# Topic 12 Notes

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# 12 Laplace transform

## 12.1 Introduction

The Laplace transform takes a function of time and transforms it to a function of a complex variable $s$. Because the transform is invertible, no information is lost and it is reasonable to think of a function $f(t)$ and its Laplace transform $F(s)$ as two views of the same phenomenon. Each view has its uses and some features of the phenomenon are easier to understand in one view or the other.

We can use the Laplace transform to transform a linear time invariant system from the time domain to the $s$-domain. This leads to the system function $G(s)$ for the system –this is the same system function used in the Nyquist criterion for stability.

One important feature of the Laplace transform is that it can transform analytic problems to algebraic problems. We will see examples of this for differential equations.

## 12.2 A brief introduction to linear time invariant systems

Let's start by defining our terms.

**Signal.** A signal is any function of time.

**System.** A system is some machine or procedure that takes one signal as input does something with it and produces another signal as output.

**Linear system.** A linear system is one that acts linearly on inputs. That is, $f_1(t)$ and $f_2(t)$ are inputs to the system with outputs $y_1(t)$ and $y_2(t)$ respectively, then the input $f_1 + f_2$ produces the output $y_1 + y_2$ and, for any constant $c$, the input $cf_1$ produces output $cy_1$.

This is often phrased in one sentence as input $c_1 f_1 + c_2 f_2$ produces output $c_1 y_1 + c_2 y_2$, i.e. linear combinations of inputs produces a linear combination of the corresponding outputs.

**Time invariance.** Suppose a system takes input signal $f(t)$ and produces output signal $y(t)$. The system is called time invariant if the input signal $g(t) = f(t - a)$ produces output signal $y(t - a)$.

**LTI.** We will call a linear time invariant system an LTI system.

**Example 12.1.** Consider the constant coefficient differential equation

$$3y'' + 8y' + 7y = f(t)$$

This equation models a damped harmonic oscillator, say a mass on a spring with a damper, where $f(t)$ is the force on the mass and $y(t)$ is its displacement from equilibrium. If we consider $f$ to be the input and $y$ the output, then this is a linear time invariant (LTI) system.

**Example 12.2.** There are many variations on this theme. For example, we might have the LTI system

$$3y'' + 8y' + 7y = f'(t),$$

where we call $f(t)$ the input signal and $y(t)$ the output signal.

## 12.3 Laplace transform

**Definition.** The Laplace transform of a function $f(t)$ is defined by the integral

$$\mathcal{L}(f;s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,dt,$$

for those $s$ where the integral converges. Here $s$ is allowed to take complex values.

**Important note.** The Laplace transform is only concerned with $f(t)$ for $t \geq 0$. Generally, speaking we can require $f(t) = 0$ for $t < 0$.

**Standard notation.** Where the notation is clear, we will use an upper case letter to indicate the Laplace transform, e.g, $\mathcal{L}(f;s) = F(s)$.

The Laplace transform we defined is sometimes called the one-sided Laplace transform. There is a two-sided version where the integral goes from $-\infty$ to $\infty$.

### 12.3.1 First examples

Let's compute a few examples. We will also put these results in the Laplace transform table at the end of these notes.

**Example 12.3.** Let $f(t) = \mathrm{e}^{at}$. Compute $F(s) = \mathcal{L}(f;s)$ directly. Give the region in the complex $s$-plane where the integral converges.

$$\begin{aligned}\mathcal{L}(\mathrm{e}^{at};s) &= \int_0^\infty \mathrm{e}^{at}\mathrm{e}^{-st}\,dt = \int_0^\infty \mathrm{e}^{(a-s)t}\,dt = \left.\frac{\mathrm{e}^{(a-s)t}}{a-s}\right|_0^\infty \\ &= \begin{cases} \frac{1}{s-a} & \text{if } \mathrm{Re}(s) > \mathrm{Re}(a) \\ \text{divergent} & \text{otherwise} \end{cases}\end{aligned}$$

The last formula comes from plugging $\infty$ into the exponential. This is 0 if $\mathrm{Re}(a-s) < 0$ and undefined otherwise.

**Example 12.4.** Let $f(t) = b$. Compute $F(s) = \mathcal{L}(f;s)$ directly. Give the region in the complex $s$-plane where the integral converges.

$$\begin{aligned}\mathcal{L}(b;s) &= \int_0^\infty b\mathrm{e}^{-st}\,dt = \left.\frac{b\mathrm{e}^{-st}}{-s}\right|_0^\infty \\ &= \begin{cases} \frac{b}{s} & \text{if } \mathrm{Re}(s) > 0 \\ \text{divergent} & \text{otherwise} \end{cases}\end{aligned}$$

The last formula comes from plugging $\infty$ into the exponential. This is 0 if $\mathrm{Re}(-s) < 0$ and undefined otherwise.

