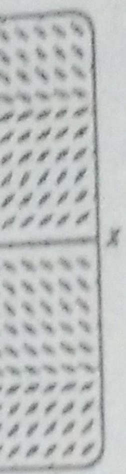


4

field i

Sketching a direction fiel by hand is straightforward but time consuming; it is probably one of those tasks about which an argument can be made for doing it once or twice in a lifetime, but it is overall most efficientl carried out by means of computer software. Before calculators, PCs, and software the method of isoclines was used to facilitate sketching a direction fiel by hand. For the DE $dy/dx = f(x, y)$, any member of the family of curves $f(x, y) = c$, c a constant, is called an **isocline**. Lineal elements drawn through points on a specifi isocline, say, $f(x, y) = c_1$ all have the same slope c_1. In Problem 15 in Exercises 2.1 you have your two opportunities to sketch a direction fiel by hand.

2.1.2 AUTONOMOUS FIRST-ORDER DEs

Autonomous First-Order DEs In Section 1.1 we divided the class of ordinary differential equations into two types: linear and nonlinear. We now consider briefly another kind of classification of ordinary differential equations, a classifica tion that is of particular importance in the qualitative investigation of differential equations. An ordinary differential equation in which the independent variable does not appear explicitly is said to be **autonomous.** If the symbol x denotes the independent variable, then an autonomous first-order differential equation can be written as $f(y, y') = 0$ or in normal form as

$$\frac{dy}{dx} = f(y). \tag{2}$$

We shall assume throughout that the function f in (2) and its derivative f' are continuous functions of y on some interval I. The first-order equation

$$\frac{dy}{dx} = 1 + y^2 \quad \text{and} \quad \frac{dy}{dx} = 0.2xy$$

are autonomous and nonautonomous, respectively.

Many differential equations encountered in applications or equations that are models of physical laws that do not change over time are autonomous. As we have already seen in Section 1.3, in an applied context, symbols other than y and x are routinely used to represent the dependent and independent variables. For example, if t represents time then inspection of

$$\frac{dA}{dt} = kA, \qquad \frac{dx}{dt} = kx(n + 1 - x), \qquad \frac{dT}{dt} = k(T - T_m), \qquad \frac{dA}{dt} = 6 - \frac{1}{100}A,$$

where k, n, and T_m are constants, shows that each equation is time independent. Indeed, *all* of the first-order differential equations introduced in Section 1.3 are time independent and so are autonomous.

Critical Points The zeros of the function f in (2) are of special importance. We say that a real number c is a **critical point** of the autonomous differential equation (2) if it is a zero of f—that is, $f(c) = 0$. A critical point is also called an **equilibrium point** or **stationary point.** Now observe that if we substitute the constant function $y(x) = c$ into (2), then both sides of the equation are zero. This means:

If c is a critical point of (2), then $y(x) = c$ is a constant solution of the autonomous differential equation.

A constant solution $y(x) = c$ of (2) is called an **equilibrium solution**; equilibria are the *only* constant solutions of (2).

42. **Chemical Reactions** When certain kinds of chemicals are combined, the rate at which the new compound is formed is modeled by the autonomous differential equation

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X),$$

where $k > 0$ is a constant of proportionality and $\beta > \alpha > 0$. Here $X(t)$ denotes the number of grams of the new compound formed in time t.

(a) Use a phase portrait of the differential equation to predict the behavior of $X(t)$ as $t \to \infty$.

(b) Consider the case when $\alpha = \beta$. Use a phase po of the differential equation to predict the beh of $X(t)$ as $t \to \infty$ when $X(0) < \alpha$. When $X(0)$

(c) Verify that an explicit solution of the DE in the when $k = 1$ and $\alpha = \beta$ is $X(t) = \alpha - 1/(t$ Find a solution that satisfies $X(0) = \alpha/2$. Then a solution that satisfies $X(0) = 2\alpha$. Graph t two solutions. Does the behavior of the solutio $t \to \infty$ agree with your answers to part (b)?

2.2 SEPARABLE EQUATIONS

REVIEW MATERIAL

- Basic integration fosmulas (See inside front cover)
- Techniques of integration: integration by parts and partial fraction decomposition
- See also the *Student Resource Manual.*

INTRODUCTION We begin our study of how to solve differential equations with the simplest of all differential equations: first-order equations with separable variables. Because the method in this section and many techniques for solving differential equations involve integration, you are urged to refresh your memory on important formulas (such as $\int du/u$) and techniques (such as integration by parts) by consulting a calculus text.

Solution by Integration Consider the first-order differential equation dy/dx $f(x, y)$. When f does not depend on the variable y, that is, $f(x, y) = g(x)$, the differen tial equation

$$\frac{dy}{dx} = g(x) \qquad (1$$

can be solved by integration. If $g(x)$ is a continuous function, then integrating bot sides of (1) gives $y = \int g(x)\,dx = G(x) + c$, where $G(x)$ is an antiderivative (indefi nite integral) of $g(x)$. For example, if $dy/dx = 1 + e^{2x}$, then its solution i $y = \int(1 + e^{2x})\,dx$ or $y = x + \frac{1}{2}e^{2x} + c$.

A Definition Equation (1), as well as its method of solution, is just a special case when the function f in the normal form $dy/dx = f(x, y)$ can be factored into a function of x times a function of y.

DEFINITION 2.2.1 Separable Equation

A first-order di ferential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables.**

For example, the equations

are separable and nonseparable, respectively. In the first equation we c
$f(x, y) = y^2xe^{3x+4y}$ as

$$f(x, y) = y^2xe^{3x+4y} = \overset{g(x)}{\overset{\downarrow}{(xe^{3x})}}\overset{h(y)}{\overset{\downarrow}{(y^2e^{4y})}},$$

but in the second equation there is no way of expressing $y + \sin x$ as a pr
function of x times a function of y.

Observe that by dividing by the function $h(y)$, we can write a separable
$dy/dx = g(x)h(y)$ as

$$p(y)\frac{dy}{dx} = g(x),$$

where, for convenience, we have denoted $1/h(y)$ by $p(y)$. From this last fo
see immediately that (2) reduces to (1) when $h(y) = 1$.

Now if $y = \phi(x)$ represents a solution of (2), we must have $p(\phi(x))\phi'($
and therefore

$$\int p(\phi(x))\phi'(x)\,dx = \int g(x)\,dx.$$

But $dy = \phi'(x)\,dx$, and so (3) is the same as

$$\int p(y)\,dy = \int g(x)\,dx \quad \text{or} \quad H(y) = G(x) + c,$$

where $H(y)$ and $G(x)$ are antiderivatives of $p(y) = 1/h(y)$ and $g(x)$, respec

Method of Solution Equation (4) indicates the procedure for solving
equations. A one-parameter family of solutions, usually given implicitly, i
by integrating both sides of $p(y)\,dy = g(x)\,dx$.

Note There is no need to use two constants in the integration of a
equation, because if we write $H(y) + c_1 = G(x) + c_2$, then the difference c
be replaced by a single constant c, as in (4). In many instances throughout
ters that follow, we will relabel constants in a manner convenient to a giver
For example, multiples of constants or combinations of constants can son
replaced by a single constant.

EXAMPLE 1 **Solving a Separable DE**

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, fro
follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$
$$\ln|y| = \ln|1 + x| + c_1$$
$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \qquad \leftarrow \text{laws of exponents}$$
$$= |1 + x|\,e^{c_1} \qquad \leftarrow \begin{cases} |1 + x| = 1 + x, \\ |1 + x| = -(1 + x \end{cases}$$
$$= \pm e^{c_1}(1 + x).$$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

ALTERNATIVE SOLUTION Because each integral results in a logarithm, a judicious choice for the constant of integration is $\ln|c|$ rather than c. Rewriting the second line of the solution as $\ln|y| = \ln|1 + x| + \ln|c|$ enables us to combine the terms on the right-hand side by the properties of logarithms. From $\ln|y| = \ln|c(1 + x)|$ we immediately get $y = c(1 + x)$. Even if the indefinite integrals are not *all* logarithms, it may still be advantageous to use $\ln|c|$. However, no firm rule can be given. ≡

In Section 1.1 we saw that a solution curve may be only a segment or an arc of the graph of an implicit solution $G(x, y) = 0$.

EXAMPLE 2 Solution Curve

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}, \quad y(4) = -3.$

SOLUTION Rewriting the equation as $y\,dy = -x\,dx$, we get

$$\int y\,dy = -\int x\,dx \quad \text{and} \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

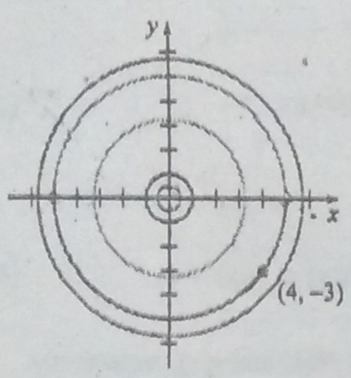


FIGURE 2.2.1 Solution curve for the IVP in Example 2

We can write the result of the integration as $x^2 + y^2 = c^2$ by replacing the constant $2c_1$ by c^2. This solution of the differential equation represents a family of concentric circles centered at the origin.

