3.1 LINEAR MODELS

REVIEW MATERIAL

- A differential equation as a mathematical model in Section 1.3
- Reread "Solving a Linear First-Order Equation" on page 56 in Section 2.3

INTRODUCTION In this section we solve some of the linear first-order models tha introduced in Section 1.3.

Growth and Decay The initial-value problem

$$\frac{dx}{dt} = kx, \quad x(t_0) = x_0,$$

where k is a constant of proportionality, serves as a model for diverse involving either growth or decay. We saw in Section 1.3 that in biologi tions the rate of growth of certain populations (bacteria, small animals) periods of time is proportional to the population present at time t. Knowi ulation at some arbitrary initial time t_0, we can then use the solution of (1 the population in the future—that is, at times $t > t_0$. The constant of pr ity k in (1) can be determined from the solution of the initial-value proble subsequent measurement of x at a time $t_1 > t_0$. In physics and chemistry in the form of a *first-o der reaction*—that is, a reaction whose rate, o dx/dt is directly proportional to the amount x of a substance that is un or remaining at time t. The decomposition, or decay, of U-238 (ura radioactivity into Th-234 (thorium) is a first-order reaction

EXAMPLE 1 **Bacterial Growth**

A culture initially has P_0 number of bacteria. At $t = 1$ h the number of bacter sured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteri sent at time t, determine the time necessary for the number of bacteria to tri

SOLUTION We first solve the differential equation in (1), with the symbol x by P. With $t_0 = 0$ the initial condition is $P(0) = P_0$. We then use the empiric vation that $P(1) = \frac{3}{2}P_0$ to determine the constant of proportionality k.

Notice that the differential equation $dP/dt = kP$ is both separable ar When it is put in the standard form of a linear first-order DE

$$\frac{dP}{dt} - kP = 0,$$

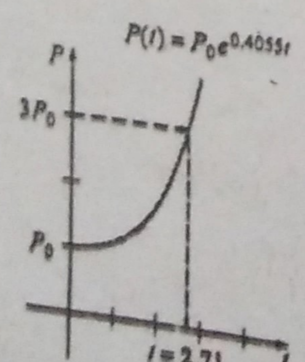
we can see by inspection that the integrating factor is e^{-kt}. Multiplying both the equation by this term and integrating gives, in turn,

$$\frac{d}{dt}[e^{-kt}P] = 0 \quad \text{and} \quad e^{-kt}P = c.$$

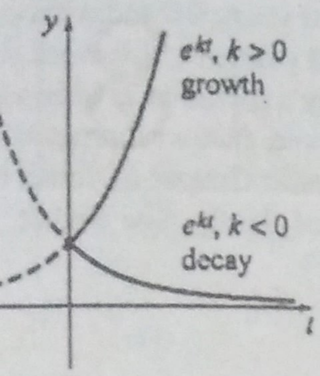
Therefore $P(t) = ce^{kt}$. At $t = 0$ it follows that $P_0 = ce^0 = c$, so $P(t) = P$ $t = 1$ we have $\frac{3}{2}P_0 = P_0e^k$ or $e^k = \frac{3}{2}$. From the last equation $k = \ln\frac{3}{2} = 0.4055$, so $P(t) = P_0e^{0.4055t}$. To find the time at which the number teria has tripled, we solve $3P_0 = P_0e^{0.4055t}$ for t. It follows that $0.4055t =$

$$t = \frac{\ln 3}{0.4055} \approx 2.71 \text{ h}.$$

See Figure 3.1.1.



GURE 3.1.1 Time in which pulation triples in Example 1



.1.2 Growth ($k > 0$) and
0)

Notice in Example 1 that the actual number P_0 of bacteria present at time $t = 0$ played no part in determining the time required for the number in the culture to triple. The time necessary for an initial population of, say, 100 or 1,000,000 bacteria to triple is still approximately 2.71 hours.

As shown in Figure 3.1.2, the exponential function e^{kt} increases as t increases for $k > 0$ and decreases as t increases for $k < 0$. Thus problems describing growth (whether of populations, bacteria, or even capital) are characterized by a positive value of k, whereas problems involving decay (as in radioactive disintegration) yield a negative k value. Accordingly, we say that k is either a **growth constant** ($k > 0$) or a **decay constant** ($k < 0$).

Half-Life In physics the **half-life** is a measure of the stability of a radioactive substance. The half-life is simply the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate, or transmute, into the atoms of another element. The longer the half-life of a substance, the more stable it is. For example, the half-life of highly radioactive radium, Ra-226, is about 1700 years. In 1700 years one-half of a given quantity of Ra-226 is transmuted into radon, Rn-222. The most commonly occurring uranium isotope, U-238, has a half-life of approximately 4,500,000,000 years. In about 4.5 billion years, one-half of a quantity of U-238 is transmuted into lead, Pb-206.