**Example 12.5.** Let $f(t) = t$. Compute $F(s) = \mathcal{L}(f;s)$ directly. Give the region in the complex $s$-plane where the integral converges.

$$\begin{aligned}\mathcal{L}(t;s) &= \int_0^\infty t\mathrm{e}^{-st}\,dt \;=\; \left.\frac{t\mathrm{e}^{-st}}{-s} - \frac{\mathrm{e}^{-st}}{s^2}\right|_0^\infty \\ &= \begin{cases} \frac{1}{s^2} & \text{if } \mathrm{Re}(s) > 0 \\ \text{divergent} & \text{otherwise} \end{cases}\end{aligned}$$

**Example 12.6.** Compute

$$\mathcal{L}(\cos(\omega t)).$$

*Solution:* We use the formula

$$\cos(\omega t) = \frac{\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}}{2}.$$

So,

$$\mathcal{L}(\cos(\omega t); s) = \frac{1/(s - i\omega) + 1/(s + i\omega)}{2} = \frac{s}{s^2 + \omega^2}.$$

### 12.3.2 Connection to Fourier transform

The Laplace and Fourier transforms are intimately connected. In fact, the Laplace transform is often called the Fourier-Laplace transform. To see the connection we'll start with the Fourier transform of a function $f(t)$.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,dt.$$

If we assume $f(t) = 0$ for $t < 0$, this becomes

$$\hat{f}(\omega) = \int_0^\infty f(t)\mathrm{e}^{-i\omega t}\,dt. \tag{1}$$

Now if $s = i\omega$ then the Laplace transform is

$$\mathcal{L}(f;s) = \mathcal{L}(f;i\omega) = \int_0^\infty f(t)\mathrm{e}^{-i\omega t}\,dt \tag{2}$$

Comparing these two equations we see that $\hat{f}(\omega) = \mathcal{L}(f;i\omega)$. We see the transforms are basically the same things using different notation –at least for functions that are 0 for $t < 0$.

## 12.4 Exponential type

The Laplace transform is defined when the integral for it converges. Functions of exponential type are a class of functions for which the integral converges for all $s$ with $\mathrm{Re}(s)$ large enough.

**Definition.** We say that $f(t)$ has exponential type $a$ if there exists an $M$ such that $|f(t)| < M\mathrm{e}^{at}$ for all $t \geq 0$.

**Note.** As we've defined it, the exponential type of a function is not unique. For example, a function of exponential type 2 is clearly also of exponential type 3. It's nice, but not always necessary, to find the smallest exponential type for a function.

**Theorem.** If $f$ has exponential type $a$ then $\mathcal{L}(f)$ converges absolutely for $\mathrm{Re}(s) > a$.

*Proof.* We prove absolute convergence by bounding

$$|f(t)\mathrm{e}^{-st}|.$$

The key here is that $\mathrm{Re}(s) > a$ implies $\mathrm{Re}(a-s) < 0$. So, we can write

$$\int_0^\infty |f(t)\mathrm{e}^{-st}|\,dt \leq \int_0^\infty |M\mathrm{e}^{(a-s)t}|\,dt = \int_0^\infty M\mathrm{e}^{\mathrm{Re}(a-s)t}\,dt$$

The last integral clearly converges when $\mathrm{Re}(a-s) < 0$. QED

**Example 12.7.** Here is a list of some functions of exponential type.

$$\begin{aligned} f(t) = \mathrm{e}^{at}: &\quad |f(t)| < 2\mathrm{e}^{\mathrm{Re}(a)t} && \text{(exponential type Re}(a)) \\ f(t) = 1: &\quad |f(t)| < 2 = 2\mathrm{e}^{0\cdot t} && \text{(exponential type 0)} \\ f(t) = \cos(\omega t): &\quad |f(t)| \leq 1 && \text{(exponential type 0)} \end{aligned}$$

In the above, all of the inequalities are for $t \geq 0$.

For $f(t) = t$, it is clear that for any $a > 0$ there is an $M$ depending on $a$ such that $|f(t)| \leq M\mathrm{e}^{at}$ for $t \geq 0$. In fact, it is a simple calculus exercise to show $M = 1/(ae)$ works. So, $f(t) = t$ has exponential type $a$ for any $a > 0$.

The same is true of $t^n$. It's worth pointing out that this follows because, if $f$ has exponential type $a$ and $g$ has exponential type $b$ then $fg$ has exponential type $a + b$. So, if $t$ has exponential type $a$ then $t^n$ has exponential type $na$.

## 12.5 Properties of Laplace transform

We have already used the linearity of Laplace transform when we computed $\mathcal{L}(\cos(\omega t))$. Let's officially record it as a property.

**Property 1.** The Laplace transform is linear. That is, if $a$ and $b$ are constants and $f$ and $g$ are functions then

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g). \tag{3}$$

(The proof is trivial –integration is linear.)

**Property 2.** A key property of the Laplace transform is that, with some technical details,

Laplace transform transforms derivatives in $t$ to multiplication by $s$ (plus some details).