Now when $x = 4$, $y = -3$, so $16 + 9 = 25 = c^2$. Thus the initial-value problem determines the circle $x^2 + y^2 = 25$ with radius 5. Because of its simplicity we can solve this implicit solution for an explicit solution that satisfies the initial condition. We saw this solution as $y = \phi_2(x)$ or $y = -\sqrt{25 - x^2}, -5 < x < 5$ in Example 3 of Section 1.1. A solution curve is the graph of a differentiable function. In this case the solution curve is the lower semicircle, shown in dark blue in Figure 2.2.1 containing the point $(4, -3)$. ≡

≡ **Losing a Solution** Some care should be exercised in separating variables, since the variable divisors could be zero at a point. Specificall , if r is a zero of the function $h(y)$, then substituting $y = r$ into $dy/dx = g(x)h(y)$ makes both sides zero; in other words, $y = r$ is a constant solution of the differential equation. But after variables are separated, the left-hand side of $\dfrac{dy}{h(y)} = g(x)\,dx$ is undefined at r. As a consequence, $y = r$ might not show up in the family of solutions that are obtained after integration and simplification. Recall that such a solution is called a singular solution.

EXAMPLE 3 Losing a Solution

Solve $\dfrac{dy}{dx} = y^2 - 4$.

SOLUTION We put the equation in the form

$$\frac{dy}{y^2 - 4} = dx \quad \text{or} \quad \left[\frac{\frac{1}{4}}{y - 2} - \frac{\frac{1}{4}}{y + 2}\right] dy = dx. \tag{5}$$

The second equation in (5) is the result of using partial fractions on the left-hand side of the first equation. Integrating and using the laws of logarithms give

$$\frac{1}{4}\ln|y - 2| - \frac{1}{4}\ln|y + 2| = x + c_1$$

$$\ln\left|\frac{y - 2}{y + 2}\right| = 4x + c_2 \quad \text{or} \quad \frac{y - 2}{y + 2} = \pm e^{4x + c_2}.$$

Here we have replaced $4c_1$ by c_2. Finally, after replacing $\pm e^{c_2}$ by c and solving the last equation for y, we get the one-parameter family of solutions

$$y = 2\frac{1 + ce^{4x}}{1 - ce^{4x}}. \tag{6}$$

Now if we factor the right-hand side of the differential equation as $dy/dx = (y - 2)(y + 2)$, we know from the discussion of critical points in Section 2.1 that $y = 2$ and $y = -2$ are two constant (equilibrium) solutions. The solution $y = 2$ is a member of the family of solutions defined by (6) corresponding to the value $c = 0$. However, $y = -2$ is a singular solution; it cannot be obtained from (6) for any choice of the parameter c. This latter solution was lost early on in the solution process. Inspection of (5) clearly indicates that we must preclude $y = \pm 2$ in these steps.

EXAMPLE 4 An Initial-Value Problem

Solve $(e^{2y} - y)\cos x \dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0.$

SOLUTION Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y}\,dy = \frac{\sin 2x}{\cos x}\,dx.$$

Before integrating, we use termwise division on the left-hand side and the trigonometric identity $\sin 2x = 2 \sin x \cos x$ on the right-hand side. Then

$$\int (e^y - ye^{-y})\,dy = 2\int \sin x\,dx$$

yields

$$e^y + ye^{-y} + e^{-y} = -2\cos x + c. \tag{7}$$

The initial cond[illegible] = 0 when $x = 0$ implies $c = 4$. Thus a solution of the initial-value problem

$$e^y + ye^{-y} + e^{-y} = 4 - 2\cos x. \tag{8}$$

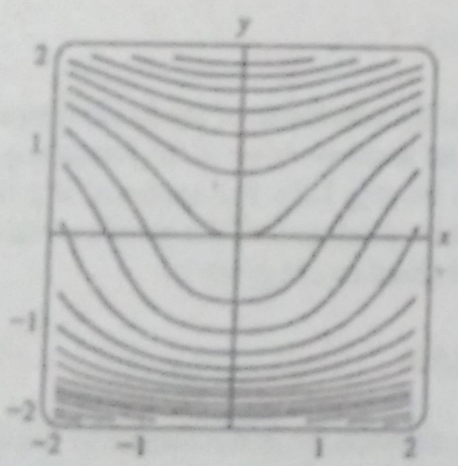
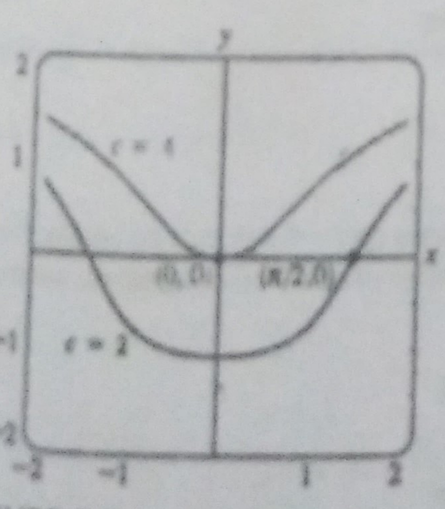


FIGURE 2.2.2 Level curves of $G(x, y) = e^y + ye^{-y} + e^{-y} + 2\cos x$



GURE 2.2.3 Level curves [illegible] 2 and $c = 4$

Use of Computers The *Remarks* at the end of Section 1.1 mentioned that it may be difficult to use an implicit solution $G(x, y) = 0$ to find an explicit solution $y = \phi(x)$. Equation (8) shows that the task of solving for y in terms of x may present more problems than just the drudgery of symbol pushing—sometimes it simply cannot be done! Implicit solutions such as (8) are somewhat frustrating; neither the graph of the equation nor an interval over which a solution satisfying $y(0) = 0$ is defined is apparent. The problem of "seeing" what an implicit solution looks like can be overcome in some cases by means of technology. One way* of proceeding is to use the contour plot application of a computer algebra system (CAS). Recall from multivariate calculus that for a function of two variables $z = G(x, y)$ the *two-dimensional curves* defined by $G(x, y) = c$, where c is constant, are called the *level curves* of the function. With the aid of a CAS, some of the level curves of the function $G(x, y) = e^y + ye^{-y} + e^{-y} + 2\cos x$ have been reproduced in Figure 2.2.2. The family of solutions defined by (7) is the level curves $G(x, y) = c$. Figure 2.2.3 illustrates the level curve $G(x, y) = 4$, which is the particular solution (8), in blue color. The other curve in Figure 2.2.3 is the level curve $G(x, y) = 2$, which is the member of the family $G(x, y) = c$ that satisfies $y(\pi/2) = 0$.

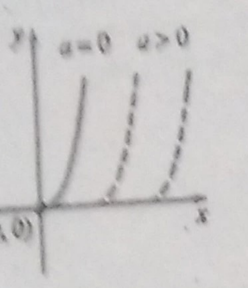
If an initial condition leads to a particular solution by yielding a specific value of the parameter c in a family of solutions for a first-order differential equation, there is

*In Section 2.6 we will discuss several other ways of proceeding that are based on the concept of a a natural inclination for most students (and instructors) to relax and be content. However, a solution of an initial-value problem might not be unique. We saw in Example 4 of Section 1.2 that the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0 \tag{9}$$

has at least two solutions, $y = 0$ and $y = \frac{1}{16}x^4$. We are now in a position to solve the equation. Separating variables and integrating $y^{-1/2}\,dy = x\,dx$ gives

$$2y^{1/2} = \frac{x^2}{2} + c_1 \quad \text{or} \quad y = \left(\frac{x^2}{4} + c\right)^2, \quad c \geq 0.$$



2.4 Piecewise-define
9)

When $x = 0$, then $y = 0$, so necessarily, $c = 0$. Therefore $y = \frac{1}{16}x^4$. The trivial solution $y = 0$ was lost by dividing by $y^{1/2}$. In addition, the initial-value problem (9) possesses infinitely many more solutions, since for any choice of the parameter $a \geq 0$ the piecewise-defined functio

$$y = \begin{cases} 0, & x < a \\ \frac{1}{16}(x^2 - a^2)^2, & x \geq a \end{cases}$$

satisfies both the di ferential equation and the initial condition. See Figure 2.2.4.

Solutions Defined by Integrals If g is a function continuous on an open interval I containing a, then for every x in I,

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x).$$

You might recall that the foregoing result is one of the two forms of the fundamental theorem of calculus. In other words, $\int_a^x g(t)\,dt$ is an antiderivative of the function g. There are times when this form is convenient in solving DEs. For example, if g is continuous on an interval I containing x_0 and x, then a solution of the simple initial-value problem $dy/dx = g(x)$, $y(x_0) = y_0$, that is defined on I is given by

$$y(x) = y_0 + \int_{x_0}^x g(t)\,dt$$

You should verify that $y(x)$ defined in this manner satisfies the initial condition. Since an antiderivative of a continuous function g cannot always be expressed in terms of elementary functions, this might be the best we can do in obtaining an explicit solution of an IVP. The next example illustrates this idea.

