Solutions of Equations in One Variable

Accelerating Convergence

Numerical Analysis (9th Edition)

R L Burden & J D Faires

Beamer Presentation Slides
prepared by
John Carroll
Dublin City University

Outline

Outline

1. Aitken's Δ^2 Method
2. Steffensen's Method

Outline

Outline

Outline

1. Aitken's Δ^2 Method
2. Steffensen's Method
3. Application of Steffensen's Method
4. Rate of Convergence

Accelerating Convergence: Aitken's Δ^2 Method

Improving the Rate of Convergence

- We have seen from an earlier result ▸ Linear Convergence that it is rare to have the luxury of quadratic convergence.
- We now consider a technique called **Aitken's $\boldsymbol{\Delta^2}$ method** that can be used to accelerate the convergence of a sequence that is linearly convergent, regardless of its origin or application.

Accelerating Convergence: Aitken's Δ^2 Method

Constructing Aitken's Δ^2 Method

- Suppose $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p.
- To motivate the construction of a sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ that converges more rapidly to p than does $\{p_n\}_{n=0}^{\infty}$, let us first assume that the signs of

$$p_n - p, \qquad p_{n+1} - p \quad \text{and} \quad p_{n+2} - p$$

agree and that n is sufficiently large that

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

Accelerating Convergence: Aitken's Δ^2 Method

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

Constructing Aitken's Δ^2 Method (Cont'd)

Solve for p:

$$\begin{aligned}
(p_{n+1} - p)^2 &\approx (p_{n+2} - p)(p_n - p) \\
\Rightarrow p_{n+1}^2 - 2p_{n+1}p + p^2 &\approx p_{n+2}p_n - (p_n + p_{n+2})p + p^2 \\
\Rightarrow (p_{n+2} + p_n - 2p_{n+1})\, p &\approx p_{n+2}p_n - p_{n+1}^2 \\
\Rightarrow p &\approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}
\end{aligned}$$

Accelerating Convergence: Aitken's Δ^2 Method

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}$$

Constructing Aitken's Δ^2 Method (Cont'd)

Adding and subtracting the terms p_n^2 and $2p_np_{n+1}$ in the numerator and grouping terms appropriately gives

$$\begin{aligned} p &\approx \frac{p_np_{n+2} - 2p_np_{n+1} + p_n^2 - p_{n+1}^2 + 2p_np_{n+1} - p_n^2}{p_{n+2} - 2p_{n+1} + p_n} \\ &= \frac{p_n(p_{n+2} - 2p_{n+1} + p_n) - (p_{n+1}^2 - 2p_np_{n+1} + p_n^2)}{p_{n+2} - 2p_{n+1} + p_n} \\ &= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}. \end{aligned}$$

Accelerating Convergence: Aitken's Δ^2 Method

Aitken's Δ^2 Method

Aitken's $\mathbf{\Delta^2}$ method is based on the assumption that the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$, defined by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

converges more rapidly to p than does the original sequence $\{p_n\}_{n=0}^{\infty}$.

Accelerating Convergence: Aitken's Δ^2 Method

Example: Computing the Iterations

- The sequence $\{p_n\}_{n=1}^{\infty}$, where $p_n = \cos(1/n)$, converges linearly to $p = 1$.
- Determine the first 5 terms of the sequence given by Aitken's Δ^2 method.

- In order to determine a term $\hat{p}_n$ of the Aitken's Δ^2 method sequence, we need to have the terms p_n, p_{n+1}, and p_{n+2} of the original sequence.
- So to determine $\hat{p}_5$ we need the first 7 terms of $\{p_n\}$.
- These are given in the following table.

Accelerating Convergence: Aitken's Δ^2 Method

Computing $\hat{p}_n$ from p_n, p_{n+1}, and p_{n+2}

n	p_n	$\hat{p}_n$
1	0.54030	0.96178
2	0.87758	0.98213
3	0.94496	0.98979
4	0.96891	0.99342
5	0.98007	0.99541
6	0.98614	
7	0.98981	

It certainly appears that $\{\hat{p}_n\}_{n=1}^{\infty}$ converges more rapidly to $p = 1$ than does $\{p_n\}_{n=1}^{\infty}$.

Accelerating Convergence: Aitken's Δ^2 Method

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \qquad \text{for } n \geq 0$$

Writing the formula using Δ Notation

The numerator and denominator of the formula (above) may be expressed using the Δ notation for forward differences ▸ Definition of Δ since

$$\begin{aligned}
\Delta p_n &= p_{n+1} - p_n \\
\Delta^2 p_n &= \Delta(p_{n+1} - p_n) = \Delta p_{n+1} - \Delta p_n \\
&= (p_{n+2} - p_{n+1}) - (p_{n+1} - p_n) \\
&= p_{n+2} - 2p_{n+1} + p_n
\end{aligned}$$

Aitken's Δ^2 Method

$$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} \qquad \text{for } n \geq 0$$

Accelerating Convergence: Aitken's Δ^2 Method

- We have stated that the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$, converges to p more rapidly than does the original sequence $\{p_n\}_{n=0}^{\infty}$, ...
- but we have not said what is meant by the term "more rapid" convergence.
- The following theorem (stated without proof) explains and justifies this terminology.