This is proved in the following theorem.

**Theorem.** If $f(t)$ has exponential type $a$ and Laplace transform $F(s)$ then

$$\mathcal{L}(f'(t); s) = sF(s) - f(0), \text{ valid for } \mathrm{Re}(s) > a. \tag{4}$$

*Proof.* We prove this using integration by parts.

$$\mathcal{L}(f'; s) = \int_0^\infty f'(t)\mathrm{e}^{-st}\,dt = f(t)\mathrm{e}^{-st}\big|_0^\infty + \int_0^\infty sf(t)\mathrm{e}^{-st}\,dt = -f(0) + sF(s).$$

In the last step we used the fact that at $t = \infty$, $f(t)\mathrm{e}^{-st} = 0$, which follows from the assumption about exponential type.

Equation 4 gives us formulas for all derivatives of $f$.

$$\mathcal{L}(f''; s) = s^2 F(s) - sf(0) - f'(0) \tag{5}$$

$$\mathcal{L}(f'''; s) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0) \tag{6}$$

*Proof.* For Equation 5:

$$\mathcal{L}(f''; s) = \mathcal{L}((f')'; s) = s\mathcal{L}(f'; s) - f'(0) = s\,(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0). \text{ QED}$$

The proof Equation 6 is similar. Also, similar statements hold for higher order derivatives.

**Note.** There is a further complication if we want to consider functions that are discontinuous at the origin or if we want to allow $f(t)$ to be a generalized function like $\delta(t)$. In these cases $f(0)$ is not defined, so our formulas are undefined. The technical fix is to replace 0 by $0^-$ in the definition and all of the formulas for Laplace transform. You can learn more about this by taking 18.031.

**Property 3. Theorem.** If $f(t)$ has exponential type $a$, then $F(s)$ is an analytic function for $\mathrm{Re}(s) > a$ and

$$F'(s) = -\mathcal{L}(tf(t); s). \tag{7}$$

*Proof.* We take the derivative of $F(s)$. The absolute convergence for $\mathrm{Re}(s)$ large guarantees that we can interchange the order of integration and taking the derivative.

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)\mathrm{e}^{-st}\,dt = \int_0^\infty -tf(t)\mathrm{e}^{-st}\,dt = \mathcal{L}(-tf(t); s).$$

This proves Equation 7.

Equation 7 is called the $s$-derivative rule. We can extend it to more derivatives in $s$: Suppose $\mathcal{L}(f; s) = F(s)$. Then,

$$\mathcal{L}(tf(t); s) = -F'(s) \tag{8}$$

$$\mathcal{L}(t^n f(t); s) = (-1)^n F^{(n)}(s) \tag{9}$$

Equation 8 is the same as Equation 7 above. Equation 9 follows from this.

**Example 12.8.** Use the s-derivative rule and the formula $\mathcal{L}(1; s) = 1/s$ to compute the Laplace transform of $t^n$ for $n$ a positive integer.

*Solution:* Let $f(t) = 1$ and $F(s) = \mathcal{L}(f; s)$. Using the s-derivative rule we get

$$\mathcal{L}(t; s) = \mathcal{L}(tf; s) = -F'(s) = \frac{1}{s^2}$$

$$\mathcal{L}(t^2; s) = \mathcal{L}(t^2 f; s) = (-1)^2 F''(s) = \frac{2}{s^3}$$

$$\mathcal{L}(t^n; s) = \mathcal{L}(t^n f; s) = (-1)^n F^n(s) = \frac{n!}{s^{n+1}}$$

**Property 4.** $t$-shift rule. As usual, assume $f(t) = 0$ for $t < 0$. Suppose $a > 0$. Then,

$$\mathcal{L}(f(t-a); s) = \mathrm{e}^{-as} F(s) \tag{10}$$

*Proof.* We go back to the definition of the Laplace transform and make the change of variables $\tau = t - a$.

$$\begin{aligned}\mathcal{L}(f(t-a); s) &= \int_0^\infty f(t-a)\mathrm{e}^{-st}\,dt = \int_a^\infty f(t-a)\mathrm{e}^{-st}\,dt \\ &= \int_0^\infty f(\tau)\mathrm{e}^{-s(\tau+a)}\,d\tau = \mathrm{e}^{-sa}\int_0^\infty f(\tau)\mathrm{e}^{-s\tau}\,d\tau = \mathrm{e}^{-sa}F(s).\end{aligned}$$

The properties in Equations 3-10 will be used in examples below. They are also in the table at the end of these notes.

## 12.6 Differential equations

**Coverup method.** We are going to use partial fractions and the coverup method. We will assume you have seen partial fractions. If you don't remember them well or have never seen the coverup method, you should read the note *Partial fractions and the coverup method* posted with the class notes.

