2.2 INTRODUCTION TO FINITE ELEMENT APPROXIMATIONS

In contrast to the finite difference schemes wherein the domain of interest is replaced by a set of discrete points, in the finite element method the domain is divided into subdomains commonly referred to as *finite elements*. The unknown function, u say, is represented within each element by an interpolating polynomial which is continuous along with its derivatives to a specified order within the element. As indicated earlier, a similar polynomial approach may be used to obtain finite difference schemes (Salvadori and Baron, 1961). Generally, the interpolating function is of lower-order continuity between elements than within an element. Thus the fundamental building block in the finite element method is the subdomain or element.

Although the finite difference schemes can be presented using Taylor series in a rather straightforward manner, the theory behind the finite element method is relatively abstract. It will be necessary to introduce concepts from functional analysis and variational methods in order to formulate the algebraic equations that are analogous to the finite difference formulas. In addition, one must overcome the conceptual problem of assembling information obtained on an element by element basis into a global representation of a problem.

2.2.1 Method of Weighted Residuals

There are several avenues that will lead to the same finite element formulation. A conceptually simple, yet mathematically rigorous, approach can be formulated using the method of weighted residuals. Although in our discussion of finite element techniques we will consider primarily two special cases of the method of weighted residuals (MWR)—the Galerkin and collocation methods—other schemes arising out of MWR are found in the engineering literature. A discussion of the application of MWR in engineering science can be found in Finlayson (1972).

In the method of weighted residuals, the desired function $u(\cdot)$ is replaced by a finite series approximation $\hat{u}(\cdot)$,

$$u(\cdot) \approx \hat{u}(\cdot) = \sum_{j=1}^{N} U_j \phi_j(\cdot). \tag{2.2.1}$$

In general, the set of functions $\phi_j(\cdot)$, $j = 1,2,\ldots,N$, can be defined over both the time and space domain and U_j, $j = 1,2,\ldots,N$, are undetermined coefficients. Equation (2.2.1) may be written $\hat{u}(\cdot) = U_0\phi_0(\cdot) + \sum_{j=1}^{N} U_j\phi_j(\cdot)$, where $\phi_j(\cdot)$ satisfy the homogeneous boundary conditions. In the finite element method, the functions of $\phi_j(\cdot)$ are chosen to be polynomials that satisfy certain of the boundary conditions imposed on the problem. These functions are variously denoted as shape functions, basis functions, and interpolation functions, depending upon the discipline in which the method is being applied.

Even when the basis functions are chosen to satisfy all boundary conditions imposed on a problem, they will not normally satisfy the PDE as well. Substitution of $\hat{u}(\cdot)$ into the PDE, $Lu - f = 0$, results in a residual, R,

$$L\hat{u}(\cdot) - f = R(\cdot). \tag{2.2.2}$$

The objective is to select the undetermined coefficients U_j such that this residual is minimized in some sense. A straightforward scheme would be to set the integral of $R(\cdot)$ to zero:

$$\int_t \int_V R(\cdot)\, dV\, dt = 0. \tag{2.2.3}$$

This scheme, however, generates only one equation for the N unknown coefficients U_j. It can be suitably modified by introducing weighting functions $w_i(\cdot), i = 1, 2, \ldots, N$.

Setting the integral of each weighted residual to zero yields n independent equations

$$\int_t \int_V R(\cdot) w_i(\cdot)\, dV\, dt = 0, \qquad i = 1, 2, \ldots, N. \tag{2.2.4}$$

In theory, at least, (2.2.4) can be solved for the N unknown coefficients. Equation (2.2.4) is the general equation describing the MWR, and a multiplicity of schemes arise out of this one expression through the definition of the weighting functions w_i.

Among the MWR family of methods, the Galerkin, subdomain, and collocation schemes are most commonly encountered in engineering practice. The one-dimensional weighting functions for each of these are illustrated in Figure 2.4. It is interesting to note how the support of these functions changes in each method. As would be expected, the computational effort required in formulating the coefficient matrices is related to the support of the function. Let us now examine the specific form of the general MWR equation (2.2.4) that arises for each scheme.

Galerkin Method* The Galerkin method results when the weighting function is chosen to be the basis function, as defined in (2.2.1). Thus we obtain

$$\int_t \int_V R(\cdot) \phi_i(\cdot)\, dV\, dt = 0, \qquad i = 1, 2, \ldots, N. \tag{2.2.5}$$

*Although this method is generally attributed to the Russian engineer Galerkin through a publication appearing in 1915, Mikhlin (1964) reports that a similar scheme was reported earlier by Bubnov in 1913.

