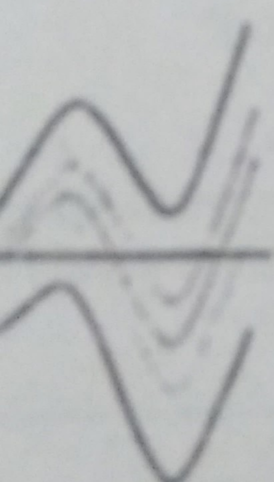
# Introduction to Differential Equations



The words *differential* and *equations* certainly suggest solving some kin equation that contains derivatives $y', y'', \ldots$. Analogous to a course in a trigonometry, in which a good amount of time is spent solving equation $x^2 + 5x + 4 = 0$ for the unknown number $x$, in this course *one* of our ta to solve differential equations such as $y'' + 2y' + y = 0$ for an unknown $y = \phi(x)$.



The preceding paragraph tells something, but not the complete story the course you are about to begin. As the course unfolds, you will see th more to the study of differential equations than just mastering methods t mathematicians over past centuries devised to solve them.

But first things first. In order to read, stu , and be conversant in a s subject, you have to master some of the terminology of that discipline. T thrust of the first two sections of this chapte . In the last section we brief the link between differential equations and the real world. Practical ques such as

*How fast does a disease spread? How fast does a population chang* involve rates of change or derivatives. And so the mathematical descrip *mathematical model*—of phenomena, experiments, observations, or the be a differential equation.

# EFINITIONS AND TERMINOLOGY

## VIEW MATERIAL

The definition of the derivativ
Rules of differentiation
Derivative as a rate of change
Connection between the first derivative and increasing/decreasin
Connection between the second derivative and concavity

TRODUCTION The derivative $dy/dx$ of a function $y = \phi(x)$ is itself another function $\phi'(x)$ nd by an appropriate rule. The exponential function $y = e^{0.1x^2}$ is differentiable on the interval $\infty, \infty)$, and, by the Chain Rule, its first derivative is $dy/dx = 0.2xe^{0.1x^2}$. If we replace $e^{0.1x^2}$ on the ht-hand side of the last equation by the symbol $y$, the derivative becomes

$$\frac{dy}{dx} = 0.2xy. \tag{1}$$

w imagine that a friend of yours simply hands you equation (1)—you have no idea how it was nstructed—and asks, *What is the function represented by the symbol y?* You are now face to face h one of the basic problems in this course:

*How do you solve such an equation for the function $y = \phi(x)$?*

**A Definition** The equation that we made up in (1) is called a **differential equation**. Before proceeding any further, let us consider a more precise definition of this concept.

**DEFINITION 1.1.1 Differential Equation**

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation (DE)**.

To talk about them, we shall classify differential equations according to **type, order,** and **linearity.**

**Classification by Type** If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a *single* independent variable, it is said to be an **ordinary differential equation (ODE)**. An equation involving partial derivatives of one or more unknown functions of two or more independent variables is called a **partial differential equation (PDE)**. Our first example illustrates several of each type of differential equation.

**EXAMPLE 1** **Types of Differential Equations**

(a) The equations

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y \tag{2}$$

are examples of ordinary differential equations.

**(b)** The following equations are partial differential equations:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Notice in the third equation that there are two unknown functions and two indepe dent variables in the PDE. This means $u$ and $v$ must be functions of *two or mo* independent variables.

**Notation** Throughout this text ordinary derivatives will be written by usi either the **Leibniz notation** $dy/dx, d^2y/dx^2, d^3y/dx^3, \ldots$ or the **prime notation** $y', y$ $y''', \ldots$. By using the latter notation, the first two differential equations in (2) can written a little more compactly as $y' + 5y = e^x$ and $y'' - y' + 6y = 0$. Actually, th prime notation is used to denote only the first three derivatives; the fourth derivati is written $y^{(4)}$ instead of $y''''$. In general, the $n$th derivative of $y$ is written $d^ny/dx^n$ $y^{(n)}$. Although less convenient to write and to typeset, the Leibniz notation has an ad vantage over the prime notation in that it clearly displays both the dependent an independent variables. For example, in the equation

unknown function
or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

it is immediately seen that the symbol $x$ now represents a dependent variable, whereas the independent variable is $t$. You should also be aware that in physical sciences and engineering, Newton's **dot notation** (derogatorily referred to by some as the "flyspeck" notation) is sometimes used to denote derivatives with respect to time $t$. Thus the differential equation $d^2s/dt^2 = -32$ becomes $\ddot{s} = -32$. Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, with the subscript notation the second equation in (3) becomes $u_{xx} = u_{tt} - 2u_t$.

**Classification by Order** The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation. For example,

second order　　first order

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

is a second-order ordinary differential equation. In Example 1, the first and third equations in (2) are first-order ODEs, whereas in (3) the first two equations are second-order PDEs. First-order ordinary differential equations are occasionally written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$. For example, if we assume that $y$ denotes the dependent variable in $(y - x)\,dx + 4x\,dy = 0$, then $y' = dy/dx$, so by dividing by the differential $dx$, we get the alternative form $4xy' + y = x$.

In symbols we can express an $n$th-order ordinary differential equation in one dependent variable by the general form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{4}$$

where $F$ is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the

*Except for this introductory section, only ordinary differential equations are considered in *A First Course in Differential Equations with Modeling Applications*, Tenth Edition. In that text the word *equation* and the abbreviation DE refer only to ODEs. Partial differential equations or PDEs are considered in the expanded volume *Differential Equations with Boundary-Value Problems*, Eighth Edition.

where $f$ is a real-valued conti... Thus when it suits our purposes, we shall use ...

$$\frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations. For examp... the normal form of the first-order equation $4xy' + y = x$ is $y' = (x - y)/4x$; the norm... form of the second-order equation $y'' - y' + 6y = 0$ is $y'' = y' - 6y$. See (*iv*) in t... *Remarks*.

**Classification by Linearity** An $n$th-order ordinary differential equation ... is said to be linear if $F$ is linear in $y, y', \ldots, y^{(n)}$. This means that an $n$th-order ODE linear when (4) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \quad (6)$$

Two important special cases of (6) are linear first-order ($n = 1$) and linear secon... order ($n = 2$) DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \text{and} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \quad (...$$

In the additive combination on the left-hand side of equation (6) we see that the cha... acteristic two properties of a linear ODE are as follows:

- The dependent variable $y$ and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, that is, the power of each term involving $y$ is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable $x$.

A nonlinear ordinary differential equation is simply one that is not linear. Nonline... functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cann... appear in a linear equation.

**EXAMPLE 2** **Linear and Nonlinear ODEs**

**(a)** The equations

$$(y - x)dx + 4xy\,dy = 0, \quad y'' - 2y' + y = 0, \quad x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, *linear* first-, second-, and third-order ordinary differential equations. W... have just demonstrated that the first equation is linear in the variable $y$ by writing it... the alternative form $4xy' + y = x$.

