7
B-Spline Approximation

B-spline methods for curves and surfaces were first proposed in the 1940s but were seriously developed only in the 1970s, by several researchers, most notably R. Riesenfeld. They have been studied extensively, have been considerably extended since the 1970s, and much is currently known about them. The designation "B" stands for Basis, so the full name of this approach to curve and surface design is the basis spline. This chapter discusses the important types of B-spline curves and surfaces, including the most versatile one, the nonuniform rational B-spline (NURBS, Section 7.14).

The B-spline curve overcomes the main disadvantages of the Bézier curve which are (1) the degree of the Bézier curve depends on the number of control points, (2) it offers only global control, and (3) individual segments are easy to connect with C^1 continuity, but C^2 is difficult to obtain. The B-spline curve features local control and any desired degree of continuity. To obtain C^n continuity, the individual spline segments have to be polynomials of degree n. The B-spline curve is an approximating curve and is therefore defined by control points. However, in addition to the control points, the user has to specify the values of certain quantities called "knots." They are real numbers that offer additional control over the shape of the curve. The basic approach taken in the first part of this chapter ignores the knots, but they are introduced in Section 7.8 and their effect on the curve is explored.

There are several types of B-splines. In the *uniform* (also called periodic) B-spline (Sections 7.1 and 7.2), the knot values are uniformly spaced and all the weight functions have the same shape and are shifted with respect to each other. In the *nonuniform* B-spline (Section 7.11), the knots are specified by the user and the weight functions are generally different. There is also an *open uniform* B-spline (Section 7.10), where the knots are not uniform but are specified in a simple way. In a *rational* B-spline (Section 7.14), the weight functions are in the form of a ratio of two polynomials. In a *nonrational* B-spline, they are polynomials in t. The B-spline is an approximating curve based on control points, but there is also an *interpolating* version that passes through the points (Section 7.7). Section 7.4 shows how tension can be added to the B-spline.

B-splines are mathematically more sophisticated than other types of splines, so we start with a gentle introduction. We first use basic assumptions to derive the expressions for the quadratic and cubic uniform B-splines directly and without mentioning knots. We then show how to extend the derivations to uniform B-splines of any order. Following this, we discuss a different, recursive formulation of the weight functions of the uniform, open uniform, and nonuniform B-splines.

7.1 The Quadratic Uniform B-Spline

We start with the quadratic uniform B-spline. We assume that $n+1$ control points, $\mathbf{P}_0$, $\mathbf{P}_1,\ldots,$ $\mathbf{P}_n$, are given and we want to construct a spline curve where each segment $\mathbf{P}_i(t)$ is a quadratic parametric polynomial based on three points, $\mathbf{P}_{i-1}$, $\mathbf{P}_i$, and $\mathbf{P}_{i+1}$. We require that the segments connect with C^1 continuity (only cubic and higher-degree polynomial segments can have C^2 or higher continuities) and that the entire curve has local control. To achieve all this, we have to give up something and we elect to give up the requirement that a segment will pass through its first and last control points. We denote the start and end points of segment $\mathbf{P}_i(t)$ by $\mathbf{K}_i$ and $\mathbf{K}_{i+1}$, respectively and we call them *joint points*, or just *joints*. These points are still unknown and will have to be determined. Figure 7.1a shows two quadratic segments $\mathbf{P}_1(t)$ and $\mathbf{P}_2(t)$ defined by the four control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. The first segment goes from joint $\mathbf{K}_1$ to joint $\mathbf{K}_2$ and the second segment goes from joint $\mathbf{K}_2$ to joint $\mathbf{K}_3$, where the joints are drawn tentatively and will have to be determined and redrawn. Note that each segment is defined by three control points, so its control polygon has two edges. The first spline segment is defined only by $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$, so any changes in $\mathbf{P}_3$ will not affect it. This is how local control is achieved in a B-spline.

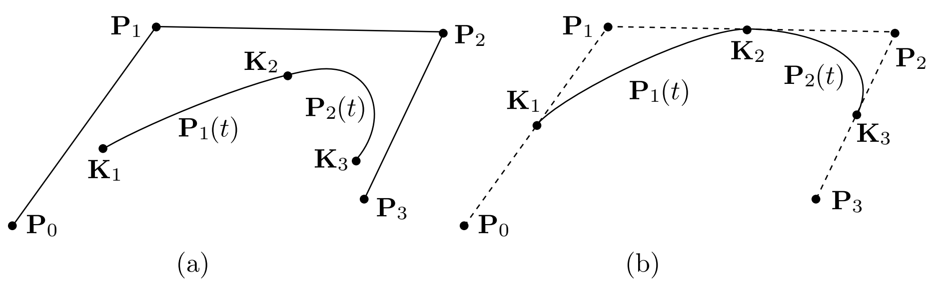


Figure 7.1: The Quadratic Uniform B-Spline.

We use the usual notation for the two segments

$$\mathbf{P}_i(t) = (t^2, t, 1)\mathbf{M}\begin{pmatrix}\mathbf{P}_{i-1}\\ \mathbf{P}_i\\ \mathbf{P}_{i+1}\end{pmatrix}, \quad i = 1, 2, \tag{7.1}$$

where $\mathbf{M}$ is the 3×3 basis matrix whose nine elements have to be calculated. We define three functions $a(t)$, $b(t)$, and $c(t)$ by:

$$\begin{aligned}(t^2,t,1)\mathbf{M} &= (t^2,t,1)\begin{pmatrix} a_2 & b_2 & c_2\\ a_1 & b_1 & c_1\\ a_0 & b_0 & c_0\end{pmatrix}\\ &= (a_2t^2+a_1t+a_0, b_2t^2+b_1t+b_0, c_2t^2+c_1t+c_0)\\ &= \big(a(t),b(t),c(t)\big).\end{aligned} \tag{7.2}$$

The nine elements of $\mathbf{M}$ are determined from the following three requirements:

1. The two segments should meet at a common joint and their tangent vectors should be equal at that point. This is expressed as

$$\mathbf{P}_1(1)=\mathbf{P}_2(0),\quad \mathbf{P}_1^t(1)=\mathbf{P}_2^t(0) \tag{7.3}$$

and produces the explicit equations (where a dot indicates differentiation with respect to t)

$$\begin{aligned} a(1)\mathbf{P}_0+b(1)\mathbf{P}_1+c(1)\mathbf{P}_2 &= a(0)\mathbf{P}_1+b(0)\mathbf{P}_2+c(0)\mathbf{P}_3,\\ \dot a(1)\mathbf{P}_0+\dot b(1)\mathbf{P}_1+\dot c(1)\mathbf{P}_2 &= \dot a(0)\mathbf{P}_1+\dot b(0)\mathbf{P}_2+\dot c(0)\mathbf{P}_3.\end{aligned}$$

Since the control points $\mathbf{P}_i$ are arbitrary and can be any points, we can rewrite these two equations in the form

$$\begin{aligned} a(1)&=0, & \dot a(1)&=0, &&\text{for } \mathbf{P}_0,\\ b(1)&=a(0), & \dot b(1)&=\dot a(0), &&\text{for } \mathbf{P}_1,\\ c(1)&=b(0), & \dot c(1)&=\dot b(0), &&\text{for } \mathbf{P}_2,\\ 0&=c(0), & 0&=\dot c(0), &&\text{for } \mathbf{P}_3.\end{aligned}$$

Using the notation of Equation (7.2), this can be written

$$\begin{aligned} a_2+a_1+a_0&=0, & 2a_2+a_1&=0,\\ b_2+b_1+b_0&=a_0, & 2b_2+b_1&=0,\\ c_2+c_1+c_0&=b_0, & 2c_2+c_1&=0,\\ 0&=c_0, & 0&=c_1.\end{aligned} \tag{7.4}$$

This requirement produces eight equations for the nine unknown matrix elements.

2. The entire curve should be independent of the particular coordinate system used, which implies that the weight functions of each segment should be barycentric, i.e., $a(t)+b(t)+c(t)\equiv 1$. This condition can be written explicitly as

$$a_2+b_2+c_2=0,\quad a_1+b_1+c_1=0,\quad a_0+b_0+c_0=1, \tag{7.5}$$

and these add three more equations.

We now have 11 equations for the nine unknowns, but it is easy to show that only nine of the 11 are independent. The sum of the first two of Equations (7.5) equals the sum of the three equations in the right column of Equation (7.4). Taking this into account, the equations can be solved uniquely, yielding

$$\begin{aligned} a_2 &= 1/2, \quad a_1 = -1, \quad a_0 = 1/2, \\ b_2 &= -1, \quad b_1 = 1, \quad b_0 = 1/2, \\ c_2 &= 1/2, \quad c_1 = 0, \quad c_0 = 0. \end{aligned}$$

The general quadratic B-spline segment, Equation (7.1), can now be written as

$$\begin{aligned} \mathbf{P}_i(t) &= \frac{1}{2}(t^2, t, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_{i-1} \\ \mathbf{P}_i \\ \mathbf{P}_{i+1} \end{pmatrix} \\ &= \frac{1}{2}(t^2 - 2t + 1)\mathbf{P}_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)\mathbf{P}_i + \frac{t^2}{2}\mathbf{P}_{i+1}, \quad i = 1, 2. \end{aligned} \tag{7.6}$$

We are now in a position to determine the start and end points, $\mathbf{K}_i$ and $\mathbf{K}_{i+1}$ of segment i. They are

$$\mathbf{K}_i = \mathbf{P}_i(0) = \frac{1}{2}(\mathbf{P}_{i-1} + \mathbf{P}_i), \quad \mathbf{K}_{i+1} = \mathbf{P}_i(1) = \frac{1}{2}(\mathbf{P}_i + \mathbf{P}_{i+1}).$$

Thus, the quadratic spline segment starts in the middle of the straight segment $\mathbf{P}_{i-1}\mathbf{P}_i$ and ends at the middle of the straight segment $\mathbf{P}_i\mathbf{P}_{i+1}$, as shown in Figure 7.1b.

The tangent vector of the general quadratic B-spline segment is easily obtained from Equation (7.6). It is

$$\mathbf{P}_i^t(t) = \frac{1}{2}(2t, 1, 0) \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_{i-1} \\ \mathbf{P}_i \\ \mathbf{P}_{i+1} \end{bmatrix} = (t-1)\mathbf{P}_{i-1} + (-2t+1)\mathbf{P}_i + t\mathbf{P}_{i+1}. \tag{7.7}$$

The tangent vectors at both ends of the segment are therefore

$$\mathbf{P}^t(0) = \mathbf{P}_i - \mathbf{P}_{i-1}, \quad \mathbf{P}^t(1) = \mathbf{P}_{i+1} - \mathbf{P}_i,$$

i.e., each of them points in the direction of one of the edges of the control polygon of the spline segment.

Since a quadratic spline segment is a polynomial of degree 2, we require continuity of the first derivative only. It is easy to show that the second derivative of our segment is $\mathbf{P}_{i-1} - 2\mathbf{P}_i + \mathbf{P}_{i+1}$. It is constant for a segment but is different for different segments.

Equation (8.4) of Section 8.2 shows a relation between the quadratic B-spline and Bézier curves. A similar relation between the corresponding cubic curves is illustrated in Section 7.5.

Example: Given the four control points $\mathbf{P}_0 = (1,0)$, $\mathbf{P}_1 = (1,1)$, $\mathbf{P}_2 = (2,1)$, and $\mathbf{P}_3 = (2,0)$ (Figure 7.2), the first quadratic spline segment is obtained from Equation (7.6)

$$\begin{aligned}\mathbf{P}_1(t) &= \frac{1}{2}(t^2,t,1)\begin{pmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}\\ &= \frac{1}{2}(t^2-2t+1)(1,0)+\frac{1}{2}(-2t^2+2t+1)(1,1)+\frac{t^2}{2}(2,1)\\ &= (t^2/2+1,-t^2/2+t+1/2).\end{aligned}$$

It starts at joint $\mathbf{K}_1 = \mathbf{P}_1(0) = (1,\frac{1}{2})$ and ends at joint $\mathbf{K}_2 = \mathbf{P}_1(1) = (\frac{3}{2},1)$.

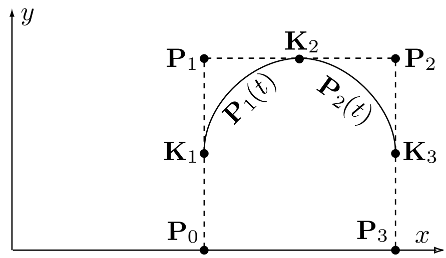


Figure 7.2: A Quadratic Uniform B-Spline Example.

The tangent vector of this segment is obtained from Equation (7.7)

$$\begin{aligned}\mathbf{P}^t_1(t) &= \frac{1}{2}(2t,1,0)\begin{pmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}\\ &= (t-1)(1,0)+(-2t+1)(1,1)+t(2,1)\\ &= (t,1-t).\end{aligned}$$

Thus, the first segment starts going in direction $\mathbf{P}^t_1(0) = (0,1)$ (straight up) and ends going in direction $\mathbf{P}^t_1(1) = (1,0)$ (to the right).

◇ **Exercise 7.1:** Calculate the second segment, its tangent vector, and joint point $\mathbf{K}_3$.