**Example 12.9.** Solve $y'' - y = \mathrm{e}^{2t}$, $y(0) = 1$, $y'(0) = 1$ using Laplace transform.

*Solution:* Call $\mathcal{L}(y) = Y$. Apply the Laplace transform to the equation gives

$$(s^2Y - sy(0) - y'(0)) - Y = \frac{1}{s-2}$$

A little bit of algebra now gives

$$(s^2 - 1)Y = \frac{1}{s-2} + s + 1.$$

So

$$Y = \frac{1}{(s-2)(s^2-1)} + \frac{s+1}{s^2-1} = \frac{1}{(s-2)(s^2-1)} + \frac{1}{s-1}$$

Use partial fractions to write

$$Y = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{1}{s-1}.$$

The coverup method gives $A = 1/3$, $B = -1/2$, $C = 1/6$.

We recognize

$$\frac{1}{s-a}$$

as the Laplace transform of $\mathrm{e}^{at}$, so

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{t} + C\mathrm{e}^{-t} + \mathrm{e}^{t} = \frac{1}{3}\mathrm{e}^{2t} - \frac{1}{2}\mathrm{e}^{t} + \frac{1}{6}\mathrm{e}^{-t} + \mathrm{e}^{t}.$$

**Example 12.10.** Solve $y'' - y = 1$, $y(0) = 0$, $y'(0) = 0$.

*Solution:* The rest (zero) initial conditions are nice because they will not add any terms to the algebra. As in the previous example we apply the Laplace transform to the entire equation.

$$s^2Y - Y = \frac{1}{s}, \text{ so } Y = \frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

The coverup method gives $A = -1$, $B = 1/2$, $C = 1/2$. So,

$$y = A + Be^t + Ce^{-t} = -1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}.$$

## 12.7 System functions and the Laplace transform

When we introduced the Nyquist criterion for stability we stated without any justification that the system was stable if all the poles of the system function $G(s)$were in the left half-plane. We also asserted that the poles corresponded to exponential modes of the system. In this section we'll use the Laplace transform to more fully develop these ideas for differential equations.

### 12.7.1 Lightning review of 18.03

**Definitions.**

**1.** $D = \frac{d}{dt}$ is called a differential operator. Applied to a function $f(t)$ we have

$$Df = \frac{df}{dt}.$$

We read $Df$ as '$D$ applied to $f$.'

**Example 12.11.** If $f(t) = t^3 + 2$ then $Df = 3t^2$, $D^2 f = 6t$.

**2.** If $P(s)$ is a polynomial then $P(D)$ is called a polynomial differential operator.

**Example 12.12.** Suppose $P(s) = s^2 + 8s + 7$. What is $P(D)$? Compute $P(D)$ applied to $f(t) = t^3 + 2t + 5$. Compute $P(D)$ applied to $g(t) = e^{2t}$.

*Solution:* $P(D) = D^2 + 8D + 7I$. (The $I$ in $7I$ is the identity operator.) To compute $P(D)f$ we compute all the terms and sum them up:

$$\begin{aligned} f(t) &= t^3 + 2t + 5 \\ Df(t) &= 3t^2 + 2 \\ D^2 f(t) &= 6t \end{aligned}$$

Therefore: $(D^2 + 8D + 7I)f = 6t + 8(3t^2 + 2) + 7(t^3 + 2t + 5) = 7t^3 + 24t^2 + 20t + 51.$

$$\begin{aligned} g(t) &= e^{2t} \\ Dg(t) &= 2e^{2t} \\ D^2 g(t) &= 4e^{2t} \end{aligned}$$

Therefore: $(D^2 + 8D + 7I)g = 4\mathrm{e}^{2t} + 8(2)\mathrm{e}^{2t} + 7\mathrm{e}^{2t} = (4 + 16 + 7)\mathrm{e}^{2t} = P(2)\mathrm{e}^{2t}$.

The substitution rule is a straightforward statement about the derivatives of exponentials.

**Theorem 12.13.** (Substitution rule) For a polynomial differential operator $P(D)$ we have

$$P(D)\mathrm{e}^{st} = P(s)\mathrm{e}^{st}. \tag{11}$$

*Proof.* This is obvious. We 'prove it' by example. Let $P(D) = D^2 + 8D + 7I$. Then

$$P(D)e^{at} = a^2\mathrm{e}^{at} + 8a\mathrm{e}^{at} + 7\mathrm{e}^{at} = (a^2 + 8a + 7)\mathrm{e}^{at} = P(a)\mathrm{e}^{at}.$$

Let's continue to work from this specific example. From it we'll be able to remind you of the general approach to solving constant coefficient differential equations.