**(b)** The equations

nonlinear term: coefficient depends on $y$ ↓    nonlinear term: nonlinear function of $y$ ↓    nonlinear term: power not 1 ↓

$$(1 - y)y' + 2y = e^x, \quad \frac{d^2y}{dx^2} + \sin y = 0, \quad \text{and} \quad \frac{d^4y}{dx^4} + y^2 = 0$$

are examples of *nonlinear* first-, second-, and fourth... tions, respectively.

**Solutions** As was stated before, one of the goals in this course is to solve find solutions of, differential equations. In the next definition we consider the c cept of a solution of an ordinary differential equation.

**DEFINITION 1.1.2 Solution of an ODE**

Any function $\phi$, defined on an interval $I$ and possessing at least $n$ derivatives that are continuous on $I$, which when substituted into an $n$th-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an $n$th-order ordinary differential equation (4) is a func tion $\phi$ that possesses at least $n$ derivatives and for which

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I.$$

We say that $\phi$ *satisfie* the differential equation on $I$. For our purposes we shall als assume that a solution $\phi$ is a real-valued function. In our introductory discussion w saw that $y = e^{0.1x^2}$ is a solution of $dy/dx = 0.2xy$ on the interval $(-\infty, \infty)$.

Occasionally, it will be convenient to denote a solution by the alternativ symbol $y(x)$.

**Interval of Definition** You cannot think *solution* of an ordinary differentia equation without simultaneously thinking *interval*. The interval $I$ in Definition 1.1.2 is variously called the **interval of definition** the **interval of existence**, the **interval of validity**, or the **domain of the solution** and can be an open interval $(a, b)$, a closed interval $[a, b]$, an infinite interval $a, \infty)$, and so on.

**EXAMPLE 3** Verification of a Solutio

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(a) $dy/dx = xy^{1/2}$; $y = \frac{1}{16}x^4$ (b) $y'' - 2y' + y = 0$; $y = xe^x$

**SOLUTION** One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every $x$ in the interval.

(a) From

*left-hand side:* $$\frac{dy}{dx} = \frac{1}{16}(4 \cdot x^3) = \frac{1}{4}x^3,$$

*right-hand side:* $$xy^{1/2} = x \cdot \left(\frac{1}{16}x^4\right)^{1/2} = x \cdot \left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3,$$

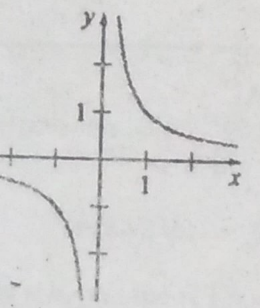
we see that each side of the equation is the same for every real number $x$. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have, for every real number $x$,

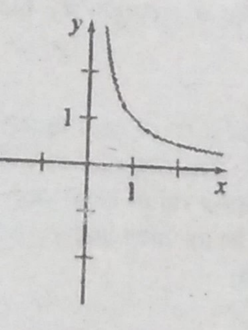
*left-hand side:* $$y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0,$$

Note, too, that in Example 3 each differential equation possesses the constant solution $y = 0$, $-\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval $I$ is said to be a **trivial solution.**

**Solution Curve** The graph of a solution $\phi$ of an ODE is called a **solution curve.** Since $\phi$ is a differentiable function, it is continuous on its interval $I$ of definition. Thus there may be a difference between the graph of the *function* $\phi$ and the graph of the *solution* $\phi$. Put another way, the domain of the function $\phi$ need not be the same as the interval $I$ of definition (or domain) of the solution $\phi$. Example 4 illustrates the difference.



...nction $y = 1/x$, $x \neq 0$



...ution $y = 1/x$, $(0, \infty)$

...1 In Example 4 the ...$1/x$ is not the same as the ...$1/x$

## EXAMPLE 4 Function versus Solution

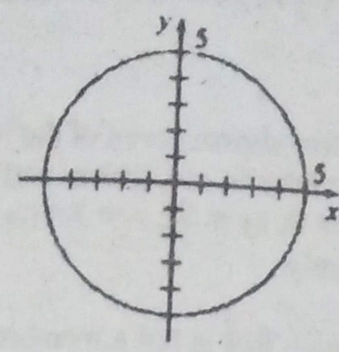
The domain of $y = 1/x$, considered simply as a *function*, is the set of all real numbers $x$ except 0. When we graph $y = 1/x$, we plot points in the $xy$-plane corresponding to a judicious sampling of numbers taken from its domain. The rational function $y = 1/x$ is discontinuous at 0, and its graph, in a neighborhood of the origin, is given in Figure 1.1.1(a). The function $y = 1/x$ is not differentiable at $x = 0$, since the $y$-axis (whose equation is $x = 0$) is a vertical asymptote of the graph.

Now $y = 1/x$ is also a solution of the linear first-order differential equation $xy' + y = 0$. (Verify.) But when we say that $y = 1/x$ is a *solution* of this DE, we mean that it is a function defined on an interval $I$ on which it is differentiable and satisfies the equation. In other words, $y = 1/x$ is a solution of the DE on *any* interval that does not contain 0, such as $(-3, -1)$, $(\frac{1}{2}, 10)$, $(-\infty, 0)$, or $(0, \infty)$. Because the solution curves defined by $y = 1/x$ for $-3 < x < -1$ and $\frac{1}{2} < x < 10$ are simply segments, or pieces, of the solution curves defined by $y = 1/x$ for $-\infty < x < 0$ and $0 < x < \infty$, respectively, it makes sense to take the interval $I$ to be as large as possible. Thus we take $I$ to be either $(-\infty, 0)$ or $(0, \infty)$. The solution curve on $(0, \infty)$ is shown in Figure 1.1.1(b).

**Explicit and Implicit Solutions** You should be familiar with the terms *explicit functions* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution.** For our purposes, let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate using the standard rules. We have just seen in the last two examples that $y = \frac{1}{16}x^4$, $y = xe^x$, and $y = 1/x$ are, in turn, explicit solutions of $dy/dx = xy^{1/2}$, $y'' - 2y' + y = 0$, and $xy' + y = 0$. Moreover, the trivial solution $y = 0$ is an explicit solution of all three equations. When we get down to the business of actually solving some ordinary differential equations, you will see that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when we attempt to solve nonlinear first-order differential equations. Often we have to be content with a relation or expression $G(x, y) = 0$ that defines a solution $\phi$ implicitly.

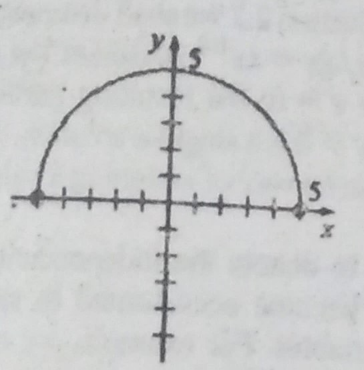
**DEFINITION 1.1.3 Implicit Solution of an ODE**

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation (4) on an interval $I$, provided that there exists at least one function $\phi$ that satisfies the relation as well as the differential equation on $I$.