Closed Quadratic B-Splines: Closed curves are sometimes needed and a closed B-spline curve is easy to construct. Given the usual $n+1$ control points, we extend them cyclically to obtain the $n+3$ points

$$\mathbf{P}_n,\ \mathbf{P}_0,\ \mathbf{P}_1,\ \mathbf{P}_2,\ \ldots,\ \mathbf{P}_{n-1},\ \mathbf{P}_n,\ \mathbf{P}_0$$

and compute the curve by applying Equation (7.6) to the $n+1$ geometry vectors

$$\begin{pmatrix}\mathbf{P}_n\\ \mathbf{P}_0\\ \mathbf{P}_1\end{pmatrix}\ \begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}\ \begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\end{pmatrix}\cdots\begin{pmatrix}\mathbf{P}_{n-2}\\ \mathbf{P}_{n-1}\\ \mathbf{P}_n\end{pmatrix}\ \begin{pmatrix}\mathbf{P}_{n-1}\\ \mathbf{P}_n\\ \mathbf{P}_0\end{pmatrix}.$$

Example: Given the four control points $\mathbf{P}_0 = (1,0)$, $\mathbf{P}_1 = (1,1)$, $\mathbf{P}_2 = (2,1)$, and $\mathbf{P}_3 = (2,0)$ of the previous example, it is easy to close the curve by calculating the two additional segments

$$\begin{aligned}
\mathbf{P}_0(t) &= \frac{1}{2}(t^2,t,1)\begin{pmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_3\\ \mathbf{P}_0\\ \mathbf{P}_1\end{pmatrix}\\
&= \frac{1}{2}(t^2-2t+1)(2,0)+\frac{1}{2}(-2t^2+2t+1)(1,0)+\frac{t^2}{2}(1,1)\\
&= (t^2/2-t+3/2, t^2/2).\\
\mathbf{P}_3(t) &= \frac{1}{2}(t^2,t,1)\begin{pmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_2\\ \mathbf{P}_3\\ \mathbf{P}_0\end{pmatrix}\\
&= \frac{1}{2}(t^2-2t+1)(2,1)+\frac{1}{2}(-2t^2+2t+1)(2,0)+\frac{t^2}{2}(1,0)\\
&= (-t^2/2+2, t^2/2-t+1/2).
\end{aligned}$$

The four segments connect the four joint points $(1,1/2)$, $(3/2,1)$, $(2,1/2)$, $(3/2,0)$ and back to $(1,1/2)$.

The B stands for "basis".

7.2 The Cubic Uniform B-Spline

This curve is again defined by $n+1$ control points and it consists of spline segments $\mathbf{P}_i(t)$, each a PC defined by four control points $\mathbf{P}_{i-1}$, $\mathbf{P}_i$, $\mathbf{P}_{i+1}$, and $\mathbf{P}_{i+2}$. The general form of segment i is therefore

$$\mathbf{P}_i(t) = (t^3,t^2,t,1)\mathbf{M}\begin{pmatrix}\mathbf{P}_{i-1}\\ \mathbf{P}_i\\ \mathbf{P}_{i+1}\\ \mathbf{P}_{i+2}\end{pmatrix}, \tag{7.8}$$

where $\mathbf{M}$ is a 4×4 matrix whose 16 elements have to be determined by translating the constraints on the curve into 16 equations and solving them. The constraints are (1) two segments should meet with C^2 continuity and (2) the entire curve should be independent of the particular coordinate system. As in the quadratic case, we give up the requirement that a segment $\mathbf{P}_i(t)$ starts and ends at control points, and we denote its extreme points by $\mathbf{K}_i$ and $\mathbf{K}_{i+1}$. These joints can be computed as soon as the expression for the segment is derived. Figure 7.3a shows a tentative design for two cubic segments.

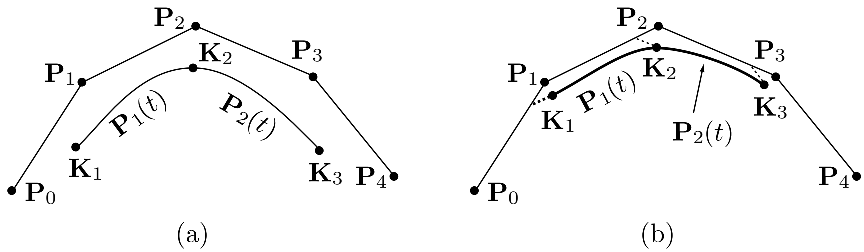


Figure 7.3: The Cubic Uniform B-Spline.

We start the derivation by writing

$$\begin{aligned}(t^3,t^2,t,1)\mathbf{M} &= (t^3,t^2,t,1)\begin{pmatrix} a_3 & b_3 & c_3 & d_3\\ a_2 & b_2 & c_2 & d_2\\ a_1 & b_1 & c_1 & d_1\\ a_0 & b_0 & c_0 & d_0\end{pmatrix}\\ &= (a_3t^3+a_2t^2+a_1t+a_0, b_3t^3+b_2t^2+b_1t+b_0,\\ &\quad c_3t^3+c_2t^2+c_1t+c_0, d_3t^3+d_2t^2+d_1t+d_0)\\ &= \big(a(t),b(t),c(t),d(t)\big).\end{aligned}$$

The first three constraints are expressed by

$$\mathbf{P}_1(1)=\mathbf{P}_2(0),\quad \mathbf{P}_1^t(1)=\mathbf{P}_2^t(0),\quad \mathbf{P}_1^{tt}(1)=\mathbf{P}_2^{tt}(0),$$

or, explicitly

$$\begin{aligned}a(1)\mathbf{P}_0+b(1)\mathbf{P}_1+c(1)\mathbf{P}_2+d(1)\mathbf{P}_3 &= a(0)\mathbf{P}_1+b(0)\mathbf{P}_2+c(0)\mathbf{P}_3+d(0)\mathbf{P}_4,\\ \dot a(1)\mathbf{P}_0+\dot b(1)\mathbf{P}_1+\dot c(1)\mathbf{P}_2+\dot d(1)\mathbf{P}_3 &= \dot a(0)\mathbf{P}_1+\dot b(0)\mathbf{P}_2+\dot c(0)\mathbf{P}_3+\dot d(0)\mathbf{P}_4,\\ \ddot a(1)\mathbf{P}_0+\ddot b(1)\mathbf{P}_1+\ddot c(1)\mathbf{P}_2+\ddot d(1)\mathbf{P}_3 &= \ddot a(0)\mathbf{P}_1+\ddot b(0)\mathbf{P}_2+\ddot c(0)\mathbf{P}_3+\ddot d(0)\mathbf{P}_4.\end{aligned}$$

Using the definitions of $a(t)$ and its relatives, this can be written explicitly as

$$\begin{gathered}a_3+a_2+a_1+a_0=0,\quad 3a_3+2a_2+a_1=0,\quad 6a_3+2a_2=0,\\ b_3+b_2+b_1+b_0=a_0,\quad 3b_3+2b_2+b_1=a_1,\quad 6b_3+2b_2=2a_2,\\ c_3+c_2+c_1+c_0=b_0,\quad 3c_3+2c_2+c_1=b_1,\quad 6c_3+2c_2=2b_2,\\ d_3+d_2+d_1+d_0=c_0,\quad 3d_3+2d_2+d_1=c_1,\quad 6d_3+2d_2=2c_2,\\ 0=d_0,\quad 0=d_1,\quad 0=2d_2.\end{gathered}\tag{7.9}$$

These are 15 equations for the 16 unknowns.

We already know from the quadratic case that the weight functions of each segment should be barycentric, i.e., $a(t)+b(t)+c(t)+d(t)\equiv 1$. This condition can be written explicitly as

$$\begin{gathered}a_3+b_3+c_3+d_3=0,\quad a_2+b_2+c_2+d_2=0,\\ a_1+b_1+c_1+d_1=0,\quad a_0+b_0+c_0+d_0=1,\end{gathered}\tag{7.10}$$

and they add four more equations. We now have 19 equations, but only 16 of them are independent, since the first three equations of Equation (7.10) can be obtained by summing the first four equations of the left column of Equation (7.9). The system of equations can therefore be uniquely solved and the solutions are

$$\begin{gathered}
a_3=-1/6,\quad a_2=1/2,\quad a_1=-1/2,\quad a_0=1/6,\\
b_3=1/2,\quad b_2=-1,\quad b_1=0,\quad b_0=2/3,\\
c_3=-1/2,\quad c_2=1/2,\quad c_1=1/2,\quad c_0=1/6,\\
d_3=1/6,\quad d_2=0,\quad d_1=0,\quad d_0=0.
\end{gathered}$$

The cubic B-spline segment can now be expressed as

$$\begin{aligned}
\mathbf{P}_i(t)&=\frac{1}{6}(t^3,t^2,t,1)\begin{pmatrix}-1&3&-3&1\\3&-6&3&0\\-3&0&3&0\\1&4&1&0\end{pmatrix}\begin{pmatrix}\mathbf{P}_{i-1}\\\mathbf{P}_i\\\mathbf{P}_{i+1}\\\mathbf{P}_{i+2}\end{pmatrix}\\
&=\frac{1}{6}(-t^3+3t^2-3t+1)\mathbf{P}_{i-1}+\frac{1}{6}(3t^3-6t^2+4)\mathbf{P}_i\\
&\quad+\frac{1}{6}(-3t^3+3t^2+3t+1)\mathbf{P}_{i+1}+\frac{t^3}{6}\mathbf{P}_{i+2}.
\end{aligned}\tag{7.11}$$

The two extreme points are therefore

$$\mathbf{K}_i=\mathbf{P}_i(0)=\frac{1}{6}(\mathbf{P}_{i-1}+4\mathbf{P}_i+\mathbf{P}_{i+1}),\ \text{and}\ \mathbf{K}_{i+1}=\mathbf{P}_i(1)=\frac{1}{6}(\mathbf{P}_i+4\mathbf{P}_{i+1}+\mathbf{P}_{i+2}).$$

In order to interpret them geometrically, we write them as

$$\begin{aligned}
\mathbf{K}_i&=\left(\frac{1}{6}\mathbf{P}_{i-1}+\frac{5}{6}\mathbf{P}_i\right)+\frac{1}{6}\left(\mathbf{P}_{i+1}-\mathbf{P}_i\right),\\
\mathbf{K}_{i+1}&=\left(\frac{1}{6}\mathbf{P}_i+\frac{5}{6}\mathbf{P}_{i+1}\right)+\frac{1}{6}\left(\mathbf{P}_{i+2}-\mathbf{P}_{i+1}\right).
\end{aligned}\tag{7.12}$$

Point $\mathbf{K}_i$ is the sum of the point $(\frac{1}{6}\mathbf{P}_{i-1}+\frac{5}{6}\mathbf{P}_i)$ and one-sixth of the vector $(\mathbf{P}_{i+1}-\mathbf{P}_i)$. Point $\mathbf{K}_{i+1}$ has a similar interpretation. Both are shown in Figure 7.3b.

◇ **Exercise 7.2:** Show another way to interpret $\mathbf{P}_i(0)$ and $\mathbf{P}_i(1)$ geometrically.

Users, especially those familiar with Bézier curves, find it counterintuitive that the B-spline curve does not start and end at its terminal control points. This "inconvenient" feature can be modified—and the curve made to start and end at its extreme points—by adding two *phantom* endpoints, $\mathbf{P}_{-1}$ and $\mathbf{P}_{n+1}$, at both ends of the curve, and placing those points at locations that would force the curve to start at $\mathbf{P}_0$ and end at $\mathbf{P}_n$. The calculation of this case is simple. The first segment starts at $\frac{1}{6}[\mathbf{P}_{-1}+4\mathbf{P}_0+\mathbf{P}_1]$. This value will equal $\mathbf{P}_0$ if we select $\mathbf{P}_{-1}=2\mathbf{P}_0-\mathbf{P}_1$. Similarly, the last segment ends at $\frac{1}{6}[\mathbf{P}_{n-1}+4\mathbf{P}_n+\mathbf{P}_{n+1}]$ and this value equals $\mathbf{P}_n$ if we select $\mathbf{P}_{n+1}=2\mathbf{P}_n-\mathbf{P}_{n-1}$.

Adding phantom points adds two segments to the curve, but this has the advantage that the tangents at the start and the end of the curve have known directions. The former is in the direction from $\mathbf{P}_0$ to $\mathbf{P}_1$ and the latter is from $\mathbf{P}_{n-1}$ to $\mathbf{P}_n$ (same as the end tangents of a Bézier curve). The tangent vector at the start of the first segment is $\frac{1}{2}\mathbf{P}_{-1} + \frac{1}{2}\mathbf{P}_1 = \mathbf{P}_1 - \mathbf{P}_0$, and similarly for the end tangent of the last segment.

The tangent vector of the general cubic B-spline segment is

$$\mathbf{P}_i^t(t) = \frac{1}{6}(-3t^2 + 6t - 3)\mathbf{P}_{i-1} + \frac{1}{6}(9t^2 - 12t)\mathbf{P}_i + \frac{1}{6}(-9t^2 + 6t + 3)\mathbf{P}_{i+1} + \frac{t^2}{2}\mathbf{P}_{i+2}.$$

As a result, the extreme tangent vectors are

$$\mathbf{P}_i^t(0) = \frac{1}{2}(\mathbf{P}_{i+1} - \mathbf{P}_{i-1}), \quad \mathbf{P}_i^t(1) = \frac{1}{2}(\mathbf{P}_{i+2} - \mathbf{P}_i). \tag{7.13}$$

They have simple geometric interpretations.

The second derivative of the cubic segment is

$$\mathbf{P}_i^{tt}(t) = \frac{1}{6}(-6t + 6)\mathbf{P}_{i-1} + \frac{1}{6}(18t - 12)\mathbf{P}_i + \frac{1}{6}(-18t + 6)\mathbf{P}_{i+1} + t\mathbf{P}_{i+2},$$

and it's easy to see that $\mathbf{P}_i^{tt}(1) = \mathbf{P}_{i+1}^{tt}(0) = \mathbf{P}_i - 2\mathbf{P}_{i+1} + \mathbf{P}_{i+2}$, which proves the C^2 continuity of this curve.