Accelerating Convergence: Aitken's Δ^2 Method

Rate of Convergence of Aitken's Δ^2 Method

Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges linearly to the limit p and that

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} < 1.$$

Then the Aitken's Δ^2 sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0.$$

Outline

Accelerating Convergence: Steffensen's Method

From Aitken's Method to Steffensen's Method

- By applying a modification of Aitken's Δ^2 method to a linearly convergent sequence obtained from fixed-point iteration, we can accelerate the convergence to quadratic.
- This procedure is known as Steffensen's method and differs slightly from applying Aitken's Δ^2 method directly to the linearly convergent fixed-point iteration sequence.

Accelerating Convergence: Steffensen's Method

From Aitken's Method to Steffensen's Method (Cont'd)

- Aitken's Δ^2 method constructs the terms in order:

$$p_0, \quad p_1 = g(p_0), \quad p_2 = g(p_1), \quad \hat{p}_0 = \{\Delta^2\}(p_0),$$
$$p_3 = g(p_2), \quad \hat{p}_1 = \{\Delta^2\}(p_1), \ldots,$$

- Steffensen's method constructs the same first four terms, p_0, p_1, p_2, and $\hat{p}_0$. However, at this step we assume that $\hat{p}_0$ is a better approximation to p than is p_2 and apply fixed-point iteration to $\hat{p}_0$ instead of p_2. Using this notation the sequence generated is

$$p_0^{(0)}, \quad p_1^{(0)} = g(p_0^{(0)}), \quad p_2^{(0)} = g(p_1^{(0)}), \quad p_0^{(1)} = \{\Delta^2\}(p_0^{(0)})$$
$$p_1^{(1)} = g(p_0^{(1)}), \ldots.$$

Algorithm for Steffensen's Method

To find a solution to $p = g(p)$ given an initial approximation p_0, tolerance TOL; maximum number of iterations N_0.

1 Set $i = 1$

2 While $i \leq N_0$ do Steps 3–6:

3 Set $p_1 = g(p_0)$; (*Compute* $p_1^{(i-1)}$.)

$p_2 = g(p_1)$; (*Compute* $p_2^{(i-1)}$.)

$p = p_0 - (p_1 - p_0)^2/(p_2 - 2p_1 + p_0)$. (*Compute* $p_0^{(i)}$.)

4 If $|p - p_0| < TOL$ then OUTPUT (p);
(*Procedure completed successfully*.) STOP.

5 Set $i = i + 1$

6 Set $p_0 = p$. (*Update* p_0)

7 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(*Procedure completed unsuccessfully*.) STOP.

Outline

Accelerating Convergence: Steffensen's Method

Example: Application of Steffensen's Method

- Earlier, we solved $x^3 + 4x^2 - 10 = 0$ using fixed-point iteration.
- One choice for $g(x)$ was obtained by letting $x^3 + 4x^2 = 10$, dividing by $x + 4$ and solving for x to obtain $x = g(x)$ where

$$g(x) = \sqrt{\frac{10}{x + 4}}$$

- With this choice of $g(x)$, we will now solve $x^3 + 4x^2 - 10 = 0$ using Steffensen's method starting with $p_0 = 1.5$.

Steffensen's Method: $g(x) = \sqrt{\frac{10}{x+4}}$

k	$p_0^{(k)}$	$p_1^{(k)}$	$p_2^{(k)}$
0	1.5	1.348399725	1.367376372
1	1.365265224	1.365225534	1.365230583
2	1.365230013		

- The iterate $p_0^{(2)} = 1.365230013$ is accurate to the ninth decimal place.
- In this example, Steffensen's method gave about the same accuracy as Newton's method applied to this polynomial.

Outline

Accelerating Convergence: Steffensen's Method

From this example, it appears that Steffensen's method gives quadratic convergence without evaluating a derivative, and the following theorem states that this is the case.

Rate of Convergence of Steffensen's Method

Suppose that $x = g(x)$ has the solution p with $g'(p) \neq 1$. If there exists a $\delta > 0$ such that $g \in C^3[p - \delta, p + \delta]$, then Steffensen's method gives quadratic convergence for any $p_0 \in [p - \delta, p + \delta]$.

The proof of this theorem can be found in [He2], pp. 90–92, or [IK], pp. 103–107 ▸ References.

Questions?

Reference Material

Thoerem (Linear Convergence)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all $x \in [a, b]$. Suppose, in addition, that g' is continuous on (a, b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a, b]$, the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1,$$

converges only linearly to the unique fixed point p in $[a, b]$.

Return to Rate of Convergence

Forward Difference Operator Δ

For a given sequence $\{p_n\}_{n=0}^{\infty}$, the forward difference Δp_n (read "delta p_n") is defined by

$$\Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0.$$

Higher powers of the operator Δ are defined recursively by

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2.$$

Return to Aitken's Method

Rate of Convergence of Steffensen's Method

Suppose that $x = g(x)$ has the solution p with $g'(p) \neq 1$. If there exists a $\delta > 0$ such that $g \in C^3[p - \delta, p + \delta]$, then Steffensen's method gives quadratic convergence for any $p_0 \in [p - \delta, p + \delta]$.

Sources for the Proof

-
-

Return to Steffensen's Method