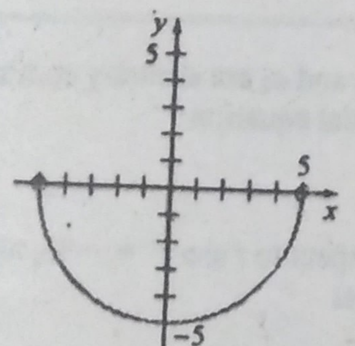

(a) implicit solution

$x^2 + y^2 = 25$




(b) explicit solution

$y_1 = \sqrt{25 - x^2}, -5 < x < 5$




(c) explicit solution

$y_2 = -\sqrt{25 - x^2}, -5 < x < 5$

**FIGURE 1.1.2** An implicit solution and two explicit solutions of (8) in Example 5

It is beyond the scope of this course to investigate the conditions relation $G(x, y) = 0$ defines a differentiable function $\phi$. So we shall the formal implementation of a method of solution leads to a relatic then there exists at least one function $\phi$ that satisfies both the re $G(x, \phi(x)) = 0$) and the differential equation on an interval $I$. If the in $G(x, y) = 0$ is fairly simple, we may be able to solve for $y$ in terms c one or more explicit solutions. See (*i*) in the *Remarks*.

**EXAMPLE 5** **Verification of an Implicit Solutio**

The relation $x^2 + y^2 = 25$ is an implicit solution of the different... equ

$$\frac{dy}{dx} = -\frac{x}{y}$$

on the open interval $(-5, 5)$. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \qquad \text{or} \qquad 2x + 2y\frac{dy}{dx} = 0.$$

Solving the last equation for the symbol $dy/dx$ gives (8). Moreo $x^2 + y^2 = 25$ for $y$ in terms of $x$ yields $y = \pm\sqrt{25 - x^2}$. The tw $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfy the relat $x^2 + \phi_1^2 = 25$ and $x^2 + \phi_2^2 = 25$) and are explicit solutions defined on $(-5, 5)$. The solution curves given in Figures 1.1.2(b) and 1.1.2(c) are the graph of the implicit solution in Figure 1.1.2(a).

Any relation of the form $x^2 + y^2 - c = 0$ *formally* satisfies (8) for any However, it is understood that the relation should always make sense in the system; thus, for example, if $c = -25$, we cannot say that $x^2 + y^2 + 2$ implicit solution of the equation. (Why not?)

Because the distinction between an explicit solution and an implic should be intuitively clear, we will not belabor the issue by always sayin an explicit (implicit) solution."

**Families of Solutions** The study of differential equations is simila integral calculus. In some texts a solution $\phi$ is sometimes referred to as of the equation, and its graph is called an **integral curve.** When evaluati derivative or indefinite integral in calculus, we use a single constant $c$ of i Analogously, when solving a first-order differential equation $F(x, y, y$ *usually* obtain a solution containing a single arbitrary constant or para solution containing an arbitrary constant represents a set $G(x, y, c) = 0$ o called a **one-parameter family of solutions.** When solving an $n$th-order equation $F(x, y, y', \ldots, y^{(n)}) = 0$, we seek an **$n$-parameter family of** $G(x, y, c_1, c_2, \ldots, c_n) = 0$. This means that *a single differential equation c an infinite number of solutions* corresponding to the unlimited number for the parameter(s). A solution of a differential equation that is free o parameters is called a **particular solution.**

**EXAMPLE 6** **Particular Solutions**

**(a)** The one-parameter family $y = cx - x\cos x$ is an explicit solution of first-order equation

$$xy' - y = x^2 \sin x$$

on the interval $(-\infty, \infty)$. (Verify.) Figure 1.1.3 shows the graphs of some solutions in this family for various choices of $c$. The solution $y = -x\cos x$ particular solution corresponding to $c = 0$, graph in the figure, is a particular solution corresponding to $c = 0$.

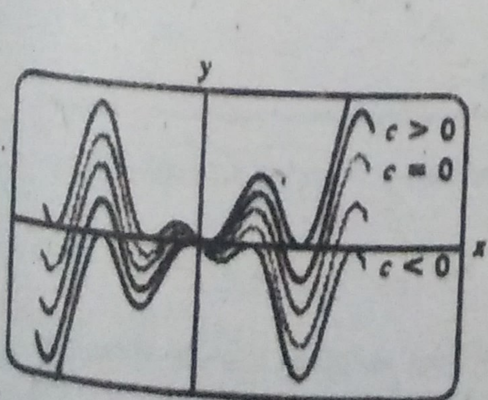



**FIGURE 1.1.3** Some solutions of DE in part (a) of Example 6

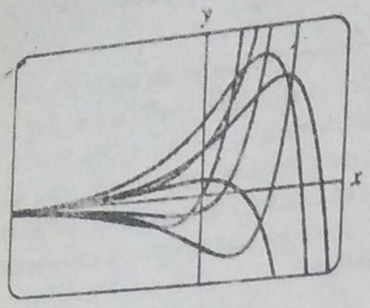



FIGURE 1.1.4 Some solutions of DE in part (b) of Example 6

**(b)** The two-parameter family $y = c_1e^x + c_2xe^x$ is an explicit solution of the second-order equation

$$y'' - 2y' + y = 0$$

in part (b) of Example 3. (Verify.) In Figure 1.1.4 we have shown seven of the "ble infinity" of solutions in the family. The solution curves in red, green, and are the graphs of the particular solutions $y = 5xe^x$ ($c_1 = 0$, $c_2 = 5$), $y = 3e^x$ ($c_1$ $c_2 = 0$), and $y = 5e^x - 2xe^x$ ($c_1 = 5$, $c_2 = 2$), respectively.

Sometimes a differential equation possesses a solution that is not a member family of solutions of the equation—that is, a solution that cannot be obtained by cializing *any* of the parameters in the family of solutions. Such an extra solution is ca **a singular solution.** For example, we have seen that $y = \frac{1}{16}x^4$ and $y = 0$ are solution the differential equation $dy/dx = xy^{1/2}$ on $(-\infty, \infty)$. In Section 2.2 we shall demonstr by actually solving it, that the differential equation $dy/dx = xy^{1/2}$ possesses the o parameter family of solutions $y = \left(\frac{1}{4}x^2 + c\right)^2$. When $c = 0$, the resulting particu solution is $y = \frac{1}{16}x^4$. But notice that the trivial solution $y = 0$ is a singular solution, sin it is not a member of the family $y = \left(\frac{1}{4}x^2 + c\right)^2$; there is no way of assigning a value the constant $c$ to obtain $y = 0$.

In all the preceding examples we used $x$ and $y$ to denote the independent ar dependent variables, respectively. But you should become accustomed to seein and working with other symbols to denote these variables. For example, we coul denote the independent variable by $t$ and the dependent variable by $x$.