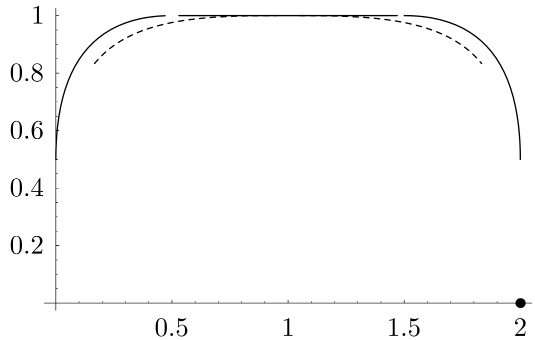
Example: We select the five points $\mathbf{P}_0 = (0,0)$, $\mathbf{P}_1 = (0,1)$, $\mathbf{P}_2 = (1,1)$, $\mathbf{P}_3 = (2,1)$, and $\mathbf{P}_4 = (2,0)$. They have simple, integer coordinates to simplify the computations. We use these points to construct two cubic B-spline segments. The first one is given by Equation (7.11)

$$\begin{aligned}\mathbf{P}_1(t) &= \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)(0,0) + \frac{1}{6}(3t^3 - 6t^2 + 4)(0,1)\\ &\quad + \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)(1,1) + \frac{t^3}{6}(2,1)\\ &= (-t^3/6 + t^2/2 + t/2 + 1/6, t^3/6 - t^2/2 + t/2 + 5/6).\end{aligned}$$

It starts at joint $\mathbf{K}_1 = \mathbf{P}_1(0) = (1/6, 5/6)$ and ends at joint $\mathbf{K}_2 = \mathbf{P}_1(1) = (1,1)$. Notice that these joint points can be verified from Equation (7.12). The tangent vector of this segment is

$$\begin{aligned}\mathbf{P}_1^t(t) &= \frac{1}{6}(-3t^2 + 6t - 3)(0,0) + \frac{1}{6}(9t^2 - 12t)(0,1)\\ &\quad + \frac{1}{6}(-9t^2 + 6t + 3)(1,1) + \frac{t^2}{2}(2,1)\\ &= (-t^2/2 + t + 1/2, t^2/2 - t + 1/2).\end{aligned}$$

The two extreme tangents are $\mathbf{P}_1^t(0) = (1/2, 1/2)$ and $\mathbf{P}_1^t(1) = (1,0)$. These can also be verified by Equation (7.13). Figure 7.4 shows this segment and its successor (the dashed curves).




```
(* B-spline example of 2 cubic segs and 3 quadr segs for 5 points *)
Clear[Pt,T,t,M3,comb,a,g1,g2,g3];
Pt={{0,0},{0,1},{1,1},{2,1},{2,0}};
(* first, 2 cubic segments (dashed) *)
T[t_]:={t^3,t^2,t,1};
M3={{-1,3,-3,1},{3,-6,3,0},{-3,0,3,0},{1,4,1,0}}/6;
comb[i_]:=(T[t].M3)[[i]] Pt[[i+a]];
g1=Graphics[{PointSize[.02], Point/@Pt}];
a=0;
g2=ParametricPlot[comb[1]+comb[2]+comb[3]+comb[4], {t,0,.95},
 Compiled->False, PlotRange->All, DisplayFunction->Identity,
 PlotStyle->AbsoluteDashing[{2,2}]];
a=1;
g3=ParametricPlot[comb[1]+comb[2]+comb[3]+comb[4], {t,0.05,1},
 Compiled->False, PlotRange->All, DisplayFunction->Identity,
 PlotStyle->AbsoluteDashing[{2,2}]];
(* Now the 3 quadratic segments (solid) *)
T[t_]:={t^2,t,1};
M2={{1,-2,1},{-2,2,0},{1,1,0}}/2;
comb[i_]:=(T[t].M2)[[i]] Pt[[i+a]];
a=0;
g4=ParametricPlot[comb[1]+comb[2]+comb[3], {t,0,.97},
 Compiled->False, PlotRange->All, DisplayFunction->Identity];
a=1;
g5=ParametricPlot[comb[1]+comb[2]+comb[3], {t,0.03,.97},
 Compiled->False, PlotRange->All, DisplayFunction->Identity];
a=2;
g6=ParametricPlot[comb[1]+comb[2]+comb[3], {t,0,1},
 Compiled->False, PlotRange->All, DisplayFunction->Identity];
Show[g2,g3,g4,g5,g6,g1, PlotRange->All, DefaultFont->{"cmr10", 10},
 DisplayFunction->$DisplayFunction];
```

Figure 7.4: Two Cubic (Dashed) and Three Quadratic (Solid) Segments of a B-spline.

◇ **Exercise 7.3:** Calculate the second spline segment $\mathbf{P}_2(t)$, its tangent vector, and joint $\mathbf{K}_3$.

◇ **Exercise 7.4:** Use the five control points of the example above to construct the three segments and determine the four joints of the *quadratic* uniform B-spline defined by the points.

Exercise 7.4 shows that the same $n+1$ control points can be used to construct a quadratic or a cubic B-spline curve (or a B-spline curve of any order up to $n+1$). This is in contrast to the Bézier curve whose order is determined by the number of control points. This is also the reason why both n and the degree of the polynomials that make up the spline segments are needed to identify a B-spline. In practice, we use n and k (the *order*) to identify a B-spline. The order is simply the degree plus 1. Thus, a B-spline defined by five control points $\mathbf{P}_0$ through $\mathbf{P}_4$ can be of order 2 (linear, with four segments), order 3 (quadratic, with three segments), order 4 (cubic, with two segments), or order 5, (quintic, with one segment).

Figure 7.5a,b,c shows how a Bézier curve, a cubic B-spline, and a quadratic B-spline, respectively, are attracted to their control polygons. We already know that these three types of curves don't have the same endpoints, so this figure is only qualitative. It only shows how the various types of curves are attracted to their control points.

Collinear Points: Segment $\mathbf{P}_2(t)$ of Exercise 7.4 depends on points $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ that are located on the line $y = 1$. This is why this segment is horizontal (and therefore straight). We conclude that the B-spline can consist of curved and straight segments connected with any desired continuity. All that's necessary in order to have a straight segment is to have enough collinear control points. In the case of a quadratic B-spline, three collinear points will result in a straight segment that will connect to its neighbors (curved or straight) with C^1 continuity. In the case of a cubic B-spline, four collinear points will result in a straight segment that will connect to its neighbors (curved or straight) with C^2 continuity, and similarly for higher-degree uniform B-splines.

A Closed Cubic B-Spline Curve: closing a cubic B-spline is similar to closing a quadratic curve. Given a set of $n+1$ control points, we extend them cyclically to obtain the $n+4$ points

$$\mathbf{P}_n,\ \mathbf{P}_0,\ \mathbf{P}_1,\ \mathbf{P}_2,\ \ldots,\ \mathbf{P}_{n-1},\ \mathbf{P}_n,\ \mathbf{P}_0,\ \mathbf{P}_1,$$

and compute the curve by applying Equation (7.11) to the $n+1$ geometry vectors

$$\begin{pmatrix}\mathbf{P}_n\\ \mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}\quad \begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\end{pmatrix}\quad \begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\\ \mathbf{P}_4\end{pmatrix}\cdots \begin{pmatrix}\mathbf{P}_{n-2}\\ \mathbf{P}_{n-1}\\ \mathbf{P}_n\\ \mathbf{P}_0\end{pmatrix}\quad \begin{pmatrix}\mathbf{P}_{n-1}\\ \mathbf{P}_n\\ \mathbf{P}_0\\ \mathbf{P}_1\end{pmatrix}.$$

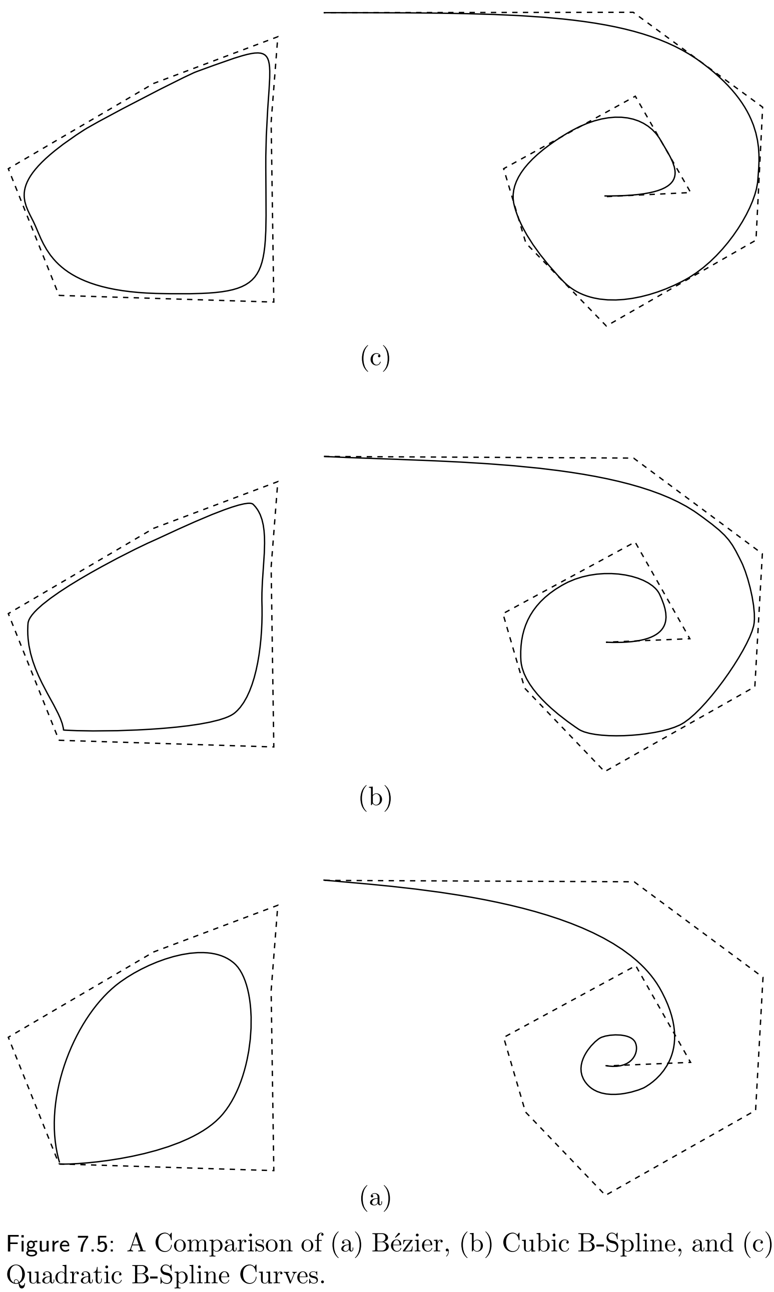


Figure 7.5: A Comparison of (a) Bézier, (b) Cubic B-Spline, and (c) Quadratic B-Spline Curves.

7.3 Multiple Control Points

It is possible to have several identical control points and a set of identical points is referred to as a multiple point. We use the uniform cubic B-spline [Equation (7.11)] as an example, but higher-degree uniform B-splines behave similarly.

We start with a double control point. Consider the cubic segment $\mathbf{P}_1(t)$ defined by the four control points $\mathbf{P}_0$, $\mathbf{P}_1 = \mathbf{P}_2$, and $\mathbf{P}_3$. Its expression is

$$\mathbf{P}_1(t) = \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)\mathbf{P}_0 + \frac{1}{6}(-3t^2 + 3t + 5)\mathbf{P}_1 + \frac{t^3}{6}\mathbf{P}_3,$$

which implies $$\mathbf{P}_1(0) = \frac{1}{6}\mathbf{P}_0 + \frac{5}{6}\mathbf{P}_1, \quad \mathbf{P}_1(1) = \frac{5}{6}\mathbf{P}_1 + \frac{1}{6}\mathbf{P}_3.$$

This segment therefore starts and ends at the same points as the general cubic segment and also has the same extreme tangent vectors. The difference is that it is strongly attracted to the double point.

Next, we consider a triple point. The five control points $\mathbf{P}_0$, $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{P}_3$, and $\mathbf{P}_4$ define the two cubic segments

$$\begin{aligned}\mathbf{P}_1(t) &= \frac{1}{6}(-t^3 + 3t^2 - 3t + 1)\mathbf{P}_0 + \frac{1}{6}(t^3 - 3t^2 + 3t + 5)\mathbf{P}_1\\ &= (1-u)\mathbf{P}_0 + u\mathbf{P}_1, \quad \text{for } u = (t^3 - 3t^2 + 3t + 5)/6,\\ \mathbf{P}_2(t) &= \frac{1}{6}(-t^3 + 6)\mathbf{P}_1 + \frac{t^3}{6}\mathbf{P}_4\\ &= (1-w)\mathbf{P}_1 + w\mathbf{P}_4, \quad \text{for } w = t^3/6.\end{aligned}$$

The parameter substitutions above show that these segments are straight (Figure 7.6). The extreme points of the two segments are

$$\begin{aligned}&\mathbf{P}_1(0) = \frac{1}{6}\mathbf{P}_0 + \frac{5}{6}\mathbf{P}_1, \quad \mathbf{P}_1(1) = \mathbf{P}_1,\\ &\mathbf{P}_2(0) = \mathbf{P}_1, \quad \mathbf{P}_2(1) = \frac{5}{6}\mathbf{P}_1 + \frac{1}{6}\mathbf{P}_4,\end{aligned}$$

showing that the segments meet at the triple control point.

In general, a cubic segment is attracted to a double control point and passes through a triple control point. A degree-4 segment is attracted to double and triple control points and passes through quadruple points, and similarly for higher-degree uniform segments.

The tangent vectors of the two cubic segments are

$$\begin{aligned}\mathbf{P}_1^t(t) &= \frac{1}{6}(-3t^2 + 6t - 3)\mathbf{P}_0 + \frac{1}{6}(3t^2 - 6t + 3)\mathbf{P}_1,\\ \mathbf{P}_2^t(t) &= -\frac{t^2}{2}\mathbf{P}_1 + \frac{t^2}{2}\mathbf{P}_4,\end{aligned}$$

yielding the extreme directions

$$\mathbf{P}_1^t(0) = \frac{1}{2}(\mathbf{P}_1 - \mathbf{P}_0), \quad \mathbf{P}_1^t(1) = 0 \cdot \mathbf{P}_0 + 0 \cdot \mathbf{P}_1 = (0,0),$$
$$\mathbf{P}_2^t(0) = (0,0), \quad \mathbf{P}_2^t(1) = \frac{1}{2}(\mathbf{P}_4 - \mathbf{P}_1).$$

Thus, the first segment starts in the direction from $\mathbf{P}_0$ to the triple point $\mathbf{P}_1$. The second segment ends going in the direction from $\mathbf{P}_1$ to $\mathbf{P}_4$. However, at the triple point, both tangents are indefinite, suggesting a cusp. It turns out that the two segments are straight lines (Figure 7.6).