**Example 12.14.** Suppose $P(s) = s^2 + 8s + 7$. Find the exponential modes of the equation $P(D)y = 0$.

*Solution:* The exponential modes are solutions of the form $y(t) = \mathrm{e}^{s_0 t}$. Using the substititution rule

$$P(D)\mathrm{e}^{s_0 t} = 0 \;\Leftrightarrow\; P(s_0) = 0.$$

That is, $y(t) = \mathrm{e}^{s_0 t}$ is a mode exactly when $s_0$ is a root of $P(s)$. The roots of $P(s)$ are $-1$, $-7$. So the modal solutions are

$$y_1(t) = \mathrm{e}^{-t} \quad \text{and} \quad y_2(t) = \mathrm{e}^{-7t}.$$

**Example 12.15.** Redo the previous example using the Laplace transform.

*Solution:* For this we solve the differential equation with arbitrary initial conditions:

$$P(D)y = y'' + 8y' + 7y = 0; \quad y(0) = c_1,\, y'(0) = c_2.$$

Let $Y(s) = \mathcal{L}(y; s)$. Applying the Laplace transform to the equation we get

$$(s^2Y(s) - sy(0) - y'(0)) + 8(sY(s) - y(0)) + 7Y(s) = 0$$

Algebra:

$$(s^2 + 8s + 7)Y(s) - sc_1 - c_2 - 8c_1 = 0 \;\Leftrightarrow\; Y = \frac{sc_1 + 8c_1 + c_2}{s^2 + 8s + 7}$$

Factoring the denominator and using partial fractions, we get

$$Y(s) = \frac{sc_1 + 8c_1 + c_2}{s^2 + 8s + 7} = \frac{sc_1 + 8c_1 + c_2}{(s+1)(s+7)} = \frac{A}{s+1} + \frac{B}{s+7}.$$

We are unconcerned with the exact values of $A$ and $B$. Taking the Laplace inverse we get

$$y(t) = A\mathrm{e}^{-t} + B\mathrm{e}^{-7t}.$$

That is, $y(t)$ is a linear combination of the exponential modes.

You should notice that the denominator in the expression for $Y(s)$ is none other than the characteristic polynomial $P(s)$.

### 12.7.2 The system function

**Example 12.16.** With the same $P(s)$ as in Example 12.12 solve the inhomogeneous DE with rest initial conditions: $P(D)y = f(t), \quad y(0) = 0, \, y'(0) = 0$.

*Solution:* Taking the Laplace transform of the equation we get

$$P(s)Y(s) = F(s).$$

Therefore

$$Y(s) = \frac{1}{P(s)} F(s)$$

We can't find $y(t)$ explicitly because $f(t)$ isn't specified.

But, we can make the following definitions and observations. Let $G(s) = 1/P(s)$. If we declare $f$ to be the input and $y$ the output of this linear time invariant system, then $G(s)$ is called the system function. So, we have

$$Y(s) = G(s) \cdot F(s). \tag{12}$$

The formula $Y = G \cdot F$ can be phrased as

$$\text{output} = \text{system function} \times \text{input}.$$

Note well, the roots of $P(s)$ correspond to the exponential modes of the system, i.e. the poles of $G(s)$ correspond to the exponential modes.

The system is called stable if the modes all decay to 0 as $t$ goes to infinity. That is, if all the poles have negative real part.

**Example 12.17.** This example is to emphasize that not all system functions are of the form $1/P(s)$. Consider the system modeled by the differential equation

$$P(D)x = Q(D)f,$$

where $P$ and $Q$ are polynomials. Suppose we consider $f$ to be the input and $x$ to be the ouput. Find the system function.

*Solution:* If we start with rest initial conditions for $x$ and $f$ then the Laplace transform gives $P(s)X(s) = Q(s)F(s)$ or

$$X(s) = \frac{Q(s)}{P(s)} \cdot F(s)$$

Using the formulation

$$\text{output} = \text{system function} \times \text{input},$$

we see that the system function is

$$G(s) = \frac{Q(s)}{P(s)}.$$

Note that when $f(t) = 0$ the differential equation becomes $P(D)x = 0$. If we make the assumption that the $Q(s)/P(s)$ is in reduced form, i.e. $P$ and $Q$ have no common zeros, then the modes of the system (which correspond to the roots of $P(s)$) are still the poles of the system function.

**Comments.** All LTI systems have system functions. They are not even all of the form $Q(s)/P(s)$. But, in the $s$-domain, the output is always the system function times the input. If the system function is not rational then it may have an infinite number of poles. Stability is harder to characterize, but under some reasonable assumptions the system will be stable if all the poles are in the left half-plane.

The system function is also called the transfer function. You can think of it as describing how the system transfers the input to the output.

## 12.8 Laplace inverse

Up to now we have computed the inverse Laplace transform by table lookup. For example, $\mathcal{L}^{-1}(1/(s-a)) = e^{at}$. To do this properly we should first check that the Laplace transform has an inverse.

We start with the bad news: Unfortunately this is not strictly true. There are many functions with the same Laplace transform. We list some of the ways this can happen.

1. If $f(t) = g(t)$ for $t \geq 0$, then clearly $F(s) = G(s)$. Since the Laplace transform only concerns $t \geq 0$, the functions can differ completely for $t < 0$.