## EXAMPLE 7 Using Different Symbols

The functions $x = c_1 \cos 4t$ and $x = c_2 \sin 4t$, where $c_1$ and $c_2$ are arbitrary constants or parameters, are both solutions of the linear differential equation

$$x'' + 16x = 0.$$

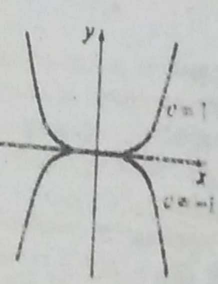
For $x = c_1 \cos 4t$ the first two derivatives with respect to $t$ are $x' = -4c_1 \sin 4t$ and $x'' = -16c_1 \cos 4t$. Substituting $x''$ and $x$ then gives

$$x'' + 16x = -16c_1 \cos 4t + 16(c_1 \cos 4t) = 0.$$

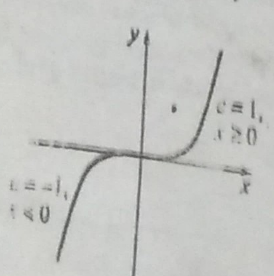
In like manner, for $x = c_2 \sin 4t$ we have $x'' = -16c_2 \sin 4t$, and so

$$x'' + 16x = -16c_2 \sin 4t + 16(c_2 \sin 4t) = 0.$$

Finally, it is straightforward to verify that the linear combination of solutions, or the two-parameter family $x = c_1 \cos 4t + c_2 \sin 4t$, is also a solution of the differential equation.




(a) two explicit solutions




(b) piecewise-defined solution

FIGURE 1.1.5 Some solutions of DE

The next example shows that a solution of a differential equation can be a piecewise-defined function

## Piecewise-Defined Solutio

The one-parameter family of quartic monomial functions $y = cx^4$ is an explicit solution of the linear first-order equatio

$$xy' - 4y = 0$$

on the interval $(-\infty, \infty)$. (Verify.) The blue and red solution curves shown in Figure 1.1.5(a) are the graphs of $y = x^4$ and $y = -x^4$ and correspond to the choices

The piecewise-defined di ferentiable function

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x > 0 \end{cases}$$

is also a solution of the differential equation but cannot be obtained from the family $y = cx^4$ by a single choice of $c$. As seen in Figure 1.1.5(b) the solution is constructed from the family by choosing $c = -1$ for $x < 0$ and $c = 1$ for $x \geq 0$.

**Systems of Differential Equations** Up to this point we have been discussing single differential equations containing one unknown function. But often in theory, as well as in many applications, we must deal with systems of differential equations. A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. For example, if $x$ and $y$ denote dependent variables and $t$ denotes the independent variable, then a system of two first-orde differential equations is given by

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y). \tag{9}$$

A **solution** of a system such as (9) is a pair of differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$, defined on a common interval $I$, that satisfy each equation of the system on this interval.

## REMARKS

(*i*) A few last words about implicit solutions of differential equations are in order. In Example 5 we were able to solve the relation $x^2 + y^2 = 25$ for $y$ in terms of $x$ to get two explicit solutions, $\phi_1(x) = \sqrt{25 - x^2}$ and $\phi_2(x) = -\sqrt{25 - x^2}$, of the differential equation (8). But don't read too much into this one example. Unless it is easy or important or you are instructed to, there is usually no need to try to solve an implicit solution $G(x, y) = 0$ for $y$ explicitly in terms of $x$. Also do not misinterpret the second sentence following Definition 1.1.3. An implicit solution $G(x, y) = 0$ can define a perfectly good differentiable function $\phi$ that is a solution of a DE, yet we might not be able to solve $G(x, y) = 0$ using analytical methods such as algebra. The solution curve of $\phi$ may be a segment or piece of the graph of $G(x, y) = 0$. See Problems 45 and 46 in Exercises 1.1. Also, read the discussion following Example 4 in Section 2.2.

(*ii*) Although the concept of a solution has been emphasized in this section, you should also be aware that a DE does not necessarily have to possess a solution. See Problem 39 in Exercises 1.1. The question of whether a solution exists will be touched on in the next section.

(*iii*) It might not be apparent whether a first-order ODE written in differential form $M(x, y)dx + N(x, y)dy = 0$ is linear or nonlinear because there is nothing in this form that tells us which symbol denotes the dependent variable. See Problems 9 and 10 in Exercises 1.1.

(*iv*) It might not seem like a big deal to assume that $F(x, y, y', \ldots, y^{(n)}) = 0$ can be solved for $y^{(n)}$, but one should be a little bit careful here. There are exceptions, and there certainly are some problems connected with this assumption. See Problems 52 and 53 in Exercises 1.1.

(*v*) You may run across the term *closed form solutions* in DE texts or in lectures in courses in differential equations. Translated, this phrase usually familiar) func... tial and logarithmic functions, ... functions.

(*vi*) If *every* solution of an $n$th-order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ on an in... val $I$ can be obtained from an $n$-parameter family $G(x, y, c_1, c_2, \ldots, c_n) = 0$ ... appropriate choices of the parameters $c_i$, $i = 1, 2, \ldots, n$, we then say that ... family is the **general solution** of the DE. In solving linear ODEs, we shall ... pose relatively simple restrictions on the coefficients of the equation; with th... restrictions one can be assured that not only does a solution exist on an inter... but also that a family of solutions yields all possible solutions. Nonlinear OD... with the exception of some first-order equations, are usually difficult or imp... sible to solve in terms of elementary functions. Furthermore, if we happen ... obtain a family of solutions for a nonlinear equation, it is not obvious wheth... this family contains all solutions. On a practical level, then, the designati... "general solution" is applied only to linear ODEs. Don't be concerned abo... this concept at this point, but store the words "general solution" in the back ... your mind—we will come back to this notion in Section 2.3 and again i... Chapter 4.

## EXERCISES 1.1

*Answers to selected odd-numbered problems begin on page AN...*

In Problems 1–8 state the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear by matching it with (6).

1. $(1 - x)y'' - 4xy' + 5y = \cos x$
2. $x\dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^4 + y = 0$
3. $t^5y^{(4)} - t^3y'' + 6y = 0$
4. $\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + u = \cos(r + u)$
5. $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$
6. $\dfrac{d^2R}{dt^2} = -\dfrac{k}{R^2}$
7. $(\sin \theta)y''' - (\cos \theta)y' = 2$
8. $\ddot{x} - \left(1 - \dfrac{\dot{x}^2}{3}\right)\dot{x} + x = 0$

In Problems 9 and 10 determine whether the given first-order differential equation is linear in the indicated dependent variable by matching it with the first differential equation given in (7).

9. $(y^2 - 1)\,dx + x\,dy = 0$; in $y$; in $x$
10. $u\,dv + (v + uv - ue^u)\,du = 0$; in $v$; in $u$

In Problems 11–14 verify that the indicated function is an explicit solution of the given differential equation. Assume an appropriate interval $I$ of definition for each solution

11. $2y' + y = 0$; $y = e^{-x/2}$
12. $\dfrac{dy}{dt} + 20y = 24$; $y = \dfrac{6}{5} - \dfrac{6}{5}e^{-20t}$
13. $y'' - 6y' + 13y = 0$; $y = e^{3x}\cos 2x$
14. $y'' + y = \tan x$; $y = -(\cos x)\ln(\sec x + \tan x)$

In Problems 15–18 verify that the indicated function $y = \phi(x)$ is an explicit solution of the given first-order differential equation. Proceed as in Example 2, by considering $\phi$ simply as a *function*, give its domain. Then by considering $\phi$ as a *solution* of the differential equation, give at least one interval $I$ of definition

15. $(y - x)y' = y - x + 8$; $y = x + 4\sqrt{x + 2}$
16. $y' = 25 + y^2$; $y = 5\tan 5x$
17. $y' = 2xy^2$; $y = 1/(4 - x^2)$
18. $2y' = y^3\cos x$; $y = (1 - \sin x)^{-1/2}$

In Problems 19 and 20 verify that the indicated expression is an implicit solution of the given first-order differential equation. Find at least one explicit solution $y = \phi(x)$ in each case.