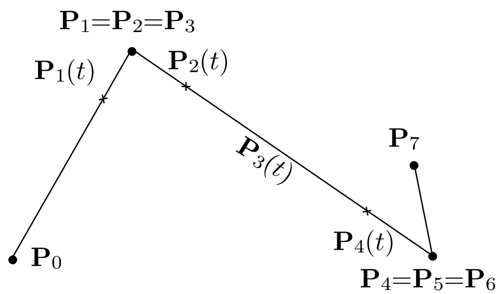


Figure 7.6: A Triple Point.

◇ **Exercise 7.5:** Given the eight control points $\mathbf{P}_0$, $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{P}_3$, $\mathbf{P}_4 = \mathbf{P}_5 = \mathbf{P}_6$, and $\mathbf{P}_7$, calculate the two cubic segments $\mathbf{P}_3(t)$ and $\mathbf{P}_4(t)$ and their start and end points (Figure 7.6).

◇ **Exercise 7.6:** Show that a cubic B-spline segment passes through its first control point if it is a triple point.

As a corollary, we deduce that a uniform cubic B-spline curve where every control point is triple is a polyline.

Example: We consider the case where both terminal points are triple and there are two other points in between. The total number of control points is eight and they satisfy $\mathbf{P}_0 = \mathbf{P}_1 = \mathbf{P}_2$ and $\mathbf{P}_5 = \mathbf{P}_6 = \mathbf{P}_7$. The five cubic spline segments are

$$\begin{aligned}
\mathbf{P}_1(t) &= \frac{1}{6}(-t^3+6)\mathbf{P}_0 + \frac{t^3}{6}\mathbf{P}_3,\\
\mathbf{P}_2(t) &= \frac{1}{6}(2t^3-3t^2-3t+5)\mathbf{P}_0 + \frac{1}{6}(-3t^3+3t^2+3t+1)\mathbf{P}_3 + \frac{t^3}{6}\mathbf{P}_4,\\
\mathbf{P}_3(t) &= \frac{1}{6}(-t^3+3t^2-3t+1)\mathbf{P}_0 + \frac{1}{6}(3t^3-6t^2+4)\mathbf{P}_3\\
&\quad + \frac{1}{6}(-3t^3+3t^2+3t+1)\mathbf{P}_4 + \frac{t^3}{6}\mathbf{P}_5, \qquad (7.14)\\
\mathbf{P}_4(t) &= \frac{1}{6}(-t^3+3t^2-3t+1)\mathbf{P}_3 + \frac{1}{6}(3t^3-6t^2+4)\mathbf{P}_4
\end{aligned}$$

$$+\frac{1}{6}(-2t^3+3t^2+3t+1)\mathbf{P}_5,$$
$$\mathbf{P}_5(t)=\frac{1}{6}(-t^3+3t^2-3t+1)\mathbf{P}_4+\frac{1}{6}(t^3-3t^2+3t+5)\mathbf{P}_5.$$

It is easy to see that they satisfy $\mathbf{P}_1(0)=\mathbf{P}_0$ and $\mathbf{P}_5(1)=\mathbf{P}_5$ and that they meet at the four points

$$\frac{5}{6}\mathbf{P}_0+\frac{1}{6}\mathbf{P}_3,\quad \frac{1}{6}\mathbf{P}_0+\frac{4}{6}\mathbf{P}_3+\frac{1}{6}\mathbf{P}_4,\quad \frac{1}{6}\mathbf{P}_3+\frac{4}{6}\mathbf{P}_4+\frac{1}{6}\mathbf{P}_5,\quad \text{and}\quad \frac{1}{6}\mathbf{P}_4+\frac{5}{6}\mathbf{P}_5.$$

If we want to keep the two extreme points as triples, we can edit this curve only by moving the two interior points $\mathbf{P}_3$ and $\mathbf{P}_4$. Moving $\mathbf{P}_4$ affects the last four segments, and moving $\mathbf{P}_3$ affects the first four segments. This type of curve is therefore similar to a Bézier curve in that it starts and ends at its extreme control points and it features only limited local control.

◇ **Exercise 7.7:** Given the eight control points $\mathbf{P}_0=\mathbf{P}_1=\mathbf{P}_2=(1,0)$, $\mathbf{P}_3=(2,1)$, $\mathbf{P}_4=(4,0)$, and $\mathbf{P}_5=\mathbf{P}_6=\mathbf{P}_7=(4,1)$, use Equation (7.14) to calculate the cubic uniform B-spline curve defined by these points and compare it to the Bézier curve defined by the points.

7.4 Cubic B-Splines with Tension

Adding a tension parameter to the uniform cubic B-spline is similar to tension in the cardinal spline (Section 5.4). We use Hermite interpolation [Equation (4.7)] to calculate a PC segment that starts and ends at the same points as a cubic B-spline and whose extreme tangent vectors point in the same directions as those of the cubic B-spline, but whose magnitudes are controlled by a tension parameter s. Substituting $\frac{1}{6}\mathbf{P}_0+\frac{4}{6}\mathbf{P}_1+\frac{1}{6}\mathbf{P}_2$ and $\frac{1}{6}\mathbf{P}_1+\frac{4}{6}\mathbf{P}_2+\frac{1}{6}\mathbf{P}_3$ for the terminal points and $s(\mathbf{P}_2-\mathbf{P}_0)$ and $s(\mathbf{P}_3-\mathbf{P}_1)$ for the extreme tangents, we write Equation (4.7) and manipulate it such that it ends up looking like a uniform cubic B-spline segment, Equation (7.11).

$$\mathbf{P}(t)=(t^3,t^2,t,1)\begin{pmatrix}2&-2&1&1\\-3&3&-2&-1\\0&0&1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}\frac{1}{6}\mathbf{P}_0+\frac{4}{6}\mathbf{P}_1+\frac{1}{6}\mathbf{P}_2\\ \frac{1}{6}\mathbf{P}_1+\frac{4}{6}\mathbf{P}_2+\frac{1}{6}\mathbf{P}_3\\ s(\mathbf{P}_2-\mathbf{P}_0)\\ s(\mathbf{P}_3-\mathbf{P}_1)\end{pmatrix}$$
$$=\frac{1}{6}\Big[\big(t^3(2-s)+t^2(2s-3)-st+1\big)\mathbf{P}_0+\big(t^3(6-s)+t^2(s-9)+4\big)\mathbf{P}_1$$
$$+\big(t^3(s-6)+t^2(9-2s)+st+1\big)\mathbf{P}_2+\big(t^3(s-2)+t^2(3-s)\big)\mathbf{P}_3\Big]$$
$$=\frac{1}{6}(t^3,t^2,t,1)\begin{pmatrix}2-s&6-s&s-6&s-2\\2s-3&s-9&9-2s&3-s\\-s&0&s&0\\1&4&1&0\end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\end{pmatrix}. \tag{7.15}$$

A quick check verifies that Equation (7.15) reduces to the uniform cubic B-spline segment, Equation (7.11), for $s = 3$. This value is therefore considered the "neutral" or "standard" value of the tension parameter s. Since s controls the length of the tangent vectors, small values of s should produce the effects of higher tension and, in the extreme, the value $s = 0$ should result in indefinite tangent vectors and in the spline segment becoming a straight line. To show this, we rewrite Equation (7.15) for $s = 0$:

$$\begin{aligned}\mathbf{P}(t) &= \frac{1}{6}(t^3, t^2, t, 1)\begin{pmatrix} 2 & 6 & -6 & -2 \\ -3 & -9 & 9 & 3 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\end{pmatrix}\\ &= \frac{1}{6}(2t^3 - 3t^2 + 1)\mathbf{P}_0 + \frac{1}{6}(6t^3 - 9t^2 + 4)\mathbf{P}_1\\ &\quad + \frac{1}{6}(-6t^3 + 9t^2 + 1)\mathbf{P}_2 + \frac{1}{6}(-2t^3 + 3t^2)\mathbf{P}_3.\end{aligned}$$

Substituting $T = 3t^2 - 2t^3$ for the parameter t changes the above expression to the form

$$\mathbf{P}(T) = \frac{1}{6}(-\mathbf{P}_0 - 3\mathbf{P}_1 + 3\mathbf{P}_2 + \mathbf{P}_3)T + \frac{1}{6}(\mathbf{P}_0 + 4\mathbf{P}_1 + \mathbf{P}_2),$$

which is a straight line from $\mathbf{P}(0) = \frac{1}{6}(\mathbf{P}_0 + 4\mathbf{P}_1 + \mathbf{P}_2)$ to $\mathbf{P}(1) = \frac{1}{6}(\mathbf{P}_1 + 4\mathbf{P}_2 + \mathbf{P}_3)$.

The tangent vector of Equation (7.15) is

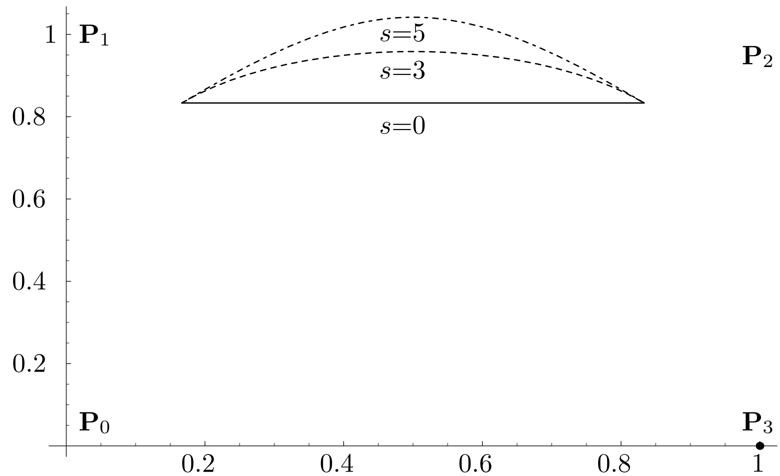
$$\begin{aligned}\mathbf{P}^t(t) &= \frac{1}{6}(3t^2, 2t, 1, 0)\begin{pmatrix} 2-s & 6-s & s-6 & s-2 \\ 2s-3 & s-9 & 9-2s & 3-s \\ -s & 0 & s & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\end{pmatrix}\\ &= \frac{1}{6}\Big[\big(3t^2(2-s) + 2t(2s-3) - s\big)\,\mathbf{P}_0 + \big(3t^2(6-s) + 2t(s-9)\big)\,\mathbf{P}_1\\ &\quad + \big(3t^2(s-6) + 2t(9-2s) + s\big)\,\mathbf{P}_2 + \big(3t^2(s-2) + 2t(3-s)\big)\,\mathbf{P}_3\Big].\end{aligned} \tag{7.16}$$

The extreme tangents are

$$\mathbf{P}^t(0) = \frac{s}{6}(\mathbf{P}_2 - \mathbf{P}_0) \quad\text{and}\quad \mathbf{P}^t(1) = \frac{s}{6}(\mathbf{P}_3 - \mathbf{P}_1).$$

Substituting $s = 0$ in Equation (7.16) yields the tangent vector for the case of infinite tension

$$\begin{aligned}\mathbf{P}^t(t) &= \frac{1}{6}\Big[6(t^2 - t)\mathbf{P}_0 + 18(t^2 - t)\mathbf{P}_1 - 18(t^2 - t)\mathbf{P}_2 - 6(t^2 - t)\mathbf{P}_3\Big]\\ &= (t^2 - t)(\mathbf{P}_0 + 3\mathbf{P}_1 - 3\mathbf{P}_2 - \mathbf{P}_3).\end{aligned} \tag{7.17}$$




```
(* Cubic B-spline with tension *)
Clear[t,s,pnts,stnp,tensMat,bsplineTensn,g1,g2,g3,g4];
pnts={{0,0},{0,1},{1,1},{1,0}};
stnp=Transpose[pnts];
tensMat={{2-s,6-s,s-6,s-2},{2s-3,s-9,9-2s,3-s},{-s,0,s,0},{1,4,1,0}};
bsplineTensn[t_]:=Module[{tmpstruc}, tmpstruc={t^3,t^2,t,1}.tensMat;
{tmpstruc.stnp[[1]],tmpstruc.stnp[[2]]}/6];
g1=ListPlot[pnts, Prolog->AbsolutePointSize[3],
 DisplayFunction->Identity];
s=0;
g2=ParametricPlot[bsplineTensn[t], {t,0,1},
 Compiled->False, DisplayFunction->Identity];
s=3;
g3=ParametricPlot[bsplineTensn[t], {t,0,1},
 Compiled->False, DisplayFunction->Identity,
 PlotStyle->AbsoluteDashing[{2,2}]];
s=5;
g4=ParametricPlot[bsplineTensn[t], {t,0,1},
 Compiled->False, DisplayFunction->Identity,
 PlotStyle->AbsoluteDashing[{1,2,2,2}]];
Show[g1,g2,g3,g4, DisplayFunction->$DisplayFunction]
```

Figure 7.7: Figure and Code for a Cubic B-Spline with Tension.

◇ **Exercise 7.8:** Since the spline segment is a straight line in this case, its tangent vector should always point in the same direction. Use Equation (7.17) to show that this is so.

Figure 7.7 illustrates the effect of tension on a cubic B-spline. Three curves are shown, corresponding to s values of 0, 3, and 5.

See also Section 6.11 for a discussion of cubic Bézier curves with tension.

> Sex alleviates tension and love causes it.
>
> —Woody Allen (as Andrew) in *A Midsummer Night's Sex Comedy* (1982)

7.5 Cubic B-Spline and Bézier Curves

Given a cubic B-spline segment $\mathbf{P}(t)$ based on the four control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, it is easy to find four control points $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ such that the Bézier curve $\mathbf{Q}(t)$ defined by them will have the same shape as $\mathbf{P}(t)$. This is done by equating the matrices of Equation (7.11) that define $\mathbf{P}(t)$ to those of Equation (6.8) that define $\mathbf{Q}(t)$:

$$\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Q}_0 \\ \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \end{pmatrix}.$$

The solutions are

$$\begin{aligned} \mathbf{Q}_0 &= \frac{1}{6}\left(\mathbf{P}_0 + 4\mathbf{P}_1 + \mathbf{P}_2\right), \\ \mathbf{Q}_1 &= \frac{1}{6}\left(4\mathbf{P}_1 + 2\mathbf{P}_2\right), \\ \mathbf{Q}_2 &= \frac{1}{6}\left(2\mathbf{P}_1 + 4\mathbf{P}_2\right), \\ \mathbf{Q}_3 &= \frac{1}{6}\left(\mathbf{P}_1 + 4\mathbf{P}_2 + \mathbf{P}_3\right). \end{aligned}$$

Equation (8.4) of Section 8.2 shows a similar relation between the quadratic B-spline and Bézier curves.