EXAMPLE 5 An Initial-Value Problem

Solve $\frac{dy}{dx} = e^{-x^2}$, $y(3) = 5$.

SOLUTION The function $g(x) = e^{-x^2}$ is continuous on $(-\infty, \infty)$, but its antiderivative is not an elementary function. Using t as dummy variable of integration, we can write

$$\int_3^x \frac{dy}{dt}\,dt = \int_3^x e^{-t^2}\,dt$$

$$y(t)\Big]_3^x = \int_3^x e^{-t^2}\,dt$$

$$y(x) - y(3) = \int_3^x e^{-t^2}\,dt$$

Using the initial condition $y(3) = 5$, we obtain the solution

$$y(x) = 5 + \int_3^x e^{-t^2}\,dt.$$

The procedure demonstrated in Example 5 works equally well on se equations $dy/dx = g(x)f(y)$ where, say, $f(y)$ possesses an elementary antider but $g(x)$ does not possess an elementary antiderivative. See Problems 29 an Exercises 2.2.

REMARKS

(i) As we have just seen in Example 5, some simple functions do not pos an antiderivative that is an elementary function. Integrals of these kind functions are called **nonelementary**. For example, $\int_3^x e^{-t^2}\,dt$ and $\int \sin x^2\,dx$ nonelementary integrals. We will run into this concept again in Section 2.3.

(ii) In some of the preceding examples we saw that the constant in the parameter family of solutions for a first-order differential equation can be beled when convenient. Also, it can easily happen that two individuals solving same equation correctly arrive at dissimilar expressions for their answers. example, by separation of variables we can show that one-parameter familie solutions for the DE $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$ are

$$\arctan x + \arctan y = c \qquad \text{or} \qquad \frac{x + y}{1 - xy} = c.$$

As you work your way through the next several sections, bear in mind that fa lies of solutions may be equivalent in the sense that one family may be obta from another by either relabeling the constant or applying algebra and trigon etry. See Problems 27 and 28 in Exercises 2.2.

EXERCISES 2.2

Answers to selected odd-numbered problems begin on page

In Problems 1–22 solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$
2. $\dfrac{dy}{dx} = (x + 1)^2$
3. $dx + e^{3x}dy = 0$
4. $dy - (y - 1)^2dx = 0$
5. $x\dfrac{dy}{dx} = 4y$
6. $\dfrac{dy}{dx} + 2xy^2 = 0$
7. $\dfrac{dy}{dx} = e^{3x+2y}$
8. $e^xy\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$
9. $y \ln x\dfrac{dx}{dy} = \left(\dfrac{y + 1}{x}\right)^2$
10. $\dfrac{dy}{dx} = \left(\dfrac{2y + 3}{4x + 5}\right)^2$
11. $\csc y\,dx + \sec^2x\,dy = 0$
12. $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$
13. $(e^y + 1)^2e^{-y}\,dx + (e^x + 1)^3e^{-x}\,dy = 0$
14. $x(1 + y^2)^{1/2}\,dx = y(1 + x^2)^{1/2}\,dy$
15. $\dfrac{dS}{dr} = kS$
16. $\dfrac{dQ}{dt} = k(Q - 70)$
17. $\dfrac{dP}{dt} = P - P^2$
18. $\dfrac{dN}{dt} + N = Nte^{t+}$
19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$
20. $\dfrac{dy}{dx} = \dfrac{xy + 2y -}{xy - 3y +}$
21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$
22. $(e^x + e^{-x})\dfrac{dy}{dx} =$

In Problems 23–28 find an explicit solution of th initial-value problem.

23. $\dfrac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1$
24. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$
25. $x^2\dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$

26. $\frac{dy}{dt} + 2y = 1, \quad y(0) = \frac{5}{2}$

27. $\sqrt{1-y^2}\, dx - \sqrt{1-x^2}\, dy = 0, \quad y(0) = \frac{\sqrt{3}}{2}$

28. $(1 + x^4)\, dy + x(1 + 4y^2)\, dx = 0, \quad y(1) = 0$

In Problems 29 and 30 proceed as in Example 5 and find an explicit solution of the given initial-value problem.

29. $\frac{dy}{dx} = ye^{-x^2}, \quad y(4) = 1$

30. $\frac{dy}{dx} = y^2 \sin x^2, \quad y(-2) = \frac{1}{3}$

In Problems 31–34 find an explicit solution of the given initial-value problem. Determine the exact interval I of definition by analytical methods. Use a graphing utility to plot the graph of the solution.

31. $\frac{dy}{dx} = \frac{2x + 1}{2y}, \quad y(-2) = -1$

32. $(2y - 2)\frac{dy}{dx} = 3x^2 + 4x + 2, \quad y(1) = -2$

33. $e^x dx - e^{-x} dy = 0, \quad y(0) = 0$

34. $\sin x\, dx + y\, dy = 0, \quad y(0) = 1$

35. (a) Find a solution of the initial-value problem consisting of the differential equation in Example 3 and each of the initial-conditions: $y(0) = 2, y(0) = -2$, and $y\left(\frac{1}{4}\right) = 1$.
(b) Find the solution of the differential equation in Example 4 when $\ln c_1$ is used as the constant of integration on the *left-hand* side in the solution and $4 \ln c_1$ is replaced by $\ln c$. Then solve the same initial-value problems in part (a).

36. Find a solution of $x\frac{dy}{dx} = y^2 - y$ that passes through the indicated points.
(a) $(0, 1)$ (b) $(0, 0)$ (c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(2, \frac{1}{4}\right)$

37. Find a singular solution of Problem 21. Of Problem 22.

38. Show that an implicit solution of
$$2x \sin^2 y\, dx - (x^2 + 10) \cos y\, dy = 0$$
is given by $\ln(x^2 + 10) + \csc y = c$. Find the constant solutions, if any, that were lost in the solution of the differential equation.

Often a radical change in the form of the solution of a differential equation corresponds to a very small change in either the initial condition or the equation itself. In Problems 39–42 fin an explicit solution of the given initial-value problem. Use a graphing utility to plot the graph of each solution. Compare each solution curve in a neighborhood of (0, 1).

39. $\frac{dy}{dx} = (y - 1)^2, \quad y(0) = 1$

40. $\frac{dy}{dx} = (y - 1)^2, \quad y(0) = 1.01$

41. $\frac{dy}{dx} = (y - 1)^2 + 0.01, \quad y(0) = 1$

42. $\frac{dy}{dx} = (y - 1)^2 - 0.01, \quad y(0) = 1$

43. Every autonomous first-orde equation $dy/dx = f(y)$ is separable. Find explicit solutions $y_1(x), y_2(x), y_3(x)$, and $y_4(x)$ of the differential equation $dy/dx = y - y^3$ that satisfy, in turn, the initial conditions $y_1(0) = 2$, $y_2(0) = \frac{1}{2}$, $y_3(0) = -\frac{1}{2}$, and $y_4(0) = -2$. Use a graphing utility to plot the graphs of each solution. Compare these graphs with those predicted in Problem 19 of Exercises 2.1. Give the exact interval of definitio for each solution.

44. (a) The autonomous first-order differential equation $dy/dx = 1/(y - 3)$ has no critical points. Nevertheless, place 3 on the phase line and obtain a phase portrait of the equation. Compute d^2y/dx^2 to determine where solution curves are concave up and where they are concave down (see Problems 35 and 36 in Exercises 2.1). Use the phase portrait and concavity to sketch, by hand, some typical solution curves.
(b) Find explicit solutions $y_1(x), y_2(x), y_3(x)$, and $y_4(x)$ of the differential equation in part (a) that satisfy, in turn, the initial conditions $y_1(0) = 4$, $y_2(0) = 2$, $y_3(1) = 2$, and $y_4(-1) = 4$. Graph each solution and compare with your sketches in part (a). Give the exact interval of definition for each solution.

In Problems 45–50 use a technique of integration or a substitution to find an explicit solution of the given differential equation or initial-value problem.

45. $\frac{dy}{dx} = \frac{1}{1 + \sin x}$

46. $\frac{dy}{dx} = \frac{\sin\sqrt{x}}{\sqrt{y}}$

47. $(\sqrt{x} + x)\frac{dy}{dx} = \sqrt{y} + y$

48. $\frac{dy}{dx} = y^{2/3} - y$

49. $\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{y}, \quad y(1) = 4$

50. $\frac{dy}{dx} = \frac{x \tan^{-1} x}{y}, \quad y(0) = 3$

Discussion Problems

51. (a) Explain why the interval of definition of the explicit solution $y = \phi_2(x)$ of the initial-value problem in Example 2 is the *open* interval $(-5, 5)$.
(b) Can any solution of the differential equation cross the x-axis? Do you think that $x^2 + y^2 = 1$ is an implicit solution of the initial-value problem $dy/dx = -x/y, y(1) = 0$?