Use a graphing utility to obtain the graph of an explicit solution. Give an interval $I$ of definition of each solution $\phi$.

19. $\frac{dX}{dt} = (X-1)(1-2X)$; $\ln\left(\frac{2X-1}{X-1}\right) = t$

20. $2xy\,dx + (x^2 - y)\,dy = 0$; $-2x^2y + y^2 = 1$

In Problems 21–24 verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval $I$ of definition for each solution

21. $\frac{dP}{dt} = P(1-P)$; $P = \frac{c_1e^t}{1+c_1e^t}$

22. $\frac{dy}{dx} + 2xy = 1$; $y = e^{-x^2}\int_0^x e^{t^2}\,dt + c_1e^{-x^2}$

23. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$; $y = c_1e^{2x} + c_2xe^{2x}$

24. $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 12x^2$;

$y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$

25. Verify that the piecewise-defined functio

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$$

is a solution of the differential equation $xy' - 2y = 0$ on $(-\infty, \infty)$.

26. In Example 5 we saw that $y = \phi_1(x) = \sqrt{25-x^2}$ and $y = \phi_2(x) = -\sqrt{25-x^2}$ are solutions of $dy/dx = -x/y$ on the interval $(-5, 5)$. Explain why the piecewise-defined functio

$$y = \begin{cases} \sqrt{25-x^2}, & -5 < x < 0 \\ -\sqrt{25-x^2}, & 0 \le x < 5 \end{cases}$$

is *not* a solution of the differential equation on the interval $(-5, 5)$.

In Problems 27–30 find values of $m$ so that the function $y = e^{mx}$ is a solution of the given differential equation.

27. $y' + 2y = 0$

28. $5y' = 2y$

29. $y'' - 5y' + 6y = 0$

30. $2y'' + 7y' - 4y = 0$

In Problems 31 and 32 find values of $m$ so that the function $y = x^m$ is a solution of the given differential equation.

31. $xy'' + 2y' = 0$

32. $x^2y'' - 7$

In Problems 33–36 use the concept that $y$ is a constant function if and only if $y'$ whether the given differential equation p solutions.

33. $3xy' + 5y = 10$

34. $y' = y^2 + 2y - 3$

35. $(y-1)y' = 1$

36. $y'' + 4y' + 6y = 10$

In Problems 37 and 38 verify that the i functions is a solution of the given syste equations on the interval $(-\infty, \infty)$.

37. $\frac{dx}{dt} = x + 3y$

$\frac{dy}{dt} = 5x + 3y$;

$x = e^{-2t} + 3e^{6t}$,

$y = -e^{-2t} + 5e^{6t}$

38. $\frac{d^2x}{dt^2} = 4y +$

$\frac{d^2y}{dt^2} = 4x -$

$x = \cos 2t +$

$y = -\cos 2t$

## Discussion Problems

39. Make up a differential equation that do any real solutions.

40. Make up a differential equation that you possesses only the trivial solution $y = ($ reasoning.

41. What function do you know from calcu its first derivative is itself? Its first c constant multiple $k$ of itself? Write e the form of a first-order differential eq solution.

42. What function (or functions) do you kno lus is such that its second derivative is its derivative is the negative of itself? Write the form of a second-order differential e solution.

43. Given that $y = \sin x$ is an explicit soluti order differential equation $\frac{dy}{dx} = \sqrt{1-}$ terval $I$ of definition [*Hint*: $I$ is *not* the inte

44. Discuss why it makes intuitive sense to the linear differential equation $y'' + 2y'$ has a solution of the form $y = A\sin t +$ $A$ and $B$ are constants. Then find specif and $B$ so that $y = A\sin t + B\cos t$ is a p tion of the DE.

(d) On the same coordinate axes, sketch the graphs of the two constant solutions found in part (a). These constant solutions partition the $xy$-plane into three regions. In each region, sketch the graph of a nonconstant solution $y = \phi(x)$ whose shape is suggested by the results in parts (b) and (c).

3. Consider the differential equation $y' = y^2 + 4$.

(a) Explain why there exist no constant solutions of the DE.

(b) Describe the graph of a solution $y = \phi(x)$. For example, can a solution curve have any relative extrema?

(c) Explain why $y = 0$ is the $y$-coordinate of a point of inflection of a solution curve

(d) Sketch the graph of a solution $y = \phi(x)$ of the differential equation whose shape is suggested by parts (a)–(c).

## Computer Lab Assignments

In Problems 59 and 60 use a CAS to compute all derivatives and to carry out the simplifications needed to verify that the indicated function is a particular solution of the given differential equation.

59. $y^{(4)} - 20y''' + 158y'' - 580y' + 841y = 0;$
$y = xe^{5x}\cos 2x$

60. $x^3y''' + 2x^2y'' + 20xy' - 78y = 0;$
$y = 20\dfrac{\cos(5\ln x)}{x} - 3\dfrac{\sin(5\ln x)}{x}$

# 1.2 INITIAL-VALUE PROBLEMS

**REVIEW MATERIAL**

- Normal form of a DE
- Solution of a DE
- Family of solutions

**INTRODUCTION** We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ also satisfies certain prescribed side conditions—that is, conditions that are imposed on the unknown function $y(x)$ and its derivatives at a point $x_0$. On some interval $I$ containing $x_0$ the problem of solving an $n$th-order differential equation subject to $n$ side conditions specified at $x_0$:

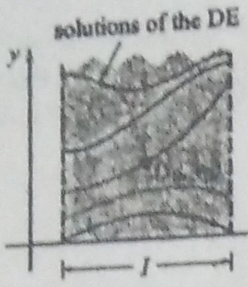
$$\text{Solve:} \qquad \frac{d^ny}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$
$$\text{Subject to:} \qquad y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}, \tag{1}$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary real constants, is called an **$n$th-order initial-value problem (IVP)**. The values of $y(x)$ and its first $n - 1$ derivatives at $x_0$, $y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called **initial conditions (IC)**.

Solving an $n$th-order initial-value problem such as (1) frequently entails first finding an $n$-parameter family of solutions of the given differential equation and then using the initial-conditions at $x_0$ to determine the $n$ constants in this family. The resulting particular solution is defined on some interval $I$ containing the initial point $x_0$.