7.6 Higher-Degree Uniform B-Splines

The methods of Sections 7.1 and 7.2 can be employed to construct uniform B-splines of higher degrees. It can be shown (see, for example, [Yamaguchi 88], p. 329) that the degree-n uniform B-spline segment is given by

$$\mathbf{P}_i(t) = (t^n, \ldots, t^2, t, 1)\mathbf{M} \begin{pmatrix} \mathbf{P}_{i-1} \\ \mathbf{P}_i \\ \mathbf{P}_{i+1} \\ \vdots \\ \mathbf{P}_{i+n-1} \end{pmatrix},$$

where the elements m_{ij} of the basis matrix $\mathbf{M}$ are

$$m_{ij} = \frac{1}{n!}\binom{n}{i}\sum_{k=j}^{n}(n-k)^i(-1)^{k-j}\binom{n+1}{k-j}.$$

Figure 7.8 shows a few examples of these matrices.

$$\mathbf{M}_1 = \frac{1}{1!}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_2 = \frac{1}{2!}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_3 = \frac{1}{3!}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_4 = \frac{1}{4!}\begin{pmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -6 & -6 & 6 & 0 \\ -4 & -12 & 12 & 4 & 0 \\ 1 & 11 & 11 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_5 = \frac{1}{5!}\begin{pmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 20 & 0 & -20 & 10 & 0 \\ 10 & 20 & -60 & 20 & 10 & 0 \\ -5 & -50 & 0 & 50 & 5 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_6 = \frac{1}{6!}\begin{pmatrix} 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ -6 & 30 & -60 & 60 & -30 & 6 & 0 \\ 15 & -45 & 30 & 30 & -45 & 15 & 0 \\ -20 & -20 & 160 & -160 & 20 & 20 & 0 \\ 15 & 135 & -150 & -150 & 135 & 15 & 0 \\ -6 & -150 & -240 & 240 & 150 & 6 & 0 \\ 1 & 57 & 302 & 302 & 57 & 1 & 0 \end{pmatrix}$$

Figure 7.8: Some Basis Matrices for Uniform B-Splines.

7.7 Interpolating B-Splines

The B-spline is an approximating curve. Its shape is determined by the control points $\mathbf{P}_i$, but the curve itself does not pass through those points. Instead, it passes through the joints $\mathbf{K}_i$. In our notation so far, we have assumed that the cubic uniform B-spline is based on $n+1$ control points and passes through $n-1$ joint points. The number of control points for the cubic curve is therefore always two more than the number of joints.

> One person's constant is another person's variable.
>
> —Susan Gerhart

This section deals with the opposite problem. We show how to employ B-splines to construct an interpolating cubic spline curve that passes through a set of $n+1$ given data points $\mathbf{K}_0$, $\mathbf{K}_1$,..., $\mathbf{K}_n$. The curve must consist of n segments and the idea is to use the $\mathbf{K}_i$ points to calculate a new set of points $\mathbf{P}_i$, then use the new points as the control points of a cubic uniform B-spline curve. To obtain n cubic segments, we need $n+3$ points and we denote them by $\mathbf{P}_{-1}$ through $\mathbf{P}_{n+1}$.

Using $\mathbf{P}_i$ as our control points, Equation (7.11) shows that the general segment $\mathbf{P}_i(t)$ terminates at $\mathbf{P}_i(1) = \frac{1}{6}[\mathbf{P}_{i-2} + 4\mathbf{P}_{i-1} + \mathbf{P}_i]$. We require that the segment ends at point $\mathbf{K}_{i-1}$, which produces the equation $\frac{1}{6}[\mathbf{P}_{i-2} + 4\mathbf{P}_{i-1} + \mathbf{P}_i] = \mathbf{K}_{i-1}$. When this equation is repeated for $0 \le i \le n$, we get a system of $n+1$ equations with the $\mathbf{P}_i$s as the unknowns. However, there are $n+3$ unknowns ($\mathbf{P}_{-1}$ through $\mathbf{P}_{n+1}$), so we need two more equations.

The required equations are obtained by considering the tangent vectors of the interpolating curve at its two ends. We denote the tangent at the start by $\mathbf{T}_1$. It is given by $\mathbf{T}_1 = \frac{1}{2}(\mathbf{P}_1 - \mathbf{P}_{-1})$, so it points in the direction from $\mathbf{P}_{-1}$ to $\mathbf{P}_1$; similarly for the end tangent $\mathbf{T}_n = \frac{1}{2}(\mathbf{P}_{n+1} - \mathbf{P}_{n-1})$. After these two relations are included, the resulting system of $n+3$ equations is

$$
n{+}3\left\{\frac{1}{6}\underbrace{\begin{pmatrix}
-3 & 0 & 3 & 0 & \ldots & 0 & 0 & 0\\
1 & 4 & 1 & 0 & \ldots & 0 & 0 & 0\\
0 & 1 & 4 & 1 & \ldots & 0 & 0 & 0\\
\vdots & & & & & & & \vdots\\
0 & 0 & 0 & 0 & \ldots & 4 & 1 & 0\\
0 & 0 & 0 & 0 & \ldots & 1 & 4 & 1\\
0 & 0 & 0 & 0 & \ldots & -3 & 0 & 3
\end{pmatrix}}_{n+3}\right.
\begin{pmatrix}\mathbf{P}_{-1}\\ \mathbf{P}_0\\ \mathbf{P}_1\\ \vdots\\ \mathbf{P}_{n-1}\\ \mathbf{P}_n\\ \mathbf{P}_{n+1}\end{pmatrix}
=
\begin{pmatrix}\mathbf{T}_1\\ \mathbf{K}_0\\ \mathbf{K}_1\\ \vdots\\ \mathbf{K}_{n-1}\\ \mathbf{K}_n\\ \mathbf{T}_n\end{pmatrix}. \tag{7.18}
$$

The user specifies the values of the two extreme tangents $\mathbf{T}_1$ and $\mathbf{T}_n$, the equations are solved, and the $\mathbf{P}_i$ points are then used in the usual way to calculate a cubic uniform B-spline that passes through the original points $\mathbf{K}_i$. This process should be compared to the similar computation of the cubic spline, Section 5.1. Specifically, Equation (7.18) should be compared with Equation (5.7).

Notice that the coefficient matrix of Equation (7.18) is not diagonally dominant because of the four ± 3's. We can, however, modify it slightly by writing the system of equations in the form

$$
n+3\left\{\frac{1}{6}\underbrace{\begin{pmatrix}
-3/2 & 0 & 3/2 & 0 & \ldots & 0 & 0 & 0\\
1 & 4 & 1 & 0 & \ldots & 0 & 0 & 0\\
0 & 1 & 4 & 1 & \ldots & 0 & 0 & 0\\
\vdots & & & & & & & \vdots\\
0 & 0 & 0 & 0 & \ldots & 4 & 1 & 0\\
0 & 0 & 0 & 0 & \ldots & 1 & 4 & 1\\
0 & 0 & 0 & 0 & \ldots & -3/2 & 0 & 3/2
\end{pmatrix}}_{n+3}\right.
\begin{pmatrix}\mathbf{P}_{-1}\\ \mathbf{P}_0\\ \mathbf{P}_1\\ \vdots\\ \mathbf{P}_{n-1}\\ \mathbf{P}_n\\ \mathbf{P}_{n+1}\end{pmatrix}
=
\begin{pmatrix}\mathbf{T}_1/2\\ \mathbf{K}_0\\ \mathbf{K}_1\\ \vdots\\ \mathbf{K}_{n-1}\\ \mathbf{K}_n\\ \mathbf{T}_n/2\end{pmatrix}. \qquad (7.19)
$$

The coefficient matrix of Equation (7.19) is columnwise diagonally dominant and is therefore nonsingular. Thus, this system of equations has a unique solution, but this system is mathematically identical to Equation (7.18), so that system of equations also has a unique solution.

Example: This is the opposite of the example on page 259. We start with $\mathbf{K}_0 = (1/6, 5/6)$, $\mathbf{K}_1 = (1, 1)$, $\mathbf{K}_2 = (11/6, 5/6)$, and the two extreme tangents $\mathbf{T}_1 = (1/2, 1/2)$ and $\mathbf{T}_2 = (1/2, -1/2)$, and set up the 5×5 system of equations

$$
\frac{1}{6}\begin{pmatrix}
-3 & 0 & 3 & 0 & 0\\
1 & 4 & 1 & 0 & 0\\
0 & 1 & 4 & 1 & 0\\
0 & 0 & 1 & 4 & 1\\
0 & 0 & -3 & 0 & 3
\end{pmatrix}
\begin{pmatrix}\mathbf{P}_{-1}\\ \mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\end{pmatrix}
=
\begin{pmatrix}(1/2, 1/2)\\ (1/6, 5/6)\\ (1, 1)\\ (11/6, 5/6)\\ (1/2, -1/2)\end{pmatrix}.
$$

This is easy to solve and the solutions are $\mathbf{P}_{-1} = (0, 0)$, $\mathbf{P}_0 = (0, 1)$, $\mathbf{P}_1 = (1, 1)$, $\mathbf{P}_2 = (2, 1)$, and $\mathbf{P}_3 = (2, 0)$, identical to the original control points of the above-mentioned example.

7.8 A Knot Vector-Based Approach

The knot vector approach to the uniform B-spline curve assumes that the curve is a weighted sum, $\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_{n,i}(t)$ of the control points with unknown weight functions that have to be determined. The method is similar to that used in deriving the Bézier curve (Section 6.2). The cubic uniform B-spline is used here as an example, but this approach can be applied to B-splines of any order. We assume that five control points are given—so that five weight functions, $B_{4,0}(t)$ through $B_{4,4}(t)$ are required—and that the curve will consist of two cubic segments. In this approach we assume that each spline segment is traced when the parameter t varies over an interval of one unit, from an integer value u to the next integer $u+1$. The u values are called the *knots* of the B-spline. Since they are the integers $0, 1, 2, \ldots$, they are uniformly distributed, hence the name *uniform* B-spline. To trace out a two-segment spline curve, t should vary in the interval $[0, 2]$.

The guiding principle is that each weight function should be a cubic polynomial, should have a maximum at the vicinity of "its" control point, and should drop to zero when away from the point. A general weight function should therefore have the bell shape shown in Figure 7.9a. To derive such a function, we write it as the union of four parts, $b_0(t)$, $b_1(t)$, $b_2(t)$, and $b_3(t)$, each a simple cubic polynomial, and each defined over one unit of t. Figure 7.9b shows how each weight $B_{4,i}(t)$ is defined over a range of five knots and is zero elsewhere

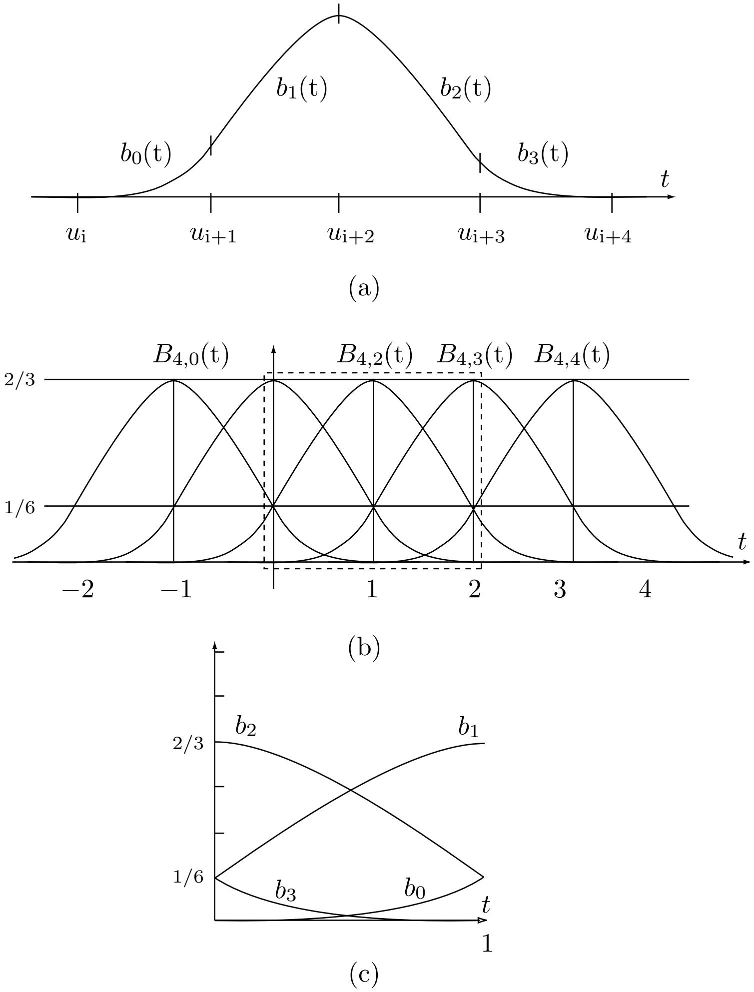


Figure 7.9: Weight Functions of the Cubic Uniform B-Spline.

The following considerations are employed to set up equations to calculate the $b_i(t)$ functions:

1. They should be barycentric.
2. They should provide C^2 continuity at the three points where they join.
3. $b_0(t)$ and its first two derivatives should be zero at the start point $b_0(0)$.
4. $b_3(t)$ and its first two derivatives should be zero at the end point $b_3(1)$.

We adopt the notation $b_i(t) = A_i t^3 + B_i t^2 + C_i t + D_i$. The conditions above yield the following equations:

1. The single equation $B_{4,0}(0) + B_{4,1}(0) + B_{4,2}(0) + B_{4,3}(0) = 1$. This is a special case of condition 1. We see later that the $b_i(t)$ functions resulting from our equations are, in fact, barycentric.