2. Suppose $f(t) = e^{at}$ and

$$g(t) = \begin{cases} f(t) & \text{for } t \neq 1 \\ 0 & \text{for } t = 1. \end{cases}.$$

That is, $f$ and $g$ are the same except we arbitrarily assigned them different values at $t = 1$. Then, since the integrals won't notice the difference at one point, $F(s) = G(s) = 1/(s-a)$. In this sense it is impossible to define $\mathcal{L}^{-1}(F)$ uniquely.

The good news is that the inverse exists as long as we consider two functions that only differ on a negligible set of points the same. In particular, we can make the following claim.

**Theorem.** Suppose $f$ and $g$ are continuous and $F(s) = G(s)$ for all $s$ with $\text{Re}(s) > a$ for some $a$. Then $f(t) = g(t)$ for $t \geq 0$.

This theorem can be stated in a way that includes piecewise continuous functions. Such a statement takes more care, which would obscure the basic point that the Laplace transform has a unique inverse up to some, for us, trivial differences.

We start with a few examples that we can compute directly.

**Example 12.18.** Let

$$f(t) = e^{at}.$$

So,

$$F(s) = \frac{1}{s-a}.$$

Show

$$f(t) = \sum \text{Res}(F(s)e^{st}) \tag{13}$$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\, ds \tag{14}$$

The sum is over all poles of $e^{st}/(s-a)$. As usual, we only consider $t > 0$.

Here, $c > \mathrm{Re}(a)$ and the integral means the path integral along the vertical line $x = c$.

*Solution:* Proving Equation 13 is straightforward: It is clear that

$$\frac{\mathrm{e}^{st}}{s-a}$$

has only one pole which is at $s = a$. Since,

$$\sum \mathrm{Res}\left(\frac{\mathrm{e}^{st}}{s-a}, a\right) = \mathrm{e}^{at}$$

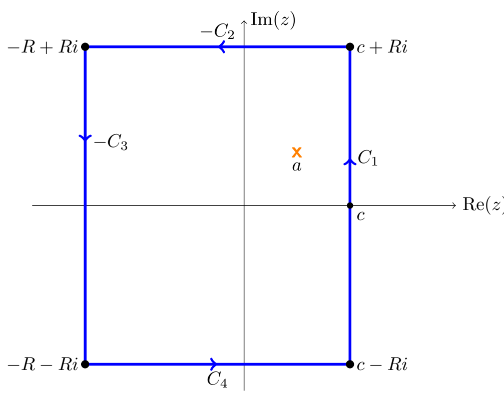
we have proved Equation 13.

Proving Equation 14 is more involved. We should first check the convergence of the integral. In this case, $s = c + iy$, so the integral is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\mathrm{e}^{st}\,ds = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{(c+iy)t}}{c+iy-a}\,i\,dy = \frac{\mathrm{e}^{ct}}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{iyt}}{c+iy-a}\,dy.$$

The (conditional) convergence of this integral follows using exactly the same argument as in the example near the end of Topic 9 on the Fourier inversion formula for $f(t) = \mathrm{e}^{at}$. That is, the integrand is a decaying oscillation, around 0, so its integral is also a decaying oscillation around some limiting value.

Now we use the contour shown below.




We will let $R$ go to infinity and use the following steps to prove Equation 14.

1. The residue theorem guarantees that if the curve is large enough to contain $a$ then

$$\frac{1}{2\pi i}\int_{C_1-C_2-C_3+C_4} \frac{\mathrm{e}^{st}}{s-a}\,ds = \sum \mathrm{Res}\left(\frac{\mathrm{e}^{st}}{s-a}, a\right) = \mathrm{e}^{at}.$$

2. In a moment we will show that the integrals over $C_2$, $C_3$, $C_4$ all go to 0 as $R \to \infty$.

3. Clearly as $R$ goes to infinity, the integral over $C_1$ goes to the integral in Equation 14

Putting these steps together we have

$$\mathrm{e}^{at} = \lim_{R\to\infty}\int_{C_1-C_2-C_3+C_4} \frac{\mathrm{e}^{st}}{s-a}\,ds = \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{e}^{st}}{s-a}\,ds.$$

Except for proving the claims in step 2, this proves Equation 14.

To verify step 2 we look at one side at a time.

$C_2$: $C_2$ is parametrized by $s = \gamma(u) = u + iR$, with $-R \le u \le c$. So,

$$\left|\int_{C_2} \frac{\mathrm{e}^{st}}{s-a}\, ds\right| = \int_{-R}^{c} \left|\frac{\mathrm{e}^{(u+iR)t}}{u+iR-a}\right| \le \int_{-R}^{c} \frac{\mathrm{e}^{ut}}{R}\, du = \frac{\mathrm{e}^{ct} - \mathrm{e}^{-Rt}}{tR}.$$

Since $c$ and $t$ are fixed, it's clear this goes to 0 as $R$ goes to infinity.

The bottom $C_4$ is handled in exactly the same manner as the top $C_2$.