**Geometric Interpretation of IVPs** The cases $n = 1$ and $n = 2$ in (1),

$$\text{Solve:} \qquad \frac{dy}{dx} = f(x, y)$$
$$\text{Subject to:} \qquad y(x_0) = y_0 \tag{2}$$



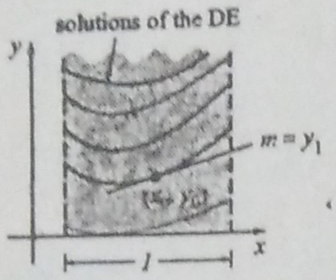

GURE 1.2.1 Solution curve of t-orde IVP

and

$$\text{Solve:} \quad \frac{d^2y}{dx^2} = f(x, y, y') \tag{3}$$

$$\text{Subject to:} \quad y(x_0) = y_0,\ y'(x_0) = y_1$$

are examples of first and second-order initial-value problems, respectively. These two problems are easy to interpret in geometric terms. For (2) we are seeking a solution $y(x)$ of the differential equation $y' = f(x, y)$ on an interval $I$ containing $x_0$ so that its graph passes through the specified point $(x_0, y_0)$. A solution curve is shown in blue in Figure 1.2.1. For (3) we want to find a solution $y(x)$ of the differential equation $y'' = f(x, y, y')$ on an interval $I$ containing $x_0$ so that its graph not only passes through $(x_0, y_0)$ but the slope of the curve at this point is the number $y_1$. A solution curve is shown in blue in Figure 1.2.2. The words *initial conditions* derive from physical systems where the independent variable is time $t$ and where $y(t_0) = y_0$ and $y'(t_0) = y_1$ represent the position and velocity, respectively, of an object at some beginning, or initial, time $t_0$.

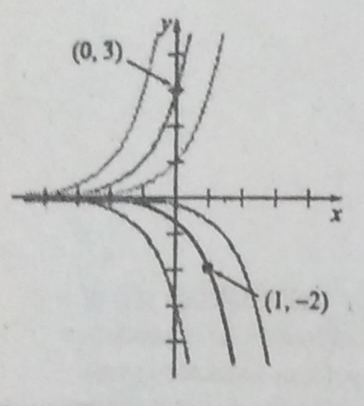



GURE 1.2.2 Solution curve of cond-order IVP

**EXAMPLE 1** **Two First-Order IVPs**

**(a)** In Problem 41 in Exercises 1.1 you were asked to deduce that $y = ce^x$ is a one-parameter family of solutions of the simple first-order equation $y' = y$. All the solutions in this family are defined on the interval $(-\infty, \infty)$. If we impose an initial condition, say, $y(0) = 3$, then substituting $x = 0$, $y = 3$ in the family determines the constant $3 = ce^0 = c$. Thus $y = 3e^x$ is a solution of the IVP

$$y' = y, \quad y(0) = 3.$$




GURE 1.2.3 Solution curves of two Ps in Example 1

**(b)** Now if we demand that a solution curve pass through the point $(1, -2)$ rather than $(0, 3)$, then $y(1) = -2$ will yield $-2 = ce$ or $c = -2e^{-1}$. In this case $y = -2e^{x-1}$ is a solution of the IVP

$$y' = y, \quad y(1) = -2.$$

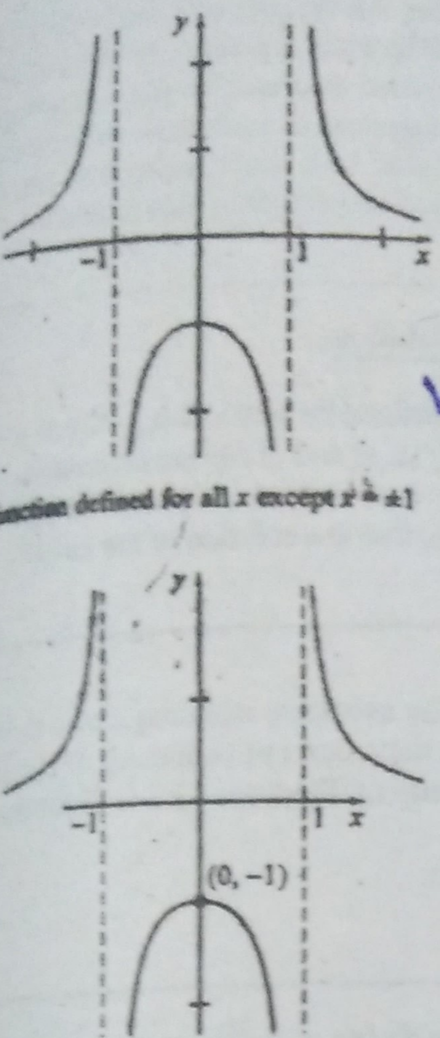
The two solution curves are shown in dark blue and dark red in Figure 1.2.3. ≡

The next example illustrates another first-order initial-value problem. In this example notice how the interval $I$ of definition of the solution $y(x)$ depends on the initial condition $y(x_0) = y_0$.

**EXAMPLE 2** **Interval $I$ of Definition of a Solutio**

In Problem 6 of Exercises 2.2 you will be asked to show that a one-parameter family of solutions of the first-order differential equation $y' + 2xy^2 = 0$ is $y = 1/(x^2 + c)$. If we impose the initial condition $y(0) = -1$, then substituting $x = 0$ and $y = -1$ into the family of solutions gives $-1 = 1/c$ or $c = -1$. Thus $y = 1/(x^2 - 1)$. We now emphasize the following three distinctions:

- Considered as a *function*, the domain of $y = 1/(x^2 - 1)$ is the set of real numbers $x$ for which $y(x)$ is defined; this is the set of all real number except $x = -1$ and $x = 1$. See Figure 1.2.4(a).
- Considered as a *solution of the differential equation* $y' + 2xy^2 = 0$, the interval $I$ of definition of $y = 1/(x^2 - 1)$ could be taken to be any interval over which $y(x)$ is defined and di ferentiable. As can be seen in Figure 1.2.4(a), the largest intervals on which $y = 1/(x^2 - 1)$ is a solution are $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.




(a) function defined for all $x$ except $x = \pm 1$

(b) solution defined on interval containing $x = 0$

FIGURE 1.2.4 Graphs of function and solution of IVP in Example 2

• Considered as a *solution of the initial-value problem* $y' + 2xy^2 = 0$, $y(0) = -1$, the interval $I$ of definition of $y = 1/(x^2 - 1)$ could be taken to be any interval over which $y(x)$ is defined, differentiable, *and* contains the initial point $x = 0$; the largest interval for which this is true is $(-1, 1)$. See the red curve in Figure 1.2.4(b).

See Problems 3–6 in Exercises 1.2 for a continuation of Example 2.

## EXAMPLE 3 Second-Order IVP

In Example 7 of Section 1.1 we saw that $x = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of $x'' + 16x = 0$. Find a solution of the initial-value problem

$$x'' + 16x = 0, \quad x\left(\frac{\pi}{2}\right) = -2, \quad x'\left(\frac{\pi}{2}\right) = 1. \tag{4}$$

**SOLUTION** We first apply $x(\pi/2) = -2$ to the given family of solutions: $c_1 \cos 2\pi + c_2 \sin 2\pi = -2$. Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, we find that $c_1 = -2$. We next apply $x'(\pi/2) = 1$ to the one-parameter family $x(t) = -2 \cos 4t + c_2 \sin 4t$. Differentiating and then setting $t = \pi/2$ and $x' = 1$ gives $8 \sin 2\pi + 4c_2 \cos 2\pi = 1$, from which we see that $c_2 = \frac{1}{4}$. Hence $x = -2 \cos 4t + \frac{1}{4} \sin 4t$ is a solution of (4).