2. Condition 2 yields the nine equations

$$\begin{array}{lll} b_0(1) = b_1(0), & \dot{b}_0(1) = \dot{b}_1(0), & \ddot{b}_0(1) = \ddot{b}_1(0), \\ b_1(1) = b_2(0), & \dot{b}_1(1) = \dot{b}_2(0), & \ddot{b}_1(1) = \ddot{b}_2(0), \\ b_2(1) = b_3(0), & \dot{b}_2(1) = \dot{b}_3(0), & \ddot{b}_2(1) = \ddot{b}_3(0). \end{array} \tag{7.20}$$

The first two derivatives of $b_i(t)$ are

$$\frac{db_i(t)}{dt} = \dot{b}_i(t) = 3A_i t^2 + 2B_i t + C_i, \qquad \frac{d^2 b_i(t)}{dt^2} = \ddot{b}_i(t) = 6A_i t + 2B_i,$$

so the nine equations above can be written explicitly as

$$\begin{array}{lll} A_0 + B_0 + C_0 + D_0 = D_1, & 3A_0 + 2B_0 + C_0 = C_1, & 6A_0 + 2B_0 = 2B_1, \\ A_1 + B_1 + C_1 + D_1 = D_2, & 3A_1 + 2B_1 + C_1 = C_2, & 6A_1 + 2B_1 = 2B_2, \\ A_2 + B_2 + C_2 + D_2 = D_3, & 3A_2 + 2B_2 + C_2 = C_3, & 6A_2 + 2B_2 = 2B_3. \end{array}$$

3. Condition 3 yields the three equations

$$D_0 = 0, \quad C_0 = 0, \quad 2B_0 = 0.$$

4. Condition 4 yields the three equations

$$A_3 + B_3 + C_3 + D_3 = 0, \quad 3A_3 + 2B_3 + C_3 = 0, \quad 6A_3 + 2B_3 = 0.$$

Thus, we end up with 16 equations that are easy to solve. Their solutions are

$$\begin{array}{rl} b_0(t) = \frac{1}{6}t^3, & b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3), \\ b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3), & b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3). \end{array} \tag{7.21}$$

The proof that the $b_i(t)$ functions are barycentric is now trivial. Figure 7.9c shows the shapes of the four weights.

Now that the weight functions are known, the entire curve can be expressed as the weighted sum $\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_{4,i}(t)$, where the weights all look the same and are shifted with respect to each other by using different ranges for t. Each weight $B_{4,i}(t)$ is nonzero only in the (open) interval (u_{i-3}, u_{i+1}) (Figure 7.9b).

Each curve segment $\mathbf{P}_i(t)$ can now be expressed as the barycentric sum of the four weighted points $\mathbf{P}_{i-3}$ through $\mathbf{P}_i$ (or, alternatively, as a linear combination of the $B_{4,i}(t)$ functions), $\mathbf{P}_i(t) = \sum_{j=-3}^{0} \mathbf{P}_{i+j} B_{4,i+j}(t)$, where $u_i \le t < u_{i+1}$. The next (crucial) step is to realize that in the range $u_i \le t < u_{i+1}$, only component b_3 of $B_{4,i-3}$ is nonzero and similarly for the other three weights (see the dashed box of Figure 7.9b). The segment can therefore be written

$$\begin{aligned}
\mathbf{P}_i(t) &= \sum_{j=3}^{0} \mathbf{P}_{i-j} b_j(t) \\
&= \frac{1}{6}\mathbf{P}_{i-3}(-t^3+3t^2-3t+1) + \frac{1}{6}\mathbf{P}_{i-2}(3t^3-6t^2+4) \\
&\quad + \frac{1}{6}\mathbf{P}_{i-1}(-3t^3+3t^2+3t+1) + \frac{1}{6}\mathbf{P}_i t^3 \qquad (7.22)\\
&= \frac{1}{6}(t^3,t^2,t,1)\begin{pmatrix} -1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_{i-3}\\ \mathbf{P}_{i-2}\\ \mathbf{P}_{i-1}\\ \mathbf{P}_i\end{pmatrix},
\end{aligned}$$

an expression identical (except for the choice of index i) to Equation (7.11). This approach to deriving the weight functions can be generalized for the nonuniform B-spline.

The dashed box of Figure 7.9b illustrates how the $B_{4,i}(t)$ weight functions blend the five control points in the two spline segments. The first weight, $B_{4,0}(t)$, goes down from 1/6 to 0 when t varies from 0 to 1. Thus, the first control point $\mathbf{P}_0$ starts by contributing 1/6 of its value to the curve, then decreases its contribution until it disappears at $t = 1$. This is why $\mathbf{P}_0$ does not contribute to the second segment. The second weight, $B_{4,1}(t)$, starts at 2/3 (when $t = 0$), goes down to 1/6 for $t = 1$, then all the way to 0 when t reaches 2. This is how the second control point $\mathbf{P}_1$ participates in the blend that generates the first two spline segments. Notice how the weight functions have their maxima at integer values of t, how only three weights are nonzero at these values, and how there are four nonzero weights for any other values of t.

Figure 7.10a shows the weight functions for the linear uniform B-spline. Each has the form of a hat, going from 0 to 1 and back to 0. They also have their maxima at integer values of t. The weight functions of the quadratic B-spline are shown in Figure 7.10b. Notice how each varies from 0 to 3/4, how they meet at a height of 1/2, and how their maxima are at half-integer values of t. The first weight, $B_{3,0}(t)$, drops from 1/2 to 0 for the first spline segment (i.e., when t varies in the interval $[0,1]$) and remains zero for the second and subsequent segments. The second weight, $B_{3,1}(t)$, climbs from 1/2 to 1, then drops back to 1/2 for the first segment. For the second segment, this weight goes down from 1/2 to 0. These diagrams provide a clear understanding of how the control points are blended by the uniform B-spline.

The general B-spline weight functions are normally denoted by $N_{ik}(t)$ and can be defined recursively. Before delving into this topic, however, we show how the uniform B-spline curve itself can be defined recursively, similar to the recursive definition of the Bézier curve [Equation (6.11)]. Given a set of $n+1$ control points $\mathbf{P}_0$ through $\mathbf{P}_n$ and a uniform knot vector $(t_0, t_1, \ldots, t_{n+k})$ (a set of equally-spaced $n+k+1$ nondecreasing real numbers), the B-spline of order k is defined as

$$\mathbf{P}(t) = \mathbf{P}_l^{(k-1)}(t), \quad \text{where} \quad t_l \le t < t_{l+1} \qquad (7.23)$$

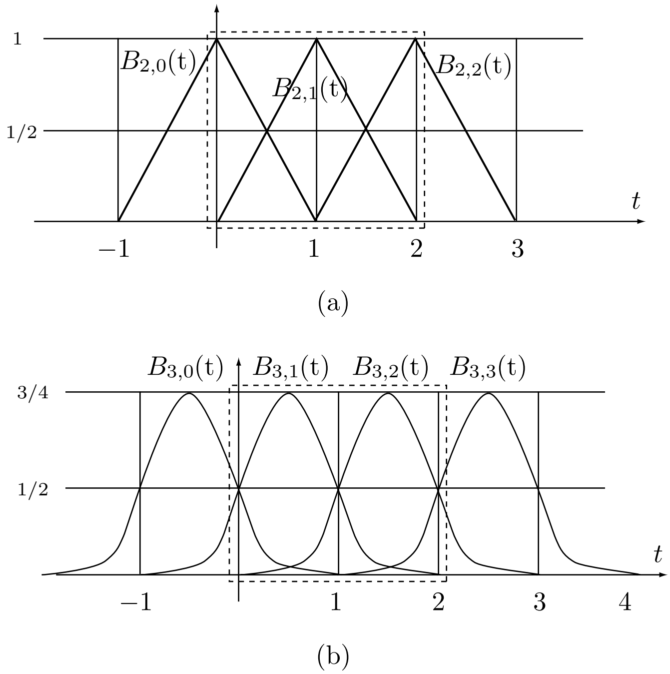


Figure 7.10: Weight Functions of the Linear and the Quadratic B-Splines.

and where the quantities $\mathbf{P}_i^{(j)}(t)$ are defined recursively by

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} \mathbf{P}_i, & \text{for } j = 0, \\ (1 - T_{ij})\mathbf{P}_{i-1}^{(j-1)}(t) + T_{ij}\mathbf{P}_i^{(j-1)}(t), & \text{for } j > 0, \end{cases}$$

and

$$T_{ij} = \frac{t - t_i}{t_{i+k-j} - t_i}.$$

Figure 7.11 is a pyramid that illustrates how the quantities $\mathbf{P}_l^{(k-1)}(t)$ are constructed recursively. Each $\mathbf{P}_i^{(j)}(t)$ in the figure is constructed as a barycentric sum of the two quantities immediately to its left. Equation (7.23) is the *geometric* definition of the uniform B-spline.

We now turn to the *algebraic* (or analytical) definition of the general (uniform and nonuniform) B-spline curve. It is defined as the weighted sum

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i N_{ik}(t),$$

where the weight functions $N_{ik}(t)$ are defined recursively by

$$N_{i1}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise}, \end{cases} \tag{7.24}$$

$$
\begin{array}{lllll}
\vdots & & & & \\
\mathbf{P}_{l-k+1} & & & & \\
 & \mathbf{P}^{(1)}_{l-k+2} & & & \\
\mathbf{P}_{l-k+2} & & \mathbf{P}^{(2)}_{l-k+3} & & \\
 & \mathbf{P}^{(1)}_{l-k+3} & & \cdot & \\
\mathbf{P}_{l-k+3} & & \cdot & & \\
\mathbf{P}_{l-k+4} & & & & \\
\cdot & & & \mathbf{P}^{(k-2)}_{l-1} & \\
\cdot & & & & \mathbf{P}^{(k-1)}_{l} \\
\cdot & & & \mathbf{P}^{(k-2)}_{l} & \\
\cdot & & & & \\
\mathbf{P}_{l-2} & & \mathbf{P}^{(2)}_{l-1} & & \\
 & \mathbf{P}^{(1)}_{l-1} & & \cdot & \\
\mathbf{P}_{l-1} & & \mathbf{P}^{(2)}_{l} & & \\
 & \mathbf{P}^{(1)}_{l} & & & \\
\mathbf{P}_{l} & & & & \\
\vdots & & & &
\end{array}
$$

Figure 7.11: Recursive Construction of $\mathbf{P}_l^{(k-1)}(t)$.

(note how the interval starts at t_i but does not reach t_{i+1}; such an interval is closed on the left and open on the right) and

$$N_{ik}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t), \quad \text{where} \quad 0 \le i \le n. \tag{7.25}$$

The weights $N_{ik}(t)$ may be tedious to calculate in the general case, where the knots t_i can be any, but are easy to calculate in the special case where the knot vector is the uniform sequence $(0, 1, \ldots, n+k)$, i.e., when $t_i = i$. Here are examples for the first few values of k.

For $k = 1$, the weight functions are defined by

$$N_{i1}(t) = \begin{cases} 1, & \text{if } t \in [i, i+1), \\ 0, & \text{otherwise.} \end{cases} \tag{7.26}$$

This results in the "step" functions shown in Figure 7.12. Notice how each step is closed on the left and open on the right and how $N_{i1}(t)$ is nonzero only in the interval $[i, i+1)$ (this interval is its *support*). It is also clear that each of them is a shifted version of its predecessor, so we can express any of them as a shifted version of the first one and write $N_{i1}(t) = N_{01}(t - i)$.

For $k = 2$, the weight functions can be calculated for any i from Equation (7.25)

$$N_{02}(t) = \frac{t - t_0}{t_1 - t_0} N_{01}(t) + \frac{t_2 - t}{t_2 - t_1} N_{11}(t)$$

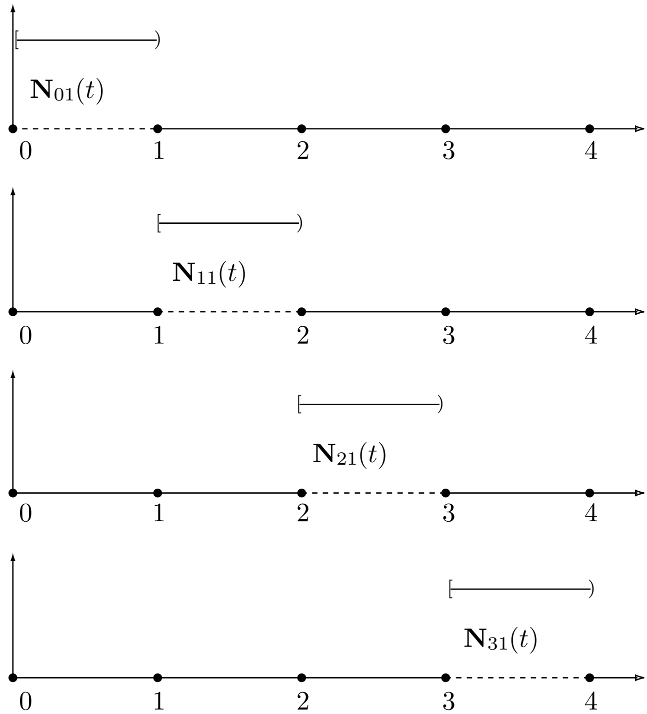


Figure 7.12: Uniform B-Spline Weight Functions for $k = 1$.

$$
\begin{aligned}
&= tN_{01}(t) + (2-t)N_{11}(t) \\
&= \begin{cases} t, & \text{when } 0 \le t < 1, \\ 2-t, & \text{when } 1 \le t < 2, \\ 0, & \text{otherwise,} \end{cases} \\
N_{12}(t) &= \frac{t-t_1}{t_2-t_1}N_{11}(t) + \frac{t_3-t}{t_3-t_2}N_{21}(t) \\
&= (t-1)N_{11}(t) + (3-t)N_{21}(t) \\
&= \begin{cases} t-1, & \text{when } 1 \le t < 2, \\ 3-t, & \text{when } 2 \le t < 3, \\ 0, & \text{otherwise,} \end{cases} \\
N_{22}(t) &= \frac{t-t_2}{t_3-t_2}N_{21}(t) + \frac{t_4-t}{t_4-t_3}N_{31}(t) \\
&= (t-2)N_{21}(t) + (4-t)N_{31}(t) \\
&= \begin{cases} t-2, & \text{when } 2 \le t < 3, \\ 4-t, & \text{when } 3 \le t < 4, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

The hat-shaped functions are shown in Figure 7.13. Notice how $N_{i2}(t)$ spans the interval $[i, i+2)$. It is also obvious that each of them is a shifted version of its predecessor, so we can express any of them as a shifted version of the first one and write $N_{i2}(t) = N_{02}(t-i)$.