EXAMPLE 2 Half-Life of Plutonium

A breeder reactor converts relatively stable uranium-238 into the isotope plutonium-239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

SOLUTION Let $A(t)$ denote the amount of plutonium remaining at time t. As in Example 1 the solution of the initial-value problem

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

is $A(t) = A_0e^{kt}$. If 0.043% of the atoms of A_0 have disintegrated, then 99.957% of the substance remains. To find the decay constant k, we use $0.99957A_0 = A(15)$—that is, $0.99957A_0 = A_0e^{15k}$. Solving for k then gives $k = \frac{1}{15} \ln 0.99957 = -0.00002867$. Hence $A(t) = A_0e^{-0.00002867t}$. Now the half-life is the corresponding value of time at which $A(t) = \frac{1}{2}A_0$. Solving for t gives $\frac{1}{2}A_0 = A_0e^{-0.00002867t}$, or $\frac{1}{2} = e^{-0.00002867t}$. The last equation yields

$$t = \frac{\ln 2}{0.00002867} \approx 24{,}180 \text{ yr.}$$

Carbon Dating About 1950, a team of scientists at the University of Chicago led by the chemist Willard Libby devised a method using a radioactive isotope of carbon as a means of determining the approximate ages of carbonaceous fossilized matter. The theory of **carbon dating** is based on the fact that the radioisotope carbon-14 is produced in the atmosphere by the action of cosmic radiation on nitrogen-14. The ratio of the amount of C-14 to the stable C-12 in the atmosphere appears to be a constant, and as a consequence the proportionate amount of the isotope present in all living organisms is the same as that in the atmosphere. When a living organism dies, the absorption of C-14, by breathing, eating, or photosynthesis, ceases. By comparing the proportionate amount of C-14, say, in a fossil with the constant amount ratio found in the atmosphere, it is possible to obtain a reasonable estimation of its age. The method is based on the knowledge of the half-life of C-14. Libby's calculated value of the half-life of C-14 was approximately 5600 years, but today the commonly accepted value of the half-life is approximately 5730 years. For his work, Libby was awarded the Nobel Prize for chemistry in 1960. Libby's method has been used to date wooden furniture found in Egyptian tombs, the woven flax wrappings of the Dead Sea Scrolls, a recently discovered copy of the Gnostic Gospel of Judas written on papyrus, and the cloth of the enigmatic Shroud of Turin. See Figure 3.1.3 and Problem 12 in Exercises 3.1.


NGS Image Collection

A page of the Gnostic

EXAMPLE 3 Age of a Fossil

A fossilized bone is found to contain 0.1% of its original amount of C-14. Determine the age of the fossil.

SOLUTION The starting point is again $A(t) = A_0e^{kt}$. To determine the value of the decay constant k we use the fact that $\frac{1}{2}A_0 = A(5730)$ or $\frac{1}{2}A_0 = A_0e^{5730k}$. The last equation implies $5730k = \ln\frac{1}{2} = -\ln 2$ and so we get $k = -(\ln 2)/5730 = -0.00012097$. Therefore $A(t) = A_0e^{-0.00012097t}$. With $A(t) = 0.001A_0$ we have $0.001A_0 = A_0e^{-0.00012097t}$ and $-0.00012097t = \ln(0.001) = -\ln 1000$. Thus

$$t = \frac{\ln 1000}{0.00012097} \approx 57{,}100 \text{ years.}$$

The date found in Example 3 is really at the border of accuracy for this method. The usual carbon-14 technique is limited to about 10 half-lives of the isotope, or roughly 60,000 years. One reason for this limitation is that the chemical analysis needed to obtain an accurate measurement of the remaining C-14 becomes somewhat formidable around the point $0.001A_0$. Also, this analysis demands the destruction of a rather large sample of the specimen. If this measurement is accomplished indirectly, based on the actual radioactivity of the specimen, then it is very difficult to distinguish between the radiation from the specimen and the normal background radiation.* But recently the use of a particle accelerator has enabled scientists to separate the C-14 from the stable C-12 directly. When the precise value of the ratio of C-14 to C-12 is computed, the accuracy can be extended to 70,000 to 100,000 years. Other isotopic techniques, such as using potassium-40 and argon-40, can give dates of several million years. Nonisotopic methods based on the use of amino acids are also sometimes possible.

Newton's Law of Cooling/Warming In equation (3) of Section 1.3 we saw that the mathematical formulation of Newton's empirical law of cooling/warming of an object is given by the linear first-order di ferential equation

$$\frac{dT}{dt} = k(T - T_m), \tag{2}$$

where k is a constant of proportionality, $T(t)$ is the temperature of the object for $t > 0$, and T_m is the ambient temperature—that is, the temperature of the medium around the object. In Example 4 we assume that T_m is constant.

EXAMPLE 4 Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

*The number of disintegrations per minute per gram of carbon is recorded by using a Geiger counter. The lower level of detectability is about 0.1 disintegrations per minute per gram.

EXERCISES 3.1

Answers to selec

Growth and Decay

1. The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years, how long will it take to triple? To quadruple?

2. Suppose it is known that the population of the community in Problem 1 is 10,000 after 3 years. What was the initial population P_0? What will be the population in 10 years? How fast is the population growing at $t = 10$?

3. The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. What will be the population in 30 years? How fast is the population growing at $t = 30$?

4. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?

5. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay?

6. Initially 100 milligrams of a radioactive substance was present. After 6 hours the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at time t, find the amount remaining after 24 hours.

7. Determine the half-life of the radioactive substance described in Problem 6.

8. (a) Consider the initial-value problem $dA/dt = kA$, $A(0) = A_0$ as the model for the decay of a radioactive substance. Show that, in general, the half-life T of the substance is $T = -(\ln 2)/k$.

   (b) Show that the solution of the initial-value problem in part (a) can be written $A(t) = A_0 2^{-t/T}$.

   (c) If a radioactive substance has the half-life T given in part (a), how long will it take an initial amount A_0 of the substance to decay to $\frac{1}{8}A_0$?

9. When a vertical beam of light passes through a transparent medium, the rate at which its intensity I decreases is proportional to $I(t)$

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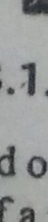

FIGURE 3.1.

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