### Existence and Uniqueness

Two fundamental questions arise in considering an initial-value problem:

*Does a solution of the problem exist?*
*If a solution exists, is it unique?*

For the first-order initial-value problem (2) we ask

**Existence** { *Does the differential equation $dy/dx = f(x, y)$ possess solutions?* *Do any of the solution curves pass through the point $(x_0, y_0)$?*

**Uniqueness** { *When can we be certain that there is precisely one solution curve passing through the point $(x_0, y_0)$?*

Note that in Examples 1 and 3 the phrase "*a* solution" is used rather than "*the* solution" of the problem. The indefinite article "a" is used deliberately to suggest the possibility that other solutions may exist. At this point it has not been demonstrated that there is a single solution of each problem. The next example illustrates an initial-value problem with two solutions.

## EXAMPLE 4 An IVP Can Have Several Solutions

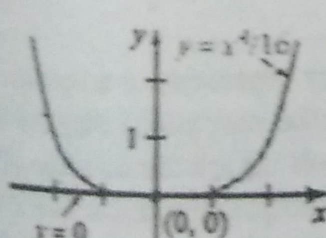



FIGURE 1.2.5 Two solutions curves of the same IVP in Example 4

Each of the functions $y = 0$ and $y = \frac{1}{16}x^4$ satisfies the differential equatio $dy/dx = xy^{1/2}$ and the initial condition $y(0) = 0$, so the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$

has at least two solutions. As illustrated in Figure 1.2.5, the graphs of both functions shown in red and blue pass through the same point $(0, 0)$.

Within the safe confine of a formal course in differential equations one can b fairly confiden that *most* differential equations will have solutions and that solutions initial-value problems will *probably* be unique. Real life, however, is not so idylli Therefore it is desirable to know in advance of trying to solve an initial-value proble on $R$ enables us to say that not only does a solution exist on some interval $I_0$ containing $x_0$, but it is the *only* solution satisfying $y(x_0) = y_0$. However, Theorem 1.2.1 does not give any indication of the sizes of intervals $I$ and $I_0$; *the interval $I$ of definition need not be as wide as the region $R$, and the interval $I_0$ of existence and uniqueness may not be as large as $I$.* The number $h > 0$ that defines the interval $I_0$: $(x_0 - h, x_0 + h)$ could be very small, so it is best to think that the solution $y(x)$ *is unique in a local sense*—that is, a solution defined near the point $(x_0, y_0)$. See Problem 50 in Exercises 1.2.

## REMARKS

(*i*) The conditions in Theorem 1.2.1 are sufficient but not necessary. This means that when $f(x, y)$ and $\partial f/\partial y$ are continuous on a rectangular region $R$, it must always follow that a solution of (2) exists and is unique whenever $(x_0, y_0)$ is a point interior to $R$. However, if the conditions stated in the hypothesis of Theorem 1.2.1 do not hold, then anything could happen: Problem (2) *may* still have a solution and this solution *may* be unique, or (2) may have several solutions, or it may have no solution at all. A rereading of Example 5 reveals that the hypotheses of Theorem 1.2.1 do not hold on the line $y = 0$ for the differential equation $dy/dx = xy^{1/2}$, so it is not surprising, as we saw in Example 4 of this section, that there are two solutions defined on a common interval $-h < x < h$ satisfying $y(0) = 0$. On the other hand, the hypotheses of Theorem 1.2.1 do not hold on the line $y = 1$ for the differential equation $dy/dx = |y - 1|$. Nevertheless it can be proved that the solution of the initial-value problem $dy/dx = |y - 1|, y(0) = 1$, is unique. Can you guess this solution?

(*ii*) You are encouraged to read, think about, work, and then keep in mind Problem 49 in Exercises 1.2.

(*iii*) Initial conditions are prescribed at a *single* point $x_0$. But we are also interested in solving differential equations that are subject to conditions specifie on $y(x)$ or its derivative at *two* different points $x_0$ and $x_1$. Conditions such as

$$y(1) = 0, \quad y(5) = 0 \qquad \text{or} \qquad y(\pi/2) = 0, \quad y'(\pi) = 1$$

and called **boundary conditions.** A differential equation together with boundary conditions is called a **boundary-value problem (BVP).** For example,

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$$

is a boundary-value problem. See Problems 39–44 in Exercises 1.2.

When we start to solve differential equations in Chapter 2 we will solve only first-order equations and first-order initial-value problems. The mathematical description of many problems in science and engineering involve second-order IVPs or two-point BVPs. We will examine some of these problems in Chapters 4 and 5.

## EXERCISES 1.2

*Answers to selected odd-numbered problems begin on page ANS-1.*

In Problems 1 and 2, $y = 1/(1 + c_1e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

1. $y(0) = -\frac{1}{3}$
2. $y(-1) = 2$

In Problems 3–6, $y = 1/(x^2 + c)$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition. Give the largest interval $I$ over which the solution is defined

3. $y(2) = \frac{1}{3}$
4. $y(-2) = \frac{1}{2}$
5. $y(0) = 1$
6. $y(\frac{1}{2}) = -4$

In Problems 7–10, $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find

Q7→14



tion of the second-order IVP consisting of this differ- equation and the given initial conditions.

(0) = −1, $x'(0) = 8$

$(\pi/2) = 0$, $x'(\pi/2) = 1$

$(\pi/6) = \frac{1}{2}$, $x'(\pi/6) = 0$

$(\pi/4) = \sqrt{2}$, $x'(\pi/4) = 2\sqrt{2}$

roblems 11–14, $y = c_1e^x + c_2e^{-x}$ is a two-parameter ly of solutions of the second-order DE $y'' - y = 0$. Find ution of the second-order IVP consisting of this differ- l equation and the given initial conditions.

$y(0) = 1$, $y'(0) = 2$

$y(1) = 0$, $y'(1) = e$

$y(-1) = 5$, $y'(-1) = -5$

$y(0) = 0$, $y'(0) = 0$

roblems 15 and 16 determine by inspection at least two tions of the given first-order IV .

$y' = 3y^{2/3}$, $y(0) = 0$

$xy' = 2y$, $y(0) = 0$

Problems 17–24 determine a region of the $xy$-plane for ch the given differential equation would have a unique ution whose graph passes through a point $(x_0, y_0)$ in the ion.

$\frac{dy}{dx} = y^{2/3}$

18. $\frac{dy}{dx} = \sqrt{xy}$

$x\frac{dy}{dx} = y$

20. $\frac{dy}{dx} - y = x$

$(4 - y^2)y' = x^2$

22. $(1 + y^3)y' = x^2$

$(x^2 + y^2)y' = y^2$

24. $(y - x)y' = y + x$

Problems 25–28 determine whether Theorem 1.2.1 guar- tees that the differential equation $y' = \sqrt{y^2 - 9}$ pos- sses a unique solution through the given point.

. (1, 4)

26. (5, 3)

. (2, −3)

28. (−1, 1)

. (a) By inspection find a one-parameter family of solu- tions of the differential equation $xy' = y$. Verify that each member of the family is a solution of the initial-value problem $xy' = y$, $y(0) = 0$.