52. (a) If $a > 0$, discuss the differences, if any, between the solutions of the initial-value problems consisting of the differential equation $dy/dx = x/y$ and

definition of the *solution* $y = \phi(x)$ in part (b). Use a graphing utility or a CAS to graph the solution curve for the IVP on this interval.

(a) Use a CAS and the concept of level curves to plot representative graphs of members of the family of solutions of the differential equation $\frac{dy}{dx} = \frac{x(4-x)}{y(-2+y)}$. Experiment with different numbers of level curves as well as various rectangular regions in the xy-plane until your result resembles Figure 2.2.6.

(b) On separate coordinate axes, plot the graph of the implicit solution corresponding to the initial condition $y(0) = \frac{3}{2}$. Use a colored pencil to mark off that segment of the graph that corresponds to the solution curve of a solution ϕ that satisfies the initial condition. With the aid of a root-finding application of a CAS, determine the approximate largest interval I of definition of the solution ϕ. [*Hint*: First find the points on the curve in part (a) where the tangent is vertical.]

(c) Repeat part (b) for the initial condition $y(0) = -2$.

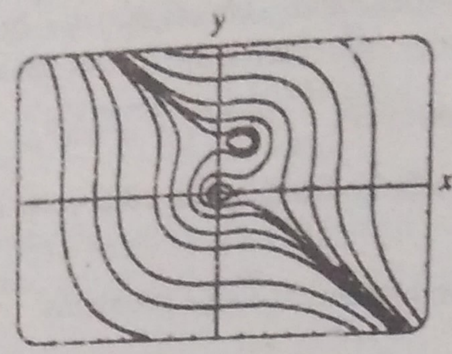


FIGURE 2.2.6 Level curves in Problem 60

2.3 LINEAR EQUATIONS

REVIEW MATERIAL

- Review the definitions of linear DEs in (6) and (7) of Section 1.1

INTRODUCTION We continue our quest for solutions of first-order differential equations by next examining linear equations. Linear differential equations are an especially "friendly" family of differential equations, in that, given a linear equation, whether first order or a higher-order kin, there is always a good possibility that we can find some sort of solution of the equation that we can examine.

A Definition The form of a linear first-order DE was given in (7) of Section 1.1. This form, the case when $n = 1$ in (6) of that section, is reproduced here for convenience.

DEFINITION 2.3.1 Linear Equation

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (1)$$

is said to be a linear equation in the variable y.

Standard Form By dividing both sides of (1) by the lead coefficient $a_1(x)$, we obtain a more useful form, the **standard form**, of a linear equation:

$$\frac{dy}{dx} + P(x)y = f(x). \qquad (2)$$

We seek a solution of (2) on an interval I for which both coefficient functions P and f are continuous.

Before we examine a general procedure for solving equations of form (1), note that in some instances (2) can be solved by separation of variables. For example, you should verify that the equations

We match each equation with (2). In the first equation $P(x) = 2x$, $f(x) = 0$ and in the second $P(x) = -1$, $f(x) = 5$.

$$\frac{dy}{dx} + 2xy = 0 \quad \text{and} \quad \frac{dy}{dx} = y + 5$$

are both linear and separable, but that the linear equation

$$\frac{dy}{dx} + y = x$$

is not separable.

Method of Solution The method for solving (2) hinges on a remarkable fact that the *left-hand side* of the equation can be recast into the form of the exact derivative of a product by multiplying the both sides of (2) by a special function $\mu(x)$. It is relatively easy to find the function $\mu(x)$ because we want

$$\overbrace{\frac{d}{dx}[\mu(x)y]}^{\text{product}} = \overbrace{\mu\frac{dy}{dx} + \frac{d\mu}{dx}y}^{\text{product rule}} = \overbrace{\mu\frac{dy}{dx} + \mu P y}^{\text{left hand side of (2) multiplied by } \mu(x)}.$$

these must be equal

The equality is true provided that

$$\frac{d\mu}{dx} = \mu P.$$

The last equation can be solved by separation of variables. Integrating

$$\frac{d\mu}{\mu} = P\,dx \quad \text{and solving} \quad \ln|\mu(x)| = \int P(x)dx + c_1$$

See Problem 50 in Exercises 2.3

gives $\mu(x) = c_2 e^{\int P(x)dx}$. Even though there are an infinite choices of $\mu(x)$ (all constant multiples of $e^{\int P(x)dx}$), all produce the same desired result. Hence we can simplify life and choose $c_2 = 1$. The function

$$\mu(x) = e^{\int P(x)dx}$$

is called an **integrating factor** for equation (2).

Here is what we have so far: We multiplied both sides of (2) by (3) and, by construction, the left-hand side is the derivative of a product of the integrating factor and y:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = e^{\int P(x)dx}f(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x)$$

Finally, we discover why (3) is called an *integrating* factor. We can integrate both sides of the last equation,

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}f(x) + c$$

and solve for y. The result is a one-parameter family of solutions of (2):

$$y = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx + ce^{-\int P(x)dx}$$

We emphasize that you should not memorize formula (4). The following procedure should be worked through each time.

SOLVING A LINEAR FIRST-ORDER EQUATION

(*i*) Remember to put a linear equation into the standard form (2).

(*ii*) From the standard form of the equation identify $P(x)$ and then find th integrating factor $e^{\int P(x)dx}$. No constant need be used in evaluating the indefinite integral $\int P(x)dx$.

(*iii*) Multiply the both sides of the standard form equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the product of the integrating factor $e^{\int P(x)dx}$ and y:

$$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} f(x).$$

(*iv*) Integrate both sides of the last equation and solve for y.

EXAMPLE 1 Solving a Linear Equation

Solve $\dfrac{dy}{dx} - 3y = 0$.

SOLUTION This linear equation can be solved by separation of variables. Alternatively, since the differential equation is already in standard form (2), we identify $P(x) = -3$, and so the integrating factor is $e^{\int(-3)dx} = e^{-3x}$. We then multiply the given equation by this factor and recognize that

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}\cdot 0 \qquad \text{is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 0.$$

Integration of the last equation,

$$\int \frac{d}{dx}[e^{-3x}y]\,dx = \int 0\,dx$$

then yields $e^{-3x}y = c$ or $y = ce^{3x}$, $-\infty < x < \infty$. ≡

EXAMPLE 2 Solving a Linear Equation

Solve $\dfrac{dy}{dx} - 3y = 6$.

SOLUTION This linear equation, like the one in Example 1, is already in standard form with $P(x) = -3$. Thus the integrating factor is again e^{-3x}. This time multiplying the given equation by this factor gives

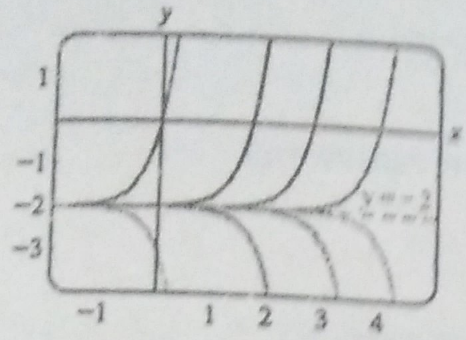
$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x} \qquad \text{and so} \qquad \frac{d}{dx}[e^{-3x}y] = 6e^{-3x}.$$

Integrating the last equation,

$$\int \frac{d}{dx}[e^{-3x}y]\,dx = 6\int e^{-3x}\,dx \qquad \text{gives} \qquad e^{-3x}y = -6\left(\frac{e^{-3x}}{3}\right) + c,$$

or $y = -2 + ce^{3x}$, $-\infty < x < \infty$. ≡

When a_1, a_0, and g in (1) are constants, the differential equation is autonomous. In Example 2 you can verify from the normal form $dy/dx = 3(y + 2)$ that -2 is a critical point and that it is unstable (a repeller). Thus a solution curve with an



GURE 2.3.1 Solution curves of DE Example 2

initial point either above or below the graph of the equilibrium solution $y = -2$ pushes away from this horizontal line as x increases. Figure 2.3.1, obtained with the aid of a graphing utility, shows the graph of $y = -2$ along with some additional solution curves.

General Solution Suppose again that the functions P and f in (2) are continuous on a common interval I. In the steps leading to (4) we showed that *if* (2) has a solution on I, then it must be of the form given in (4). Conversely, it is a straightforward exercise in differentiation to verify that any function of the form given in (4) is a solution of the differential equation (2) on I. In other words, (4) is a one-parameter family of solutions of equation (2) and *every solution of (2) defined on I is a member of this family.* Therefore we call (4) the **general solution** of the differential equation on the interval I. (See the *Remarks* at the end of Section 1.1.) Now by writing (2) in the normal form $y' = F(x, y)$, we can identify $F(x, y) = -P(x)y + f(x)$ and $\partial F/\partial y = -P(x)$. From the continuity of P and f on the interval I we see that F and $\partial F/\partial y$ are also continuous on I. With Theorem 1.2.1 as our justification, we conclude that there exists one and only one solution of the initial-value problem

$$\frac{dy}{dx} + P(x)y = f(x), \quad y(x_0) = y_0 \tag{5}$$

defined on *some* interval I_0 containing x_0. But when x_0 is in I, finding a solution of (5) is just a matter of finding an appropriate value of c in (4)—that is, to each x_0 in I there corresponds a distinct c. In other words, the interval I_0 of existence and uniqueness in Theorem 1.2.1 for the initial-value problem (5) is the entire interval I.