$C_3$: $C_3$ is parametrized by $s = \gamma(u) = -R + iu$, with $-R \le u \le R$. So,

$$\left|\int_{C_3} \frac{\mathrm{e}^{st}}{s-a}\, ds\right| = \int_{-R}^{R} \left|\frac{\mathrm{e}^{(-R+iu)t}}{-R+iu-a}\right| \le \int_{-R}^{R} \frac{\mathrm{e}^{-Rt}}{R+a}\, du = \frac{\mathrm{e}^{-Rt}}{R+a} \int_{-R}^{R} du = \frac{2R\mathrm{e}^{-Rt}}{R+a}.$$

Since $a$ and $t > 0$ are fixed, it's clear this goes to 0 as $R$ goes to infinity.

**Example 12.19.** Repeat the previous example with $f(t) = t$ for $t > 0$, $F(s) = 1/s^2$.

This is similar to the previous example. Since $F$ decays like $1/s^2$ we can actually allow $t \ge 0$

**Theorem 12.20.** Laplace inversion 1. Assume $f$ is continuous and of exponential type $a$. Then for $c > a$ we have

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\mathrm{e}^{st}\, ds. \tag{15}$$

As usual, this formula holds for $t > 0$.

*Proof.* The proof uses the Fourier inversion formula. We will just accept this theorem for now. Example 12.18 above illustrates the theorem.

**Theorem 12.21.** Laplace inversion 2. Suppose $F(s)$ has a finite number of poles and decays like $1/s$ (or faster). Define

$$f(t) = \sum \mathrm{Res}(F(s)\mathrm{e}^{st}, p_k), \text{ where the sum is over all the poles } p_k. \tag{16}$$

Then $\mathcal{L}(f; s) = F(s)$

*Proof.* Proof given in class. To be added here. The basic ideas are present in the examples above, though it requires a fairly clever choice of contours.

The integral inversion formula in Equation 15 can be viewed as writing $f(t)$ as a 'sum' of exponentials. This is extremely useful. For example, for a linear system if we know how the system responds to input $f(t) = \mathrm{e}^{at}$ for all $a$, then we know how it responds to any input by writing it as a 'sum' of exponentials.

## 12.9 Delay and feedback.

Let $f(t) = 0$ for $t < 0$. Fix $a > 0$ and let $h(t) = f(t-a)$. So, $h(t)$ is a delayed version of the signal $f(t)$. The Laplace property Equation 10 says

$$H(s) = \mathrm{e}^{-as}F(s),$$

where $H$ and $F$ are the Laplace transforms of $h$ and $f$ respectively.

Now, suppose we have a system with system function $G(s)$. (Again, called the open loop system.) As before, can feed the output back through the system. But, instead of just multiplying the output by a scalar we can delay it also. This is captured by the feedback factor $k\mathrm{e}^{-as}$.

The system function for the closed loop system is

$$G_{CL}(s) = \frac{G}{1 + k\mathrm{e}^{-as}G}$$

Note even if you start with a rational function the system function of the closed loop with delay is not rational. Usually it has an infinite number of poles.

**Example 12.22.** Suppose $G(s) = 1$, $a = 1$ and $k = 1$ find the poles of $G_{CL}(s)$.

*Solution:*

$$G_{CL}(s) = \frac{1}{1 + \mathrm{e}^{-s}}.$$

So the poles occur where $\mathrm{e}^{-s} = -1$, i.e. at $in\pi$, where $n$ is an odd integer. There are an infinite number of poles on the imaginary axis.

**Example 12.23.** Suppose $G(s) = 1$, $a = 1$ and $k = 1/2$ find the poles of $G_{CL}(s)$. Is the closed loop system stable?

*Solution:*

$$G_{CL}(s) = \frac{1}{1 + \mathrm{e}^{-s}/2}.$$

So the poles occur where $\mathrm{e}^{-s} = -2$, i.e. at $-\log(2) + in\pi$, where $n$ is an odd integer. Since $-\log(2) < 0$, there are an infinite number of poles in the left half-plane. With all poles in the left half-plane, the system is stable.

**Example 12.24.** Suppose $G(s) = 1$, $a = 1$ and $k = 2$ find the poles of $G_{CL}(s)$. Is the closed loop system stable?

*Solution:*

$$G_{CL}(s) = \frac{1}{1 + 2\mathrm{e}^{-s}}.$$

So the poles occur where $\mathrm{e}^{-s} = -1/2$, i.e. at $\log(2) + in\pi$, where $n$ is an odd integer. Since $\log(2) > 0$, there are an infinite number of poles in the right half-plane. With poles in the right half-plane, the system is not stable.