(b) Explain part (a) by determining a region $R$ in the $xy$-plane for which the differential equation $xy' = y$ would have a unique solution through a point $(x_0, y_0)$ in $R$.

(c) Verify that the piecewise-defined functio

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

satisfies the condition $y(0) = 0$. Determine whether this function is also a solution of the initial-value problem in part (a).

30. (a) Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$.

(b) Since $f(x, y) = 1 + y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region $R$ in Theorem 1.2.1 can be taken to be the entire $xy$-plane. Use the family of solutions in part (a) to find an explicit solution of the first-order initial-value problem $y' = 1 + y^2$, $y(0) = 0$. Even though $x_0 = 0$ is in the interval $(-2, 2)$, explain why the solution is not defined on this interval.

(c) Determine the largest interval $I$ of definition for the solution of the initial-value problem in part (b).

31. (a) Verify that $y = -1/(x + c)$ is a one-parameter family of solutions of the differential equation $y' = y^2$.

(b) Since $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region $R$ in Theorem 1.2.1 can be taken to be the entire $xy$-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Then find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval $I$ of definition for the solution of each initial-value problem.

(c) Determine the largest interval $I$ of definition for the solution of the first-order initial-value problem $y' = y^2$, $y(0) = 0$. [*Hint*: The solution is not a member of the family of solutions in part (a).]

32. (a) Show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(1) = 1$, is $y = 1/(2 - x)$.

(b) Then show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(3) = -1$, is $y = 1/(2 - x)$.

(c) Are the solutions in parts (a) and (b) the same?

33. (a) Verify that $3x^2 - y^2 = c$ is a one-parameter family of solutions of the differential equation $y\,dy/dx = 3x$.

(b) By hand, sketch the graph of the implicit solution $3x^2 - y^2 = 3$. Find all explicit solutions $y = \phi(x)$ of the DE in part (a) defined by this relation. Give the interval $I$ of definition of each explicit solution

(c) The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$, but which of the explicit solutions in part (b) satisfies $y(-2) = 3$?

34. (a) Use the family of solutions in part (a) of Problem 33 to find an implicit solution of the initial-value

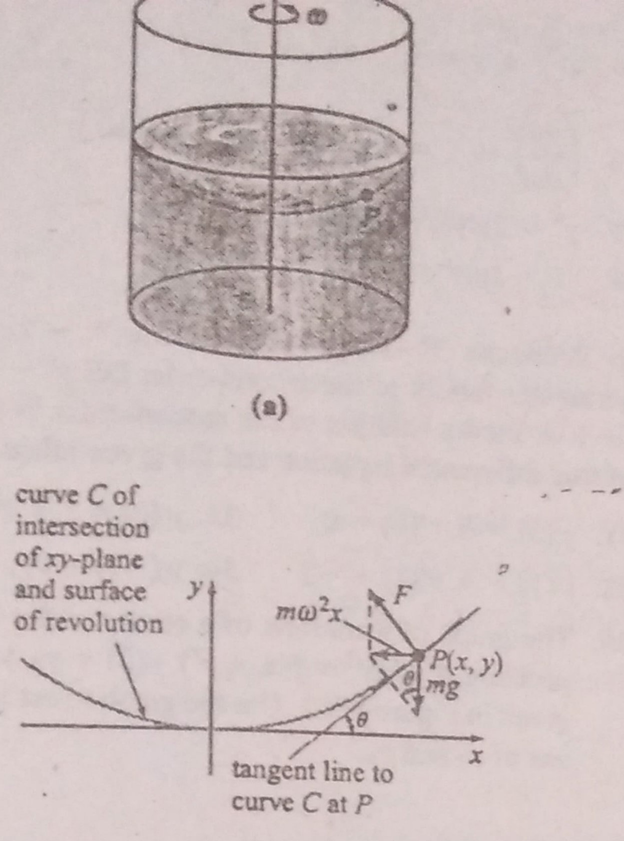



FIGURE 1.3.24 Rotating fluid in Problem 3

35. **Falling Body** In Problem 23, suppose $r = R + s$, where $s$ is the distance from the surface of the Earth to the falling body. What does the differential equation obtained in Problem 23 become when $s$ is very small in comparison to $R$? [*Hint*: Think binomial series for

$$(R + s)^{-2} = R^{-2}(1 + s/R)^{-2}.]$$

*virga* refers to falling raindrops or ic evaporate before they reach the ground. typical raindrop is spherical. Starting which we can designate as $t = 0$, the rai $r_0$ falls from rest from a cloud and begir

(a) If it is assumed that a raindrop evap manner that its shape remains spheri makes sense to assume that the rate at drop evaporates—that is, the rate at mass—is proportional to its surface this latter assumption implies that th the radius $r$ of the raindrop decrease Find $r(t)$. [*Hint*: See Problem 51 in

(b) If the positive direction is downwa mathematical model for the velocity raindrop at time $t > 0$. Ignore air re Use the form of Newton's second (17).]

37. **Let It Snow** The "snowplow problem and appears in many differential equatio was probably made famous by Ralph F

*One day it started snowing at a he rate. A snowplow started out at noon the first hour and 1 mile the second h did it start snowing?*

Find the textbook *Differential Equations,* Agnew, McGraw-Hill Book Co., and th construction and solution of the mather

38. Reread this section and classify each model as linear or nonlinear.

## CHAPTER 1 IN REVIEW

*Answers to selected odd-numbered problems begin*

In Problems 1 and 2 fill in the blank and then write this result as a linear first-order differential equation that is free of the symbol $c_1$ and has the form $dy/dx = f(x, y)$. The symbol $c_1$ represents a constant.

1. $\dfrac{d}{dx} c_1 e^{10x} =$ ________

2. $\dfrac{d}{dx}(5 + c_1 e^{-2x}) =$ ________

In Problems 3 and 4 fill in the blank and then write this result as a linear second-order differential equation that is free of the symbols $c_1$ and $c_2$ and has the form $F(y, y'') = 0$. The symbols $c_1$, $c_2$, and $k$ represent constants.

3. $\dfrac{d^2}{dx^2}(c_1 \cos kx + c_2 \sin kx) =$ ________

4. $\dfrac{d^2}{dx^2}(c_1 \cosh kx + c_2 \sinh kx) =$ ________

In Problems 5 and 6 compute $y'$ and $y''$ and these derivatives with $y$ as a linear second-or equation that is free of the symbols $c_1$ and $c_2$ a $F(y, y' y'') = 0$. The symbols $c_1$ and $c_2$ repre

5. $y = c_1 e^x + c_2 x e^x$

6. $y = c_1 e^x \cos$

In Problems 7–12 match each of the given di tions with one or more of these solutions:

(a) $y = 0$, (b) $y = 2$, (c) $y = 2x$,

7. $xy' = 2y$

8. $y' = 2$

9. $y' = 2y - 4$

10. $xy' = y$

11. $y'' + 9y = 18$

12. $xy'' - y' =$

In Problems 13 and 14 determine by inspect solution of the given differential equation.

13. $y'' = y'$

14. $y' = y(y -$