EXAMPLE 3 General Solution

Solve $x\dfrac{dy}{dx} - 4y = x^6e^x$.

SOLUTION Dividing by x, the standard form of the given DE is

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x. \tag{6}$$

From this form we identify $P(x) = -4/x$ and $f(x) = x^5e^x$ and further observe that P and f are continuous on $(0, \infty)$. Hence the integrating factor is

we can use $\ln x$ instead of $\ln|x|$ since $x > 0$
↓
$$e^{-4\int dx/x} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}.$$

Here we have used the basic identity $b^{\log_b N} = N$, $N > 0$. Now we multiply (6) by x^{-4} and rewrite

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x \quad \text{as} \quad \frac{d}{dx}[x^{-4}y] = xe^x.$$

It follows from integration by parts that the general solution defined on the interval $(0, \infty)$ is $x^{-4}y = xe^x - e^x + c$ or $y = x^5e^x - x^4e^x + cx^4$. ≡

e you are wondering why the al $(0, \infty)$ is important in Example 3, his paragraph and the paragraph ing Example 4. ▶

Except in the case in which the lead coefficient is 1, the recasting of equation (1) into the standard form (2) requires division by $a_1(x)$. Values of x for which $a_1(x) = 0$ are called **singular points** of the equation. Singular points are potentially troublesome. Specifically, in (2), if $P(x)$ (formed by dividing $a_0(x)$ by $a_1(x)$) is discontinuous at a point, the discontinuity may carry over to solutions of the differential equation.

EXAMPLE 4 General Solution

Find the general solution of $(x^2-9)\dfrac{dy}{dx}+xy=0$.

SOLUTION We write the differential equation in standard form

$$\frac{dy}{dx}+\frac{x}{x^2-9}y=0$$

and identify $P(x)=x/(x^2-9)$. Although P is continuous on $(-\infty,-3)$, $(-3,3)$, a
$(3,\infty)$, we shall solve the equation on the first and third intervals. On these interv
the integrating factor is

$$e^{\int x\,dx/(x^2-9)}=e^{\frac{1}{2}\int 2x\,dx/(x^2-9)}=e^{\frac{1}{2}\ln|x^2-9|}=\sqrt{x^2-9}.$$

After multiplying the standard form (7) by this factor, we get

$$\frac{d}{dx}\left[\sqrt{x^2-9}\,y\right]=0.$$

Integrating both sides of the last equation gives $\sqrt{x^2-9}\,y=c$. Thus for eith
$x>3$ or $x<-3$ the general solution of the equation is $y=\dfrac{c}{\sqrt{x^2-9}}$. ≡

Notice in Example 4 that $x=3$ and $x=-3$ are singular points of the equatio
and that every function in the general solution $y=c/\sqrt{x^2-9}$ is discontinuous a
these points. On the other hand, $x=0$ is a singular point of the differential equatio
in Example 3, but the general solution $y=x^5e^x-x^4e^x+cx^4$ is noteworthy in tha
every function in this one-parameter family is continuous at $x=0$ and is define
on the interval $(-\infty,\infty)$ and not just on $(0,\infty)$, as stated in the solution. However
the family $y=x^5e^x-x^4e^x+cx^4$ defined on $(-\infty,\infty)$ cannot be considered the gen-
eral solution of the DE, since the singular point $x=0$ still causes a problem. See
Problems 45 and 46 in Exercises 2.3.

EXAMPLE 5 An Initial-Value Problem

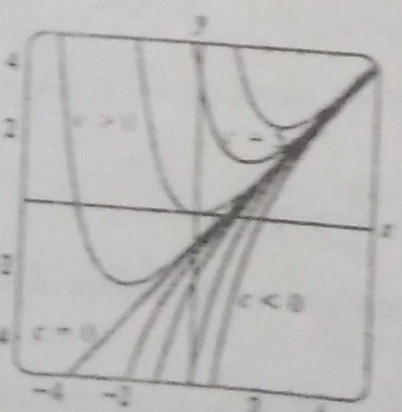
Solve $\dfrac{dy}{dx}+y=x$, $y(0)=4$.

SOLUTION The equation is in standard form, and $P(x)=1$ and $f(x)=x$ are contin-
uous on $(-\infty,\infty)$. The integrating factor is $e^{\int dx}=e^x$, so integrating

$$\frac{d}{dx}[e^xy]=xe^x$$

gives $e^xy=xe^x-e^x+c$. Solving this last equation for y yields the general solution
$y=x-1+ce^{-x}$. But from the initial condition we know that $y=4$ when $x=0$.
Substituting these values into the general solution implies that $c=5$. Hence the
solution of the problem is

$$y=x-1+5e^{-x},\quad -\infty<x<\infty. \tag{8}$$ ≡



CURE 2.3.2 Solution curves of DE
xample 5

Figure 2.3.2, obtained with the aid of a graphing utility, shows the graph of
the solution (8) in dark blue along with the graphs of other members of the one-
parameter family of solutions $y=x-1+ce^{-x}$. It is interesting to observe that as x
increases, the graphs of all members of this family are close to the graph of the solu-
tion $y=x-1$. The last solution corresponds to $c=0$ in the family and is shown in

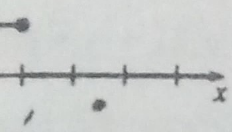
dark green in Figure 2.3.2. This asymptotic behavior of solutions is due to the fact that the contribution of ce^{-x}, $c \neq 0$, becomes negligible for increasing values of x. We say that ce^{-x} is a **transient term**, since $e^{-x} \to 0$ as $x \to \infty$. While this behavior is not characteristic of all general solutions of linear equations (see Example 2), the notion of a transient is often important in applied problems.

Transient:

Discontinuous Coefficients In applications, the coefficients $P(x)$ and $f(x)$ in (2) may be piecewise continuous. In the next example $f(x)$ is piecewise continuous on $[0, \infty)$ with a single discontinuity, namely, a (finite) jump discontinuity at $x = 1$. We solve the problem in two parts corresponding to the two intervals over which f is defined. It is then possible to piece together the two solutions at $x = 1$ so that $y(x)$ is continuous on $[0, \infty)$.

EXAMPLE 6 An Initial-Value Problem

Solve $\dfrac{dy}{dx} + y = f(x)$, $y(0) = 0$ where $f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1 \end{cases}$



3 Discontinuous $f(x)$ in

SOLUTION The graph of the discontinuous function f is shown in Figure 2.3.3. We solve the DE for $y(x)$ first on the interval $[0, 1]$ and then on the interval $(1, \infty)$. For $0 \le x \le 1$ we have

$$\frac{dy}{dx} + y = 1 \quad \text{or, equivalently,} \quad \frac{d}{dx}[e^x y] = e^x.$$

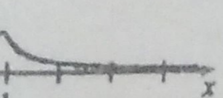
Integrating this last equation and solving for y gives $y = 1 + c_1e^{-x}$. Since $y(0) = 0$, we must have $c_1 = -1$, and therefore $y = 1 - e^{-x}$, $0 \le x \le 1$. Then for $x > 1$ the equation

$$\frac{dy}{dx} + y = 0$$

leads to $y = c_2e^{-x}$. Hence we can write

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ c_2e^{-x}, & x > 1. \end{cases}$$

By appealing to the definition of continuity at a point, it is possible to determine c_2 so that the foregoing function is continuous at $x = 1$. The requirement that $\lim_{x \to 1^+} y(x) = y(1)$ implies that $c_2e^{-1} = 1 - e^{-1}$ or $c_2 = e - 1$. As seen in Figure 2.3.4, the function



Graph of (9) in

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ (e - 1)e^{-x}, & x > 1 \end{cases} \tag{9}$$

is continuous on $(0, \infty)$.

To find c in 2nd sol. we have used continuity

It is worthwhile to think about (9) and Figure 2.3.4 a little bit; you are urged to read and answer Problem 48 in Exercises 2.3.

Functions Defined by Integrals At the end of Section 2.2 we discussed the fact that some simple continuous functions do not possess antiderivatives that are elementary functions and that integrals of these kinds of functions are called **nonelementary**. For example, you may have seen in calculus that $\int e^{-x^2} dx$ and $\int \sin x^2\, dx$ are nonelementary integrals. In applied mathematics some important functions are *defined* in terms of nonelementary integrals. Two such special functions are **the error function** and **complementary error function**:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{and} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \tag{10}$$

(*ii*) Occasionally, a first-order differential equation is not linear in one variab but in linear in the other variable. For example, the differential equation

$$\frac{dy}{dx} = \frac{1}{x + y^2}$$

is not linear in the variable y. But its reciprocal

$$\frac{dx}{dy} = x + y^2 \quad \text{or} \quad \frac{dx}{dy} - x = y^2$$

is recognized as linear in the variable x. You should verify that the integratin factor $e^{\int(-1)dy} = e^{-y}$ and integration by parts yield the explicit solutic $x = -y^2 - 2y - 2 + ce^y$ for the second equation. This expression is then a implicit solution of the first equation.