**Remark.** If $\mathrm{Re}(s)$ is large enough we can express the system function

$$G(s) = \frac{1}{1 + k\mathrm{e}^{-as}}$$

as a geometric series

$$\frac{1}{1 + k\mathrm{e}^{-as}} = 1 - k\mathrm{e}^{-as} + k^2\mathrm{e}^{-2as} - k^3\mathrm{e}^{-3as} + \ldots$$

So, for input $F(s)$, we have output

$$X(s) = G(s)F(s) = F(s) - k\mathrm{e}^{-as}F(s) + k^2\mathrm{e}^{-2as}F(s) - k^3\mathrm{e}^{-3as}F(s) + \ldots$$

Using the shift formula Equation 10, we have

$$x(t) = f(t) - kf(t-a) + k^2 f(t-2a) - k^3 f(t-3a) + \dots$$

(This is not really an infinite series because $f(t) = 0$ for $t < 0$.) If the input is bounded and $k < 1$ then even for large $t$ the series is bounded. So bounded input produces bounded output –this is also what is meant by stability. On the other hand if $k > 1$, then bounded input can lead to unbounded output –this is instability.

## 12.10 Table of Laplace transforms

### Properties and Rules

We assume that $f(t) = 0$ for $t < 0$.

| **Function** | **Transform** | |
|---|---|---|
| $f(t)$ | $F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\, dt$ | (Definition) |
| $a\, f(t) + b\, g(t)$ | $a\, F(s) + b\, G(s)$ | (Linearity) |
| $\mathrm{e}^{at} f(t)$ | $F(s-a)$ | ($s$-shift) |
| $f'(t)$ | $sF(s) - f(0)$ | |
| $f''(t)$ | $s^2 F(s) - sf(0) - f'(0)$ | |
| $f^{(n)}(t)$ | $s^n F(s) - s^{n-1} f(0) - \quad \cdots - f^{(n-1)}(0)$ | |
| $tf(t)$ | $-F'(s)$ | |
| $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ | |
| $f(t-a)$ | $\mathrm{e}^{-as} F(s)$ | ($t$-translation or $t$-shift) |
| $\int_0^t f(\tau)\, d\tau$ | $\dfrac{F(s)}{s}$ | (integration rule) |
| $\dfrac{f(t)}{t}$ | $\int_s^\infty F(\sigma)\, d\sigma$ | |

## Function Table

| Function | Transform | Region of convergence |
|---|---|---|
| $1$ | $1/s$ | $\mathrm{Re}(s) > 0$ |
| $\mathrm{e}^{at}$ | $1/(s-a)$ | $\mathrm{Re}(s) > \mathrm{Re}(a)$ |
| $t$ | $1/s^2$ | $\mathrm{Re}(s) > 0$ |
| $t^n$ | $n!/s^{n+1}$ | $\mathrm{Re}(s) > 0$ |
| $\cos(\omega t)$ | $s/(s^2+\omega^2)$ | $\mathrm{Re}(s) > 0$ |
| $\sin(\omega t)$ | $\omega/(s^2+\omega^2)$ | $\mathrm{Re}(s) > 0$ |
| $\mathrm{e}^{at}\cos(\omega t)$ | $(s-a)/((s-a)^2+\omega^2)$ | $\mathrm{Re}(s) > \mathrm{Re}(a)$ |
| $\mathrm{e}^{at}\sin(\omega t)$ | $\omega/((s-a)^2+\omega^2)$ | $\mathrm{Re}(s) > \mathrm{Re}(a)$ |
| $\delta(t)$ | $1$ | all $s$ |
| $\delta(t-a)$ | $\mathrm{e}^{-as}$ | all $s$ |
| $\cosh(kt) = \dfrac{\mathrm{e}^{kt}+\mathrm{e}^{-kt}}{2}$ | $s/(s^2-k^2)$ | $\mathrm{Re}(s) > k$ |
| $\sinh(kt) = \dfrac{\mathrm{e}^{kt}-\mathrm{e}^{-kt}}{2}$ | $k/(s^2-k^2)$ | $\mathrm{Re}(s) > k$ |
| $\dfrac{1}{2\omega^3}(\sin(\omega t) - \omega t\cos(\omega t))$ | $\dfrac{1}{(s^2+\omega^2)^2}$ | $\mathrm{Re}(s) > 0$ |
| $\dfrac{t}{2\omega}\sin(\omega t)$ | $\dfrac{s}{(s^2+\omega^2)^2}$ | $\mathrm{Re}(s) > 0$ |
| $\dfrac{1}{2\omega}(\sin(\omega t) + \omega t\cos(\omega t))$ | $\dfrac{s^2}{(s^2+\omega^2)^2}$ | $\mathrm{Re}(s) > 0$ |
| $t^n\mathrm{e}^{at}$ | $n!/(s-a)^{n+1}$ | $\mathrm{Re}(s) > \mathrm{Re}(a)$ |
| $\dfrac{1}{\sqrt{\pi t}}$ | $\dfrac{1}{\sqrt{s}}$ | $\mathrm{Re}(s) > 0$ |
| $t^a$ | $\dfrac{\Gamma(a+1)}{s^{a+1}}$ | $\mathrm{Re}(s) > 0$ |