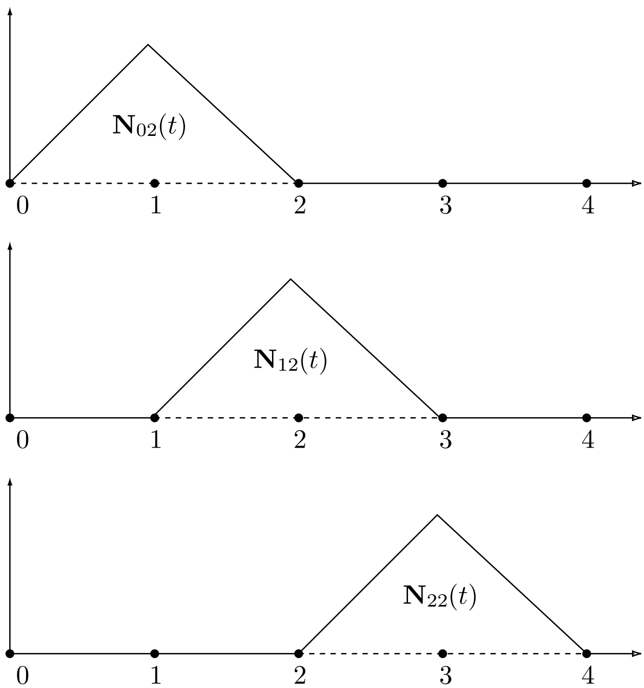


Figure 7.13: Uniform B-Spline Weight Functions for $k = 2$.

For $k = 3$, the calculations are similar:

$$
\begin{aligned}
N_{03}(t) &= \frac{t-t_0}{t_2-t_0}N_{02}(t) + \frac{t_3-t}{t_3-t_1}N_{12}(t)\\
&= \frac{t}{2}N_{02}(t) + \frac{3-t}{2}N_{12}(t)\\
&= \begin{cases} t^2/2, & \text{when } 0 \le t < 1,\\ \frac{t^2}{2}(2-t) + \frac{3-t}{2}(t-1), & \text{when } 1 \le t < 2,\\ (3-t)^2/2, & \text{when } 2 \le t < 3,\\ 0, & \text{otherwise,} \end{cases}\\
&= \begin{cases} t^2/2, & \text{when } 0 \le t < 1,\\ (-2t^2+6t-3)/2, & \text{when } 1 \le t < 2,\\ (3-t)^2/2, & \text{when } 2 \le t < 3,\\ 0, & \text{otherwise,} \end{cases}\\
N_{13}(t) &= \frac{t-t_1}{t_3-t_1}N_{12}(t) + \frac{t_4-t}{t_4-t_2}N_{22}(t)\\
&= \frac{t-1}{2}N_{12}(t) + \frac{4-t}{2}N_{22}(t)
\end{aligned}
$$

$$= \begin{cases} (t-1)^2/2, & \text{when } 1 \le t < 2, \\ (-2t^2+10t-11)/2, & \text{when } 2 \le t < 3, \\ (4-t)^2/2, & \text{when } 3 \le t < 4, \\ 0, & \text{otherwise.} \end{cases}$$

Each of these curves (Figure 7.14) is a spline whose three segments are quadratic polynomials (i.e., parabolic arcs) joined smoothly at the knots. Notice again that the support of $N_{i3}(t)$ is the interval $[i, i+3)$ and that they are shifted versions of each other, allowing us to write $N_{i3}(t) = N_{03}(t-i)$.

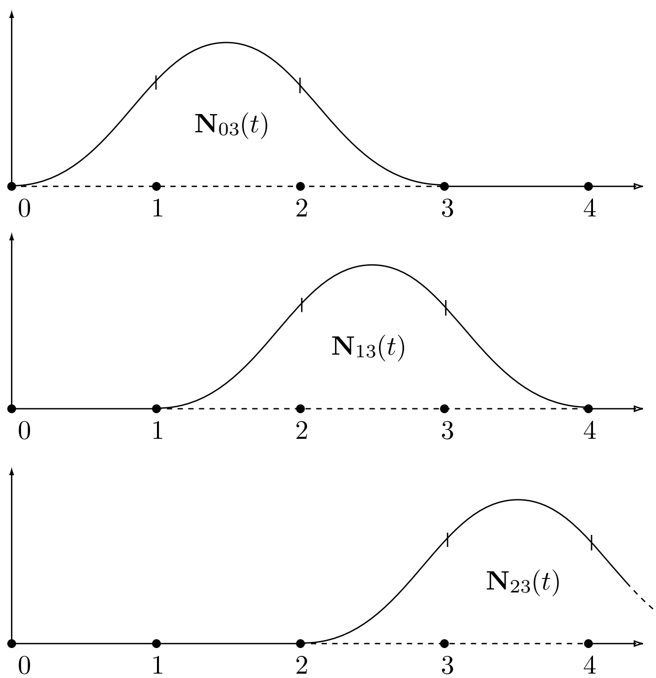


Figure 7.14: Uniform B-Spline Weight Functions for $k = 3$.

◇ **Exercise 7.9:** How can we show that the various $N_{i3}(t)$ are shifted versions of each other?

In general, the support of $N_{ik}(t)$ is the interval $[i, i+k)$ and $N_{ik}(t) = N_{0k}(t-i)$. Figure 7.15 shows how a general weight function $\mathbf{N}_{ik}(t)$ is constructed recursively. Each $\mathbf{N}_{ij}(t)$ function in this triangle is constructed as a weighted sum of the two functions immediately to its left.

The geometric and algebraic definitions of the B-spline look different but it can be shown that they are identical. The proof of this is called the Cox–DeBoor (or DeBoor–Cox) formula [DeBoor 72].

$$
\begin{array}{lllll}
\mathbf{N}_{i,1} & & & & \\
\mathbf{N}_{i+1,1} & . & & & \\
. & \;\;. & & & \\
. & & \mathbf{N}_{i,k-2} & & \\
 & & & \mathbf{N}_{i,k-1} & \\
 & & \mathbf{N}_{i+1,k-2} & & \mathbf{N}_{i,k} \\
 & & & \mathbf{N}_{i+1,k-1} & \\
. & & \mathbf{N}_{i+2,k-2} & & \\
. & \;\;. & & & \\
\mathbf{N}_{i+k-2,1} & . & & & \\
\mathbf{N}_{i+k-1,1} & & & &
\end{array}
$$

Figure 7.15: Recursive Construction of $\mathbf{N}_{i,k}(t)$.

7.9 Recursive Definitions of the B-Spline

The order k of the B-spline curve is an integer in the interval $[2, n+1]$ (it is possible to have $k = 1$, but the curve degenerates in this case to just a plot of the control points). Each blending function $N_{ik}(t)$ has support over k intervals $[t_i, t_{i+k-1})$ and is zero outside its support. The knot vector $(t_0, t_1, \ldots, t_{n+k})$ consists of $n + k + 1$ nondecreasing real numbers t_i. These values define $n + k$ subintervals $[t_i, t_{i+1})$. The two extreme values t_0 and t_n are selected based on the values of n and k. Any terms of the form $0/0$ or $x/0$ in the calculation of the blending functions are assumed to be zero. Editing the B-spline curve can be done by (1) adding, moving, or deleting control points without changing the order k, (2) changing the order k without modifying the control points, and (3) increasing the size of the knot vector. The knot vector contains $n+k+1$ values, so increasing its size implies that either n or k should be increased. Here are a few more properties of the curve:

1. Plotting the B-spline curve is done by varying the parameter t over the range of knot values $[t_{k-1}, t_{n+1})$.

2. Each segment of the curve (between two consecutive knot values) depends on k control points. This is why the curve has local control and it also implies that the maximum value of k is the number $n + 1$ of control points.

3. Any control point participates in at most k segments.

4. The curve lies inside the convex hull defined by at most k control points. This means that the curve passes close to the control points, a feature that makes it easy for a designer to place these points in order to obtain the right curve shape.

5. The blending functions $N_{ik}(t)$ are barycentric for any t in the interval $[t_{k-1}, t_{n+1})$. They are also nonnegative and, except for $k = 1$, each has one maximum.

6. The curve and its first $k - 1$ derivatives are continuous over the entire range (except that nonuniform B-splines can have discontinuities, see Figure 7.19d).

7. The entire curve can be affinely transformed by transforming the control points, then redrawing the curve from the new points.

One important difference between the B-spline and the Bézier curve is the use of a *knot vector*. This feature (which has already been mentioned) consists of a nondecreasing sequence of real numbers called *knots*. The knot vector adds flexibility to the curve and provides better control of its shape, but its use requires experience. There are three common ways to select the values in the knot vector, namely uniform, open uniform, and nonuniform. In a uniform B-spline the knot values are equally spaced. An example is $(-2, -1.5, -0.5, 0, 0.5, 1, 1.5)$, but more typical examples are a vector with normalized values between 0 and 1 $(0, 0.2, 0.4, 0.6, 0.8, 1)$ or a vector with integer values $(0, 1, 2, 3, 4, 5, 6)$. Figure 7.16 lists *Mathematica* code to calculate, print, and plot the weight functions for any set of knots.

```
(* B-spline weight functions printed and plotted *)
Clear[bspl,knt,i,k,n,t,p]
bspl[i_,k_,t_]:=If[knt[[i+k]]==knt[[i+1]],0, (* 0<=i<=n *)
 bspl[i,k-1,t] (t-knt[[i+1]])/(knt[[i+k]]-knt[[i+1]])] \
 +If[knt[[i+1+k]]==knt[[i+2]],0,
 bspl[i+1,k-1,t] (knt[[i+1+k]]-t)/(knt[[i+1+k]]-knt[[i+2]])];
bspl[i_,1,t_]:=If[knt[[i+1]]<=t<knt[[i+2]], 1, 0];
n=4; k=3; (* Note: 0<=k<=n *)
(* knt=Table[i, {i,0,n+k}]; *) (* knots for the uniform case *)
knt={0,0,0,1,2,3,3,3}; (* knots for the NONuniform case *)
(* Show the weight functions *)
Do[Print["N(",i,",",k,",",t,")=",Simplify[bspl[i,k,t]]], {i,0,n}]
(* Plot them. Plots are separated using .97 instead of 1 *)
Do[p[i+1]=Plot[bspl[i,k,t], {t,k-.97,n+.97},
 DisplayFunction->Identity], {i,0,n}]
Show[Table[p[i+1], {i,0,n}], Ticks->None,
 DisplayFunction->$DisplayFunction]
```

Figure 7.16: Code for the B-Spline Weight Functions.

7.10 Open Uniform B-Splines

The open uniform B-spline is obtained when the knot vector is uniform except at its two ends, where knot values are repeated k times. The following are simple examples:

For $n = 3$ and $k = 2$, there are $n + k + 1 = 6$ knots, e.g., $(0, 0, 1, 2, 3, 3)$.
For $n = 4$ and $k = 4$, there are $n + k + 1 = 9$ knots, e.g., $(0, 0, 0, 0, 1, 2, 2, 2, 2)$.
For $n = 3$ and $k = 2$, there are $n + k + 1 = 6$ knots, e.g., $(0, 0, 0.33, 0.67, 1, 1)$.
For $n = 4$ and $k = 4$, there are $n + k + 1 = 9$ knots, e.g., $(0, 0, 0, 0, 0.5, 1, 1, 1, 1)$.

(Notice how the last two examples are normalized.) In general, given values for n and k, we can generate an integer open knot vector by setting

$$t_i = \begin{cases} 0, & \text{for } 0 \le i < k, \\ i - k + 1, & \text{for } k \le i \le n, \\ n - k + 2, & \text{for } n < i \le n + k, \end{cases} \qquad \text{for} \quad 0 \le i \le n + k. \tag{7.27}$$

An open uniform B-spline curve starts at $\mathbf{P}_0$ and ends at $\mathbf{P}_n$. This feature makes it easy to generate closed curves of this type. The two extreme tangents of this curve point in the directions from $\mathbf{P}_0$ to $\mathbf{P}_1$ and from $\mathbf{P}_{n-1}$ to $\mathbf{P}_n$, respectively. This is why open uniform B-spline curves are similar to Bézier curves. In fact, when $k = n+1$ (i.e., when the degree of the polynomials is n), these curves have knot vectors of the form $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$ and they reduce to Bézier curves.