(*iii*) Mathematicians have adopted as their own certain words from enginee ing, which they found appropriately descriptive. The word *transient*, used ea lier, is one of these terms. In future discussions the words *input* and *output* wi occasionally pop up. The function f in (2) is called the **input** or **driving func tion**; a solution $y(x)$ of the differential equation for a given input is called th **output** or **response**.

(*iv*) The term special functions mentioned in conjunction with the error func tion also applies to the sine integral function and the Fresnel sine integra introduced in Problems 55 and 56 in Exercises 2.3. "Special Functions" i actually a well-defined branch of mathematics. More special functions ar studied in Section 6.4.

EXERCISES 2.3

Answers to selected odd-numbered problems begin on page AN

In Problems 1–24 find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

1. $\frac{dy}{dx} = 5y$
2. $\frac{dy}{dx} + 2y = 0$
3. $\frac{dy}{dx} + y = e^{3x}$
4. $3\frac{dy}{dx} + 12y = 4$
5. $y' + 3x^2y = x^2$
6. $y' + 2xy = x^3$
7. $x^2y' + xy = 1$
8. $y' = 2y + x^2 + 5$
9. $x\frac{dy}{dx} - y = x^2 \sin x$
10. $x\frac{dy}{dx} + 2y = 3$
11. $x\frac{dy}{dx} + 4y = x^3 - x$
12. $(1 + x)\frac{dy}{dx} - xy = x + x^2$
13. $x^2y' + x(x + 2)y = e^x$
14. $xy' + (1 + x)y = e^{-x} \sin 2x$
15. $y\,dx - 4(x + y^6)\,dy = 0$
16. $y\,dx = (ye^y - 2x)\,dy$
17. $\cos x\frac{dy}{dx} + (\sin x)y = 1$
18. $\cos^2x \sin x\frac{dy}{dx} + (\cos^3x)y = 1$
19. $(x + 1)\frac{dy}{dx} + (x + 2)y = 2xe^{-x}$
20. $(x + 2)^2\frac{dy}{dx} = 5 - 8y - 4xy$
21. $\frac{dr}{d\theta} + r \sec \theta = \cos \theta$
22. $\frac{dP}{dt} + 2tP = P + 4t - 2$
23. $x\frac{dy}{dx} + (3x + 1)y = e^{-3x}$
24. $(x^2 - 1)\frac{dy}{dx} + 2y = (x + 1)^2$

In Problems 25–36 solve the given initial-value prob Give the largest interval I over which the solution is def

25. $\frac{dy}{dx} = x + 5y, \quad y(0) = 3$
26. $\frac{dy}{dx} = 2x - 3y, \quad y(0) = \frac{1}{3}$
27. $xy' + y = e^x, \quad y(1) = 2$

28. $y\frac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

29. $L\frac{di}{dt} + Ri = E, \quad i(0) = i_0, \quad L, R, E, i_0$ constants

30. $\frac{dT}{dt} = k(T - T_m), \quad T(0) = T_0, \quad k, T_m, T_0$ constants

31. $x\frac{dy}{dx} + y = 4x + 1, \quad y(1) = 8$

32. $y' + 4xy = x^3e^{x^2}, \quad y(0) = -1$

33. $(x + 1)\frac{dy}{dx} + y = \ln x, \quad y(1) = 10$

34. $x(x + 1)\frac{dy}{dx} + xy = 1, \quad y(e) = 1$

35. $y' - (\sin x)y = 2\sin x, \quad y(\pi/2) = 1$

36. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

In Problems 37–40 proceed as in Example 6 to solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

37. $\frac{dy}{dx} + 2y = f(x), y(0) = 0$, where

$$f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

38. dy

53. Heart Pacemaker A heart pacemaker consists of a switch, a battery of constant voltage E_0, a capacitor with constant capacitance C, and the heart as a resistor with constant resistance R. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage E across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{RC}E.$$

Solve the DE, subject to $E(4) = E_0$.

Computer Lab Assignments

54. (a) Express the solution of the initial-value problem $y' - 2xy = -1, y(0) = \sqrt{\pi}/2$, in terms of $\text{erfc}(x)$.

(b) Use tables or a CAS to find the value of $y(2)$. Use a CAS to graph the solution curve for the IVP on $(-\infty, \infty)$.

55. (a) The **sine integral function** is defined by $\text{Si}(x) = \int_0^x (\sin t/t)\,dt$, where the integrand is defined to be 1 at $t = 0$. Express the solut the initial-value problem $x^3y' + 2x^2y$ = $y(1) = 0$ in terms of $\text{Si}(x)$.

(b) Use a CAS to graph the solution curve f for $x > 0$.

(c) Use a CAS to find the value of the abso mum of the solution $y(x)$ for $x > 0$.

56. (a) The **Fresnel sine integral** is de $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$. Express the sol of the initial-value problem $y' - (\sin$ $y(0) = 5$, in terms of $S(x)$.

(b) Use a CAS to graph the solution curve f on $(-\infty, \infty)$.

(c) It is known that $S(x) \to \frac{1}{2}$ as $x \to \infty$ and as $x \to -\infty$. What does the solution $y(x$ as $x \to \infty$? As $x \to -\infty$?

(d) Use a CAS to find the values of th maximum and the absolute minimu solution $y(x)$.

2.4 EXACT EQUATIONS

REVIEW MATERIAL

- Multivariate calculus
- Partial differentiation and partial integration
- Differential of a function of two variables

INTRODUCTION Although the simple first-order equation

$$y\,dx + x\,dy = 0$$

is separable, we can solve the equation in an alternative manner by recognizing that the expres on the left-hand side of the equality is the differential of the function $f(x, y) = xy$; that is,

$$d(xy) = y\,dx + x\,dy.$$

In this section we examine first-order equations in differential form $M(x, y)\,dx + N(x, y)\,dy = ($ applying a simple test to M and N, we can determine whether $M(x, y)\,dx + N(x, y)\,dy$ is a diffe tial of a function $f(x, y)$. If the answer is yes, we can construct f by partial integration.

Differential of a Function of Two Variables If $z = f(x, y)$ is a two variables with continuous first partial derivatives in a region R of th then its differential is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

In the special case when $f(x, y) = c$, where c is a constant, then (1) impli

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

In other words, given a one-parameter family of functions $f(x, y) = c$, we can generate a first-order differential equation by computing the differential of both sides of the equality. For example, if $x^2 - 5xy + y^3 = c$, then (2) gives the first-order DE

$$(2x - 5y)\,dx + (-5x + 3y^2)\,dy = 0. \tag{3}$$

A Definition Of course, not every first-order DE written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$ corresponds to a differential of $f(x, y) = c$. So for our purposes it is more important to turn the foregoing example around; namely, if we are given a first-order DE such as (3), is there some way we can recognize that the differential expression $(2x - 5y)\,dx + (-5x + 3y^2)\,dy$ is the differential $d(x^2 - 5xy + y^3)$? If there is, then an implicit solution of (3) is $x^2 - 5xy + y^3 = c$. We answer this question after the next definition

DEFINITION 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order di ferential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

For example, $x^2y^3\,dx + x^3y^2\,dy = 0$ is an exact equation, because its left-hand side is an exact differential:

$$d\left(\tfrac{1}{3}x^3y^3\right) = x^2y^3\,dx + x^3y^2\,dy.$$

Notice that if we make the identifications $M(x, y) = x^2y^3$ and $N(x, y) = x^3y^2$, then $\partial M/\partial y = 3x^2y^2 = \partial N/\partial x$. Theorem 2.4.1, given next, shows that the equality of the partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ is no coincidence.

THEOREM 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

PROOF OF THE NECESSITY For simplicity let us assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives for all (x, y). Now if the expression $M(x, y)\,dx + N(x, y)\,dy$ is exact, there exists some function f such that for all x in R,

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Therefore

$$M(x, y) = \frac{\partial f}{\partial x}, \qquad N(x, y) = \frac{\partial f}{\partial y},$$

and
$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

The equality of the mixed partials is a consequence of the continuity of the first partial derivatives of $M(x, y)$ and $N(x, y)$.

The sufficiency part of Theorem 2.4.1 consists of showing that there exists a function f for which $\partial f/\partial x = M(x, y)$ and $\partial f/\partial y = N(x, y)$ whenever (4) holds. The construction of the function f actually reflects a basic procedure for solving exact equations.