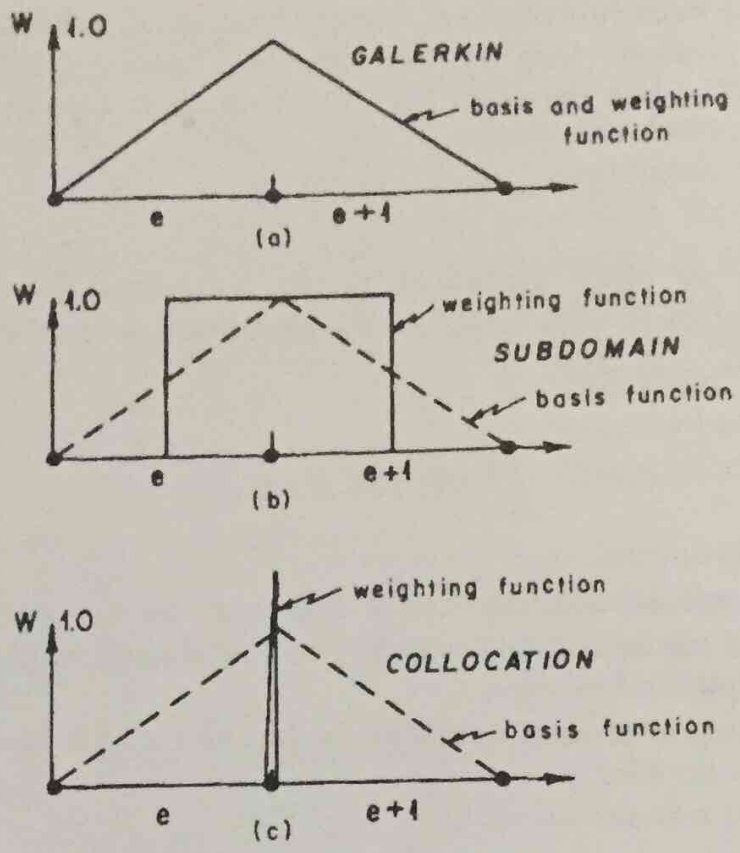

Figure 2.4. Schematic representation of the one-dimensional weighting functions for the Galerkin, subdomain and collocation methods. (It is assumed here that the chapeau function is used as a basis for all methods.)

The basis functions are formally required to be members of a complete set of functions. Because a complete set of functions can exactly represent any function of a given class, the series of (2.2.1) is inherently capable of representing the exact solution as the number of terms in the series is increased.

The requirement of completeness allows an alternative interpretation of the Galerkin formulation. A continuous function must be zero if it is orthogonal to every member of a complete set. Because (2.2.5) is, in fact, the definition of orthogonality of two functions, in this case, $R(\cdot)$ and $\phi_i(\cdot)$, the Galerkin method can be viewed as a scheme in which the residual is forced to zero in the sense that it is made orthogonal to the complete set of functions $\phi_i(\cdot)$.

In Figure 2.4*a* the weighting function (and therefore the basis function) is a hatshaped, piecewise linear function. Because of its hatlike appearance, it is sometimes called a "chapeau" function. It is often encountered in the formulation of the finite element method. The chapeau function is the simplest of the basis functions in common use. As will be demonstrated shortly, it leads to a formulation that closely resembles a finite difference scheme.

Subdomain Method.* In the subdomain method the weighting function is chosen to be unity in the subregion V_i for which it is defined and zero

*This method was apparently introduced by Biezeno and Koch (1923), two Dutch engineers.

elsewhere. In the sense of (2.2.4), this can be stated mathematically as

$$\int_V R(\cdot) w_i(\cdot)\, dV = 0, \qquad i = 1,2,\ldots,N, \tag{2.2.6a}$$

where

$$w_i = \begin{cases} 1, & (x, y, z) \text{ in } V_i^* \\ 0, & (x, y, z) \text{ not in } V_i. \end{cases} \tag{2.2.6b}$$

The one-dimensional form of w_i for this scheme appears in Figure 2.4*b*. In this example the nonoverlapping subdomains V_i, $i = 1,2,\ldots,N$, have been chosen to be equivalent to one element in width. A variant of this method in which the support for w_i and ϕ_j are the same has also been reported by Gärtner (1976). Because of the simplicity of the form of the w_i function, the integrations to be performed in this scheme are less tedious than those in Galerkin's method.

Collocation Method†. The collocation method is the simplest of the MWR schemes to implement. In this method the weighting function w_i is chosen to be the Dirac delta, that is,

$$w_i = \delta(x - x_i).$$

The Dirac delta function has the important property that‡

$$\int_t \int_V a(\cdot)\, \delta(x - x_i, y - y_i, z - z_i, t - t_i)\, dV\, dt \equiv a|_{x_i, y_i, z_i, t_i}, \tag{2.2.7}$$

where, for this discussion, we denote time by t. Thus the orthogonality constraints (2.2.4) for the collocation method

$$\int_t \int_V R(\cdot)\, \delta_i(\cdot)\, dV\, dt = 0, \qquad i = 1,2,\ldots,N, \tag{2.2.8}$$

are a mathematical statement of the requirement that the residual vanish at each collocation point (x_i, y_i, z_i, t_i). Because of the relationship (2.2.7), it is apparent that to evaluate (2.2.8) we simply calculate the value of the residual at the selected points. This requires no integration and generates the N equations

*In this definition we have assumed spatial discretization only.

†Collocation was apparently introduced by Slater (1934).

‡Note that the Dirac delta function defined over several dimensions is also written $\delta(x - x_i)\delta(y - y_i)\delta(z - z_i)\delta(t - t_i)$.