_0^1 \left(\frac{d\phi_3}{dt}\phi_3 + k\phi_3\phi_3 \right) dt \end{bmatrix}$$

$$\cdot \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \int_0^1 kT_e\phi_1\, dt \\ \int_0^1 kT_e\phi_2\, dt \\ \int_0^1 kT_e\phi_3\, dt \end{bmatrix}.$$

The integrals appearing in the coefficient matrix may now be evaluated using the bases defined in the dimensionless ξ space. A typical integration proceeds by first transforming from t to ξ.

(2.2.34)
$$\int_0^1 \left(\frac{d\phi_1(\xi)}{dt}\phi_1(\xi) + k\phi_1(\xi)\phi_1(\xi) \right) dt$$
$$= \int_{-1}^1 \left(\frac{d\phi_1(\xi)}{d\xi}\frac{d\xi}{dt}\phi_1(\xi) + k\phi_1(\xi)\phi_1(\xi) \right) \frac{dt}{d\xi}\, d\xi.$$

Because $d\xi/dt=2$, (2.2.34) becomes

(2.2.35)
$$\int_0^1 \left(\frac{d\phi_1(\xi)}{dt}\phi_1(\xi) + k\phi_1(\xi)\phi_1(\xi) \right) dt$$
$$= \int_{-1}^1 \left(\frac{d\phi_1(\xi)}{d\xi}\phi_1(\xi) + \frac{k}{2}\phi_1(\xi)\phi_1(\xi) \right) d\xi.$$

One has the option at this point to integrate directly or to employ numerical integration. Whichever route is followed, the evaluated integrals are substituted into (2.2.33) to give the final matrix equation:

$$\begin{bmatrix} -\frac{1}{2}+\frac{2k}{15} & \frac{2}{3}+\frac{k}{15} & -\frac{1}{6}-\frac{k}{30} \\ -\frac{2}{3}+\frac{k}{15} & 0+\frac{8k}{15} & \frac{2}{3}+\frac{k}{15} \\ \frac{1}{6}-\frac{k}{30} & -\frac{2}{3}+\frac{k}{15} & \frac{1}{2}+\frac{2k}{15} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6}T_e k \\ \frac{4}{6}T_e k \\ \frac{1}{6}T_e k \end{bmatrix}. \tag{2.2.36}$$

Solution of (2.2.36) yields

$$[T_1,\ T_2,\ T_3]=[1,\ 0.696,\ 0.571].$$

Now using the quadratic interpolation function to find values between nodes, we obtain

$$\begin{aligned}\hat{T}(0.25)&=T_1\phi_1(0.25)+T_2\phi_2(0.25)+T_3\phi_3(0.25)\\ &=(1)(\tfrac{3}{8})+(0.696)(\tfrac{3}{4})+(0.571)(-\tfrac{1}{8})\\ &=0.826.\end{aligned}$$

Similarly, $\hat{T}(0.75)=0.611$.

In this example the quadratic element formulation provides less accuracy at the nodes than the linear elements, but intermediate values are more accurate. In fact, the linear and quadratic elements in this case are both of accuracy $O(h^2)$ (see Gray and Pinder, 1976).

Cubic Polynomial Bases. Although there is no theoretical limit to the degree of polynomial one could generate as a basis function, the cubic is the highest degree found in general use. The reasons are threefold: (1) the higher the degree of the polynomial, the more computational effort required to solve the approximating algebraic equations arising in the MWR, (2) the computational effort required to generate the coefficient matrices using numerical integration increases as the degree of the polynomial increases, and (3) higher-degree polynomials lead to larger element sizes that may be undesirable in problems involving nonhomogeneous media. The cubic basis functions are defined in Table 2.4, illustrated in Figure 2.7*b*, and obtained in the same manner as the quadratic bases.

The cubic basis function described above interpolates along the element using a Lagrange third-order polynomial. Although this choice of function is a natural extension of the quadratic case, there is another cubic basis, known as a Hermite polynomial, which can be particularly effective in certain problems. We will turn our attention soon to the development of this basis function.

Lagrange Polynomials. The basis functions developed thus far can be easily formulated directly using Lagrange polynomials. The Lagrange polynomial, $l_i(\xi)$, is given by

$$l_i^n(\xi)=\frac{(\xi-\xi_0)\cdots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\cdots(\xi-\xi_n)}{(\xi_i-\xi_0)\cdots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\cdots(\xi_i-\xi_n)}.$$

Let us consider the case of linear basis functions where $n=1$ and only one term of the polynomial is involved. Because only two nodes are available, we know that for $\phi_i(\xi)$ with $i=-1$ we require $\xi_{i+1}=1$ and $\xi_i=-1$; similarly for $i=1$, $\xi_{i-1}=-1$, and $\xi_i=1$. Using these values of ξ_k, we obtain

$$\phi_{-1}=l_i^1|_{\xi=-1}=-\frac{\xi-\xi_{i+1}}{\xi_{i+1}-\xi_i}=\frac{1-\xi}{2}$$

and

$$\phi_1=l_i^1|_{\xi=1}=\frac{\xi-\xi_{i-1}}{\xi_i-\xi_{i-1}}=\frac{\xi+1}{2}.$$

As a second example, consider the quadratic bases. For the basis function defined for the center node of an element, $\xi=0$, the three values of ξ_k to be considered are $\xi_{i-1}=-1, \xi_i=0, \xi_{i+1}=1$. Selecting those terms of the Lagrange polynomial containing these values, we obtain

$$\phi_0=l_i^2|_{\xi=0}=\frac{(\xi-\xi_{i-1})(\xi-\xi_{i+1})}{(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})}=\frac{\xi+1}{1}\,\frac{1-\xi}{1}=1-\xi^2.$$

The basis function ϕ_{-1} involves the following values of ξ_k:

$$\xi_i=-1,\qquad \xi_{i+1}=0,\qquad \xi_{i+2}=1.$$

From the definition of l_i^n we can write

$$\phi_{-1}=l_i^2|_{\xi=-1}=\frac{(\xi-\xi_{i+1})(\xi-\xi_{i+2})}{(\xi_i-\xi_{i+1})(\xi_i-\xi_{i+2})}=\frac{(\xi-0)(\xi-1)}{(-1)(-2)}=\frac{\xi^2-\xi}{2}.$$

The function of ϕ_1 can be obtained in an analogous manner.

Hermite Polynomial Bases. Hermite polynomials are cubic splines that display, in addition to second-order derivative continuity (C^2) over the element, first-order derivative continuity (C^1) between elements. Thus Hermite interpolation not only interpolates knot values of the function but also interpolates knot values of a given number of consecutive derivatives. This property is particularly attractive for a variety of reasons when the solution