Method of Solution Given an equation in the differential form $M(x, y)\, dx + N(x, y)\, dy = 0$, determine whether the equality in (4) holds. If it does, then there exists a function f for which

$$\frac{\partial f}{\partial x} = M(x, y).$$

We can find f by integrating $M(x, y)$ with respect to x while holding y constant:

$$f(x, y) = \int M(x, y)\, dx + g(y), \tag{5}$$

where the arbitrary function $g(y)$ is the "constant" of integration. Now differentiate (5) with respect to y and assume that $\partial f/\partial y = N(x, y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\, dx + g'(y) = N(x, y).$$

This gives
$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\, dx. \tag{6}$$

Finally, integrate (6) with respect to y and substitute the result in (5). The implicit solution of the equation is $f(x, y) = c$.

Some observations are in order. First, it is important to realize that the expression $N(x, y) - (\partial/\partial y) \int M(x, y)\, dx$ in (6) is independent of x, because

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\, dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x, y)\, dx\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Second, we could just as well start the foregoing procedure with the assumption that $\partial f/\partial y = N(x, y)$. After integrating N with respect to y and then differentiating that result, we would find the analogues of (5) and (6) to be, respectivel ,

$$f(x, y) = \int N(x, y)\, dy + h(x) \quad \text{and} \quad h'(x) = M(x, y) - \frac{\partial}{\partial x}\int N(x, y)\, dy.$$

In either case *none of these formulas should be memorized.*

EXAMPLE 1 **Solving an Exact DE**

Solve $2xy\, dx + (x^2 - 1)\, dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so by Theorem 2.4.1 there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating

$$f(x, y) = x^2y + g(y).$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1.$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2y - y$, so the solution of the equation in implicit form is $x^2y - y = c$. The explicit form of the solution is easily seen to be $y = c/(1 - x^2)$ and is defined on any interval not containing either $x = 1$ or $x = -1$.

≡ **Note** The solution of the DE in Example 1 is *not* $f(x, y) = x^2y - y$. Rather, it is $f(x, y) = c$; if a constant is used in the integration of $g'(y)$, we can then write the solution as $f(x, y) = 0$. Note, too, that the equation could be solved by separation of variables.

EXAMPLE 2 Solving an Exact DE

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$.

SOLUTION The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Hence a function $f(x, y)$ exists for which

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}.$$

Now, for variety, we shall start with the assumption that $\partial f/\partial y = N(x, y)$; that is,

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x, y) = 2x\int e^{2y}\,dy - x\int \cos xy\,dy + 2\int y\,dy.$$

Remember, the reason x can come out in front of the symbol $\int$ is that in the integration with respect to y, x is treated as an ordinary constant. It follows that

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy,$$

and so $h'(x) = 0$ or $h(x) = c$. Hence a (*ii*) In some texts on differential equations the study of exact eq... precedes that of linear DEs. Then the method for finding integrating ... just discussed can be used to derive an integrating fac... $y' + P(x)y = f(x)$. By rewriting the last equation in the differenti... $(P(x)y - f(x))\,dx + dy = 0$, we see that

$$\frac{M_y - N_x}{N} = P(x)$$

From (13) we arrive at the already familiar integrating factor $e^{\int P(x)dx}$... Section 2.3.

EXERCISES 2.4

Answers to selected odd-numbered problems begin on ...

In Problems 1–20 determine whether the given differential equation is exact. If it is exact, solve it.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$
2. $(2x + y)\,dx - (x + 6y)\,dy = 0$
3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$
4. $(\sin y - y \sin x)\,dx + (\cos x + x \cos y - y)\,dy = 0$
5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$
6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$
7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$
8. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)\,dy$
9. $(x - y^3 + y^2 \sin x)\,dx = (3xy^2 + 2y \cos x)\,dy$
10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$
11. $(y \ln y - e^{-xy})\,dx + \left(\dfrac{1}{y} + x \ln y\right)dy = 0$
12. $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$
13. $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$
14. $\left(1 - \dfrac{3}{y} + x\right)\dfrac{dy}{dx} + y = \dfrac{3}{x} - 1$
15. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$
16. $(5y - 2x)y' - 2y = 0$
17. $(\tan x - \sin x \sin y)\,dx + \cos x \cos y\,dy = 0$
18. $(2y \sin x \cos x - y + 2y^2e^{xy^2})\,dx$
    $= (x - \sin^2 x - 4xye^{xy^2})\,dy$
19. $(4t^3y - 15t^2 - y)\,dt + (t^4 + 3y^2 - t)\,dy =$ ...
20. $\left(\dfrac{1}{t} + \dfrac{1}{t^2} - \dfrac{y}{t^2 + y^2}\right)dt + \left(ye^y + \dfrac{t}{t^2 + y^2}\right)d$...

In Problems 21–26 solve the given initial-valu...

21. $(x + y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1$...
22. $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0)$...
23. $(4y + 2t - 5)\,dt + (6y + 4t - 1)\,dy = 0,$ ...
24. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0, \quad y(1) = 1$
25. $(y^2 \cos x - 3x^2y - 2x)\,dx$
    $+ (2y \sin x - x^3 + \ln y)\,dy = 0, \quad y(0) = e$...
26. $\left(\dfrac{1}{1 + y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x$...

In Problems 27 and 28 find the value of k so th... differential equation is exact.

27. $(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,d$...
28. $(6xy^3 + \cos y)\,dx + (2kx^2y^2 - x \sin y)\,dy$...

In Problems 29 and 30 verify that the given ... equation is not exact. Multiply the given diffe... tion by the indicated integrating factor $\mu(x, y)$ a... the new equation is exact. Solve.

29. $(-xy \sin x + 2y \cos x)\,dx + 2x \cos x\,dy =$ ...
    $\mu(x, y) = xy$
30. $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,d$...
    $\mu(x, y) = (x + y)^{-2}$

In Problems 31–36 solve the given differenti... finding, as in Example 4, an appropriate inte...

31. $(2y^2 + 3x)\,dx + 2xy\,dy = 0$
32. $y(x + y + 1)\,dx + (x + 2y)\,dy = 0$

33. $6xy\,dx + (4y + 9x^2)\,dy = 0$

34. $\cos x\,dx + \left(1 + \frac{2}{y}\right)\sin x\,dy = 0$

35. $(10 - 6y + e^{-3x})\,dx - 2\,dy = 0$

36. $(y^2 + xy^3)\,dx + (5y^2 - xy + y^3 \sin y)\,dy = 0$

In Problems 37 and 38 solve the given initial-value prob
by finding as in Example 4, an appropriate integrating fa

37. $x\,dx + (x^2y + 4y)\,dy = 0, \quad y(4) = 0$

38. $(x^2 + y^2 - 5)\,dx = (y + xy)\,dy, \quad y(0) = 1$

39. (a) Show that a one-parameter family of solution the equation

$$(4xy + 3x^2)\,dx + (2y + 2x^2)\,dy = 0$$

is $x^3 + 2x^2y + y^2 = c$.

(b) Show that the initial conditions $y(0) = -2$ $y(1) = 1$ determine the same implicit solution.

(c) Find explicit solutions $y_1(x)$ and

2.5 SOLUTIONS BY SUBSTITUTIONS

REVIEW MATERIAL
- Techniques of integration
- Separation of variables
- Solution of linear DEs

INTRODUCTION We usually solve a differential equation by recognizing it as a certain kind of equation (say, separable, linear, or exact) and then carrying out a procedure, consisting of *equation-specific mathematical steps*, that yields a solution of the equation. But it is not uncommon to be stumped by a differential equation because it does not fall into one of the classes of equations that we know how to solve. The procedures that are discussed in this section may be helpful in this situation.

Substitutions Often the first step in solving a differential equation consis[ts] of transforming it into another differential equation by means of a **substitutio[n]**. For example, suppose we wish to transform the first-order differential equatio[n] $dy/dx = f(x, y)$ by the substitution $y = g(x, u)$, wh[ere u] is regarded as a function o[f] the variable x. If g possesses first-partial derivatives, then the Chain Rul[e]

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx} \quad \text{gives} \quad \frac{dy}{dx} = g_x(x, u) + g_u(x, u)\frac{du}{dx}.$$

If we replace dy/dx by the foregoing derivative and replace y in $f(x, y)$ by $g(x, u)$, the[n] the DE $dy/dx = f(x, y)$ becomes $g_x(x, u) + g_u(x, u)\dfrac{du}{dx} = f(x, g(x, u))$, which, solve[d] for du/dx, has the form $\dfrac{du}{dx} = F(x, u)$. If we can determine a solution $u = \phi(x)$ of thi[s] last equation, then a solution of the original differential equation is $y = g(x, \phi(x))$.

In the discussion that follows we examine three different kinds of first-orde[r] differential equations that are solvable by means of a substitution.

