Jim Lambers
MAT 417/517
Spring Semester 2013-14
Lecture 18 Notes

These notes correspond to Lesson 25 in the text.

The Finite Fourier Transforms

When solving a PDE on a finite interval $0 < x < L$, whether it be the heat equation or wave equation, it can be very helpful to use a *finite Fourier transform*. In particular, we have the *finite sine transform*

$$S_n = S[f] = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx, \quad n = 1, 2, \ldots,$$

with its *inverse sine transform*

$$S^{-1}[S_n] = f(x) = \sum_{n=1}^{\infty} S_n \sin(n\pi x/L).$$

This transform should be used with *Dirichlet* boundary conditions, that specify the value of u at $x = 0$ and $x = L$.

When *Neumann* boundary conditions are used, that specify the value of u_x at $x = 0$ and $x = L$, it is best to use the *finite cosine transform*

$$C_n = C[f] = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)\,dx, \quad n = 0, 1, 2, \ldots,$$

with its *inverse sine transform*

$$C^{-1}[C_n] = f(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(n\pi x/L).$$

Both of these transforms can be used to reduce a PDE to an ODE.

Examples of the Sine Transform

Consider the function $f(x) = 1$ on $(0, 1)$. If we apply the finite sine transform to this function, we obtain

$$\begin{aligned} S_n &= 2\int_0^1 \sin(n\pi x)\,dx \\ &= -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^1 \\ &= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}. \end{aligned}$$

Applying the inverse sine transform yields

$$1 = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin[(2n-1)\pi x].$$

Properties of the Transforms

To apply these transforms to PDEs, we must know how to transform appropriate derivatives. We have the following rules:

$$
\begin{aligned}
S[u_t] &= \frac{dS[u]}{dt}, \quad S[u_{tt}] = \frac{d^2S[u]}{dt^2}, \\
C[u_t] &= \frac{dC[u]}{dt}, \quad C[u_{tt}] = \frac{d^2C[u]}{dt^2}, \\
S[u_{xx}] &= -[n\pi/L]^2 S[u] + \frac{2n\pi}{L^2}[u(0,t) + (-1)^{n+1}u(L,t)], \\
C[u_{xx}] &= -[n\pi/L]^2 C[u] - \frac{2}{L}[u_x(0,t) + (-1)^{n+1}u_x(L,t)].
\end{aligned}
$$

The last two rules can be obtained by applying integration by parts twice.

Solving Problems via Finite Transforms

We illustrate the use of finite Fourier transforms by solving the IBVP

$$
\begin{gathered}
u_{tt} = u_{xx} + \sin(\pi x), \quad 0 < x < 1, \quad t > 0, \\
u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \\
u(x,0) = 1, \quad u_t(x,0) = 0, \quad 0 < x < 1.
\end{gathered}
$$

Because this problem has Dirichlet boundary conditions, we use the finite sine transform. From the preceding example, the transform of the initial conditions are

$$
S_n(0) = \begin{cases} \dfrac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}, \quad S_n'(0) = 0.
$$

Using the definition and aforementioned properties, we obtain the transform of the PDE,

$$
\begin{aligned}
S_1''(t) &= -\pi^2 S_1(t) + 1, \\
S_n''(t) &= -(n\pi)^2 S_n(t), \quad n = 2, 3, \ldots.
\end{aligned}
$$

The ODE for $S_1(t)$ is nonhomogeneous, and can be solved using either the method of undetermined coefficients or variation of parameters. The general solution is

$$
S_1(t) = A\cos(\pi t) + B\sin(pit) + C,
$$

where A, B and C are constants. Substituting this form of the solution into the ODE and initial conditions yields

$$
S_1(t) = \left(\frac{4}{\pi} - \frac{1}{\pi^2}\right)\cos(\pi t) + \frac{1}{\pi^2}.
$$

The ODEs for $S_n(t)$, $n > 1$, are homogeneous and can easily be solved to obtain

$$
S_n(t) = \begin{cases} \dfrac{4}{n\pi}\cos(n\pi t) & n = 3, 5, 7, \ldots, \\ 0 & n = 2, 4, 6, \ldots \end{cases}.
$$

Applying the inverse sine transform, we conclude that the solution is

$$
u(x,t) = \left[\left(\frac{4}{\pi} - \frac{1}{\pi^2}\right)\cos(\pi t) + \frac{1}{\pi^2}\right]\sin(\pi x) + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\cos[(2n+1)\pi t]\sin[(2n+1)\pi x].
$$