Homogeneous Equations If a function f possesses the property $f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α, then f is said to be a **homogeneous function o[f] degree** α. For example, $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3 since

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x, y),$$

whereas $f(x, y) = x^3 + y^3 + 1$ is not homogeneous. A first-order DE in differentia[l] form

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (1)$$

is said to be **homogeneous*** if both coefficient functions M and N are homogeneou[s] functions of the *same* degree. In other words, (1) is homogeneous if

$$M(tx, ty) = t^{\alpha}M(x, y) \quad \text{and} \quad N(tx, ty) = t^{\alpha}N(x, y).$$

In addition, if M and N are homogeneous functions of degree α, we can also write

$$M(x, y) = x^{\alpha}M(1, u) \quad \text{and} \quad N(x, y) = x^{\alpha}N(1, u), \quad \text{where } u = y/x, \qquad (2)$$

*Here the word *homogeneous* does not mean the same as it did in the *Remarks* at the end of Section 2.3. Recall that a linear first-order equation $a_1(x)y' + a_0(x)y = g(x)$ is homogeneous when $g(x) = 0$.

and

$$M(x, y) = y^\alpha M(v, 1) \quad \text{and} \quad N(x, y) = y^\alpha N(v, 1), \quad \text{where } v = x/y. \tag{3}$$

See Problem 31 in Exercises 2.5. Properties (2) and (3) suggest the substitutions that can be used to solve a homogeneous differential equation. Specificall , *either* of the substitutions $y = ux$ or $x = vy$, where u and v are new dependent variables, will reduce a homogeneous equation to a *separable* first-orde differential equation. To show this, observe that as a consequence of (2) a homogeneous equation $M(x, y)\,dx + N(x, y)\,dy = 0$ can be rewritten as

$$x^\alpha M(1, u)\,dx + x^\alpha N(1, u)\,dy = 0 \quad \text{or} \quad M(1, u)\,dx + N(1, u)\,dy = 0,$$

where $u = y/x$ or $y = ux$. By substituting the differential $dy = u\,dx + x\,du$ into the last equation and gathering terms, we obtain a separable DE in the variables u and x:

$$M(1, u)\,dx + N(1, u)[u\,dx + x\,du] = 0$$
$$[M(1, u) + uN(1, u)]\,dx + xN(1, u)\,du = 0$$

or

$$\frac{dx}{x} + \frac{N(1, u)\,du}{M(1, u) + uN(1, u)} = 0.$$

At this point we offer the same advice as in the preceding sections: Do not memorize anything here (especially the last formula); rather, *work through the procedure each time.* The proof that the substitutions $x = vy$ and $dx = v\,dy + y\,dv$ also lead to a separable equation follows in an analogous manner from (3).

EXAMPLE 1 **Solving a Homogeneous DE**

Solve $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$.

SOLUTION Inspection of $M(x, y) = x^2 + y^2$ and $N(x, y) = x^2 - xy$ shows that these coefficients are homogeneous functions of degree 2. If we let $y = ux$, then $dy = u\,dx + x\,du$, so after substituting, the given equation becomes

$$(x^2 + u^2x^2)\,dx + (x^2 - ux^2)[u\,dx + x\,du] = 0$$
$$x^2(1 + u)\,dx + x^3(1 - u)\,du = 0$$
$$\frac{1 - u}{1 + u}\,du + \frac{dx}{x} = 0$$
$$\left[-1 + \frac{2}{1 + u}\right]du + \frac{dx}{x} = 0. \quad \leftarrow \text{long division}$$

After integration the last line gives

$$-u + 2\ln|1 + u| + \ln|x| = \ln|c|$$
$$-\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| = \ln|c|. \quad \leftarrow \text{resubstituting } u = y/x$$

Using the properties of logarithms, we can write the preceding solution as

$$\ln\left|\frac{(x + y)^2}{cx}\right| = \frac{y}{x} \quad \text{or} \quad (x + y)^2 = cxe^{y/x}.$$

Although either of the indicated substitutions can be used for every homogeneous differential equation, in practice we try $x = vy$ whenever the function $M(x, y)$ is simpler than $N(x, y)$. Also it could happen that after using one substitution, we may encounter integrals that are difficult or impossible to evaluate in closed form; switching substitutions may result in an easier problem.

Bernoulli's Equation The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \tag{4}$$

where n is any real number, is called **Bernoulli's equation.** Note that for $n = 0$ and $n = 1$, equation (4) is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of form (4) to a linear equation.

EXAMPLE 2 **Solving a Bernoulli DE**

Solve $x\dfrac{dy}{dx} + y = x^2y^2$.

SOLUTION We first rewrite the equation a

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

by dividing by x. With $n = 2$ we have $u = y^{-1}$ or $y = u^{-1}$. We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx} \qquad \leftarrow \text{Chain Rule}$$

into the given equation and simplify. The result is

$$\frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor for this linear equation on, say, $(0, \infty)$ is

$$e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$

Integrating $$\frac{d}{dx}[x^{-1}u] = -1$$

gives $x^{-1}u = -x + c$ or $u = -x^2 + cx$. Since $u = y^{-1}$, we have $y = 1/u$, so a solution of the given equation is $y = 1/(-x^2 + cx)$.

Note that we have not obtained the general solution of the original nonlinear differential equation in Example 2, since $y = 0$ is a singular solution of the equation.

Reduction to Separation of Variables A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C) \tag{5}$$

can always be reduced to an equation with separable variables by means of the substitution $u = Ax + By + C$, $B \neq 0$. Example 3 illustrates the technique.

EXAMPLE 3 **An Initial-Value Problem**

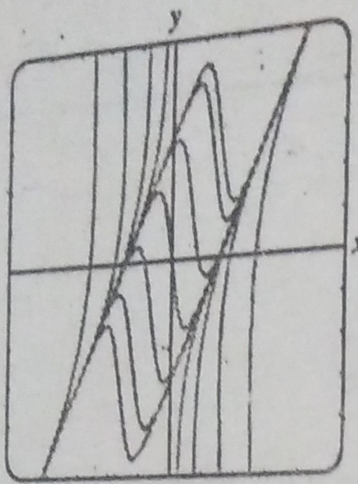
Solve $\dfrac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$

SOLUTION If we let $u = -2x + y$, then $du/dx = -2 + dy/dx$, so the differential equation is transformed into

$$\frac{du}{dx} + 2 = u^2 - 7 \qquad \text{or} \qquad \frac{du}{dx} = u^2 - 9.$$

and then integrating yields

$$\frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| = x + c_1 \quad \text{or} \quad \frac{u-3}{u+3} = e^{6x+6c_1} = ce^{6x}.$$



GURE 2.5.1 Solutions of DE in ample 3

Solving the last equation for u and then resubstituting gives the solution

$$u = \frac{3(1+ce^{6x})}{1-ce^{6x}} \quad \text{or} \quad y = 2x + \frac{3(1+ce^{6x})}{1-ce^{6x}}.$$

Finally, applying the initial condition $y(0) = 0$ to the last equation in (6) $c = -1$. Figure 2.5.1, obtained with the aid of a graphing utility, shows the gra the particular solution $y = 2x + \frac{3(1-e^{6x})}{1+e^{6x}}$ in dark blue, along with the grapl some other members of the family of solutions (6).

EXERCISES 2.5

Answers to selected odd-numbered problems begin on page ANS

ch DE in Problems 1–14 is homogeneous.

Problems 1–10 solve the given differential equation by ng an appropriate substitution.

$(x - y)\,dx + x\,dy = 0$

2. $(x + y)\,dx + x\,dy = 0$

$x\,dx + (y - 2x)\,dy = 0$

4. $y\,dx = 2(x + y)\,dy$

$(y^2 + yx)\,dx - x^2\,dy = 0$

$(y^2 + yx)\,dx + x^2\,dy = 0$

$\frac{dy}{dx} = \frac{y - x}{y + x}$

$\frac{dy}{dx} = \frac{x + 3y}{3x + y}$

$-y\,dx + (x + \sqrt{xy})\,dy = 0$

$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0$

roblems 11–14 solve the given initial-value problem.

$y^2\frac{dy}{dx} = y^3 - x^3, \quad y(1) = 2$

$(x^2 + 2y^2)\frac{dx}{dy} = xy, \quad y(-1) = 1$

$+ ye^{y/x})\,dx - xe^{y/x}\,dy = 0, \quad y(1) = 0$

$dx + x(\ln x - \ln y - 1)\,dy = 0, \quad y(1) = e$

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20 solve the given differential equation b using an appropriate substitution.

15. $x\frac{dy}{dx} + y = \frac{1}{y^2}$

16. $\frac{dy}{dx} - y = e^x y^2$

17. $\frac{dy}{dx} = y(xy^3 - 1)$

18. $x\frac{dy}{dx} - (1 + x)y = xy^2$

19. $t^2\frac{dy}{dt} + y^2 = ty$

20. $3(1 + t^2)\frac{dy}{dt} = 2ty(y^3 - 1)$

In Problems 21 and 22 solve the given initial-value problem.

21. $x^2\frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

22. $y^{1/2}\frac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

Each DE in Problems 23–30 is of the form given in (5).

In Problems 23–28 solve the given differential equation by using an appropriate substitution.

23. $\frac{dy}{dx} = (x + y + 1)^2$

24. $\frac{dy}{dx} = \frac{1 - x - y}{x + y}$

25. $\frac{dy}{dx} = \tan^2(x + y)$

26. $\frac{dy}{dx} = \sin(x + y)$

27. $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

28. $\frac{dy}{dx} = 1 + e^{y-x+5}$