Example: (1) Five control points $\mathbf{P}_0$ through $\mathbf{P}_4$ are given, implying that $n = 4$. We select order 3 (i.e., segments that are polynomials of degree 2) and use Equation (7.27) to construct the knot sequence $(0, 0, 0, 1, 2, 3, 3, 3)$. The parameter t varies from $t_{k-1} = t_2 = 0$ to $t_{n+1} = t_5 = 3$, so our curve will consist of three segments. Each of the blending functions $N_{i3}(t)$ (where $0 \le i \le n$) is nonzero over three subintervals of t and is calculated from Equations (7.24) and (7.25). The result is

$$
\begin{aligned}
N_{03}(t) &= (1-t)^2, & 0 \le t < 1,\\
N_{13}(t) &= \frac{1}{2}\begin{cases} -3t^2 + 4t, & 0 \le t < 1,\\ (2-t)^2, & 1 \le t < 2,\end{cases}\\
N_{23}(t) &= \frac{1}{2}\begin{cases} t^2, & 0 \le t < 1,\\ -2t^2 + 6t - 3, & 1 \le t < 2,\\ (3-t)^2, & 2 \le t < 3,\end{cases}\\
N_{33}(t) &= \frac{1}{2}\begin{cases} (t-1)^2, & 1 \le t < 2,\\ -3t^2 + 14t - 15, & 2 \le t < 3,\end{cases}\\
N_{43}(t) &= (t-2)^2, & 2 \le t < 3,
\end{aligned}
$$

so the three spline segments are

$$
\begin{aligned}
\mathbf{P}_1(t) &= (1-t)^2\mathbf{P}_0 + \tfrac{1}{2}t(4-3t)\mathbf{P}_1 + \tfrac{1}{2}t^2\mathbf{P}_2, & 0 \le t < 1,\\
\mathbf{P}_2(t) &= \tfrac{1}{2}(2-t)^2\mathbf{P}_1 + \tfrac{1}{2}\big[t(2-t) + (t-1)(3-t)\big]\mathbf{P}_2 + \tfrac{1}{2}(t-1)^2\mathbf{P}_3, & 1 \le t < 2,\\
\mathbf{P}_3(t) &= \tfrac{1}{2}(3-t)^2\mathbf{P}_2 + \tfrac{1}{2}(3-t)(3t-5)\mathbf{P}_3 + (t-2)^2\mathbf{P}_4, & 2 \le t < 3.
\end{aligned}
$$

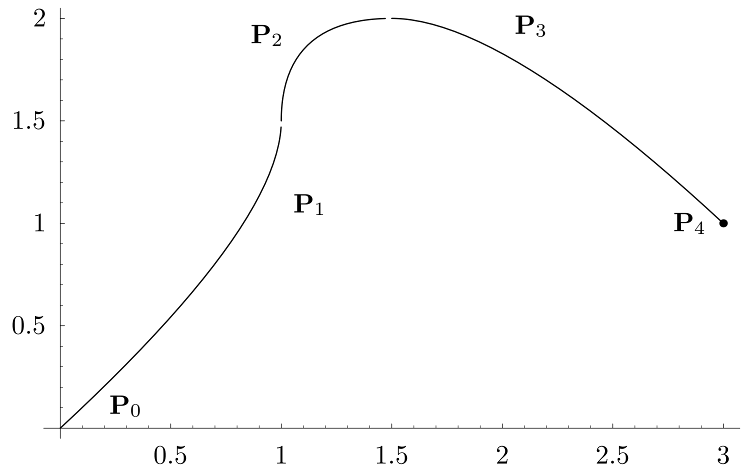
It is now easy to calculate where each segment starts and ends:

$$
\begin{gathered}
\mathbf{P}_1(0) = \mathbf{P}_0, \quad \mathbf{P}_1(1) = (\mathbf{P}_1 + \mathbf{P}_2)/2,\\
\mathbf{P}_2(1) = (\mathbf{P}_1 + \mathbf{P}_2)/2, \quad \mathbf{P}_2(2) = (\mathbf{P}_2 + \mathbf{P}_3)/2,\\
\mathbf{P}_3(2) = (\mathbf{P}_2 + \mathbf{P}_3)/2, \quad \mathbf{P}_3(3) = \mathbf{P}_4,
\end{gathered}
$$

Figure 7.17 shows a typical example of the three segments (with intentional gaps between them).

◇ **Exercise 7.10:** Show that the three spline segments provide C^1 continuity at the two interior points $\mathbf{P}_1(1) = \mathbf{P}_2(1)$ and $\mathbf{P}_2(2) = \mathbf{P}_3(2)$.

Example: (2) We again choose five control points but this time we select $k = n+1 = 5$. The curve will therefore consist of degree-4 polynomial segments. Such a segment requires five points (it has five coefficients, so five equations are needed), which is why we will end up with just one segment. Equation (7.27) is again used to construct the knot vector $(0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$. The parameter t varies from $t_{k-1} = t_4 = 0$ to




```
(* Plot a B-spline curve. Can also print the weight functions *)
Clear[bspl,knt,i,k,n,t,p,g1,g2,pnt] (* First the weight functions *)
bspl[i_,k_,t_]:=If[knt[[i+k]]==knt[[i+1]],0, (* 0<=i<=n *)
 bspl[i,k-1,t] (t-knt[[i+1]])/(knt[[i+k]]-knt[[i+1]])] \
 +If[knt[[i+1+k]]==knt[[i+2]],0,
 bspl[i+1,k-1,t] (knt[[i+1+k]]-t)/(knt[[i+1+k]]-knt[[i+2]])];
bspl[i_,1,t_]:=If[knt[[i+1]]<=t<knt[[i+2]], 1, 0];
n=4; k=3; (* Note: 0<=k<=n *)
(* knt=Table[i, {i,0,n+k}]; knots for the uniform case *)
knt={0,0,0,1,2,3,3,3}; (* knots for the open-unif or non-uniform cases *)
(* Do[Print[bspl[i,k,t]], {i,0,n}] Display the weight functions *)
pnt={{0,0},{1,1},{1,2},{2,2},{3,1}}; (* test for n+1=5 control points *)
p[t_]:=Sum[pnt[[i+1]] bspl[i,k,t], {i,0,n}] (* The curve as a weighted sum *)
g1=ListPlot[pnt, Prolog->AbsolutePointSize[3], DisplayFunction->Identity];
g2=ParametricPlot[p[t], {t,0,.97}, Compiled->False, DisplayFunction->Identity];
g3=ParametricPlot[p[t], {t,1,1.97}, Compiled->False, DisplayFunction->Identity];
g4=ParametricPlot[p[t], {t,2,3}, Compiled->False, DisplayFunction->Identity];
Show[g1,g2,g3,g4, PlotRange->All, DisplayFunction->$DisplayFunction,
 DefaultFont->{"cmr10", 10}];
```

Figure 7.17: An Open Uniform B-Spline.

$t_{n+1} = t_5 = 1$, showing again that the curve will consist of one segment. This should be a Bézier curve, because $k = n + 1$.

The calculation of the blending functions $N_{i5}(t)$ (where $0 \le i \le n$) is shown here in detail. We start with the nine functions $N_{i1}(t)$ that are calculated from Equation (7.24)

$$N_{01} = 1 \text{ when } t_0 \le t < t_1, N_{11} = 1 \text{ when } t_1 \le t < t_2, \ldots, N_{81} = 1 \text{ when } t_8 \le t < t_9.$$

Since $t_0 = t_1 = t_2 = t_3 = t_4 = 0$ and $t_5 = t_6 = t_7 = t_8 = t_9 = 1$, we conclude that

$$N_{41} = 1 \quad \text{when} \quad t \in [t_4, t_5) = [0, 1),$$

and the other eight functions $N_{i1}(t)$ are zero. The next step is to calculate the eight functions $N_{i2}(t)$ from Equation (7.25):

$$
\begin{aligned}
N_{02}(t) &= \frac{t-t_0}{t_1-t_0}N_{01} + \frac{t_2-t}{t_2-t_1}N_{11} = 0,\\
N_{12}(t) &= \frac{t-t_1}{t_2-t_1}N_{11} + \frac{t_3-t}{t_3-t_2}N_{21} = 0,\\
N_{22}(t) &= \frac{t-t_2}{t_3-t_2}N_{21} + \frac{t_4-t}{t_4-t_3}N_{31} = 0,\\
N_{32}(t) &= \frac{t-t_3}{t_4-t_3}N_{31} + \frac{t_5-t}{t_5-t_4}N_{41} = 0 + (1-t),\\
N_{42}(t) &= \frac{t-t_4}{t_5-t_4}N_{41} + \frac{t_6-t}{t_6-t_5}N_{51} = t + 0,\\
N_{52}(t) &= \frac{t-t_5}{t_6-t_5}N_{51} + \frac{t_7-t}{t_7-t_6}N_{61} = 0,\\
N_{62}(t) &= \frac{t-t_6}{t_7-t_6}N_{61} + \frac{t_8-t}{t_8-t_7}N_{71} = 0,\\
N_{72}(t) &= \frac{t-t_7}{t_8-t_7}N_{71} + \frac{t_9-t}{t_9-t_8}N_{81} = 0.
\end{aligned}
$$

Only $N_{32}(t)$ and $N_{42}(t)$ are nonzero. The seven functions $N_{i3}(t)$ are calculated similarly:

$$
\begin{aligned}
N_{03}(t) &= \frac{t-t_0}{t_2-t_0}N_{02} + \frac{t_3-t}{t_3-t_1}N_{12} = 0,\\
N_{13}(t) &= \frac{t-t_1}{t_3-t_1}N_{12} + \frac{t_4-t}{t_4-t_2}N_{22} = 0,\\
N_{23}(t) &= \frac{t-t_2}{t_4-t_2}N_{22} + \frac{t_5-t}{t_5-t_3}N_{32} = 0 + (1-t)^2,\\
N_{33}(t) &= \frac{t-t_3}{t_5-t_3}N_{32} + \frac{t_6-t}{t_6-t_4}N_{42} = t(1-t) + (1-t)t,\\
N_{43}(t) &= \frac{t-t_4}{t_6-t_4}N_{42} + \frac{t_7-t}{t_7-t_5}N_{52} = t^2 + 0,\\
N_{53}(t) &= \frac{t-t_5}{t_7-t_5}N_{52} + \frac{t_8-t}{t_8-t_6}N_{62} = 0,\\
N_{63}(t) &= \frac{t-t_6}{t_8-t_6}N_{62} + \frac{t_9-t}{t_9-t_7}N_{72} = 0.
\end{aligned}
$$

Three of the seven functions are nonzero. The six functions $N_{i4}(t)$ are

$$
\begin{aligned}
N_{04}(t) &= \frac{t-t_0}{t_3-t_0}N_{03} + \frac{t_4-t}{t_4-t_1}N_{13} = 0,\\
N_{14}(t) &= \frac{t-t_1}{t_4-t_1}N_{13} + \frac{t_5-t}{t_5-t_2}N_{23} = 0 + (1-t)^3,\\
N_{24}(t) &= \frac{t-t_2}{t_5-t_2}N_{23} + \frac{t_6-t}{t_6-t_3}N_{33} = t(1-t)^2 + 2t(1-t)^2,\\
N_{34}(t) &= \frac{t-t_3}{t_6-t_3}N_{33} + \frac{t_7-t}{t_7-t_4}N_{43} = 2t^2(1-t) + (1-t)t^2,
\end{aligned}
$$

$$N_{44}(t) = \frac{t - t_4}{t_7 - t_4} N_{43} + \frac{t_8 - t}{t_8 - t_5} N_{53} = t^3,$$
$$N_{54}(t) = \frac{t - t_5}{t_8 - t_5} N_{53} + \frac{t_9 - t}{t_9 - t_6} N_{63} = 0.$$

Four of them are nonzero. The last step is the calculation of the five functions $N_{i5}(t)$:

$$N_{05}(t) = \frac{t - t_0}{t_4 - t_0} N_{04} + \frac{t_5 - t}{t_5 - t_1} N_{14} = (1-t)^4,$$
$$N_{15}(t) = \frac{t - t_1}{t_5 - t_1} N_{14} + \frac{t_6 - t}{t_6 - t_2} N_{24} = t(1-t)^3 + 3t(1-t)^3,$$
$$N_{25}(t) = \frac{t - t_2}{t_6 - t_2} N_{24} + \frac{t_7 - t}{t_7 - t_3} N_{34} = 3t^2(1-t)^2 + 3t^2(1-t)^2,$$
$$N_{35}(t) = \frac{t - t_3}{t_7 - t_3} N_{34} + \frac{t_8 - t}{t_8 - t_4} N_{44} = 3t^3(1-t) + (1-t)t^3,$$
$$N_{45}(t) = \frac{t - t_4}{t_8 - t_4} N_{44} + \frac{t_9 - t}{t_9 - t_5} N_{54} = t^4.$$

All five are nonzero and they should look familiar (they are the Bernstein polynomials for $n = 4$). The curve consists of the single segment

$$\mathbf{P}(t) = \sum_{i=0}^{4} N_{i5}(t)\mathbf{P}_i$$
$$= (1-t)^4\mathbf{P}_0 + 4t(1-t)^3\mathbf{P}_1 + 6t^2(1-t)^2\mathbf{P}_2 + 4t^3(1-t)\mathbf{P}_3 + t^4\mathbf{P}_4,$$

which is the Bézier curve defined by the five points. The B-spline curve is again shown to be more general than the Bézier curve, since it contains the latter as a special case.

It is the multiplicity of knot values that causes the open B-spline to start and end at its extreme control points. This is easy to understand when we realize that every subinterval $[t_i, t_{i+1})$ of knots corresponds to one segment $\mathbf{P}_i(t)$ of the B-spline. When $t_i = t_{i+1}$, that segment reduces to a point. The result is that each repeat of a knot value decreases the continuity at a joint point by 1. Consider, for example, the open B-spline of order $k = 4$. The individual spline segments are degree-3 (cubic) polynomials that have C^2 continuity at their joint points. If knot t_i has multiplicity 2 (i.e., $t_i = t_{i+1}$), then segment $\mathbf{P}_i(t)$ reduces to a point and segments $\mathbf{P}_{i-1}(t)$ and $\mathbf{P}_{i+1}(t)$ meet at a joint point with C^1 continuity. If knot t_i has multiplicity 3 ($t_i = t_{i+1} = t_{i+2}$), then segments $\mathbf{P}_i(t)$ and $\mathbf{P}_{i+1}(t)$ reduce to points and segments $\mathbf{P}_{i-1}(t)$ and $\mathbf{P}_{i+2}(t)$ meet at a joint point (which in this case is a control point) with C^0 continuity. If the first knot has multiplicity 4 ($t_0 = t_1 = t_2 = t_3$), then segments $\mathbf{P}_0(t)$, $\mathbf{P}_1(t)$, and $\mathbf{P}_2(t)$ reduce to points and segment $\mathbf{P}_3(t)$ starts at that point with no continuity.

7.11 Nonuniform B-Splines

The nonuniform B-spline is more general than the uniform or open B-splines, although it is not the most general type of this curve. It is obtained when the knot values are not equally spaced. The only requirement is that the knots be nondecreasing. Adjusting the knot values (as well as having multiple values) is a feature that helps fine-tune the shape of the curve. Multiple knots can be used to pull the curve in a certain direction and to create a cusp or even a discontinuity at a join point. Nonuniform B-splines can get complex, so we limit the discussion in this section to order-4 (i.e., degree-3) nonuniform B-splines. This is not a serious limitation, as this type is the most commonly used and it makes it easier to understand the properties and behavior of the nonuniform B-spline.

In the case of order-4 nonuniform B-splines, the knot vector contains values from t_0 to t_{n+4} (there are four more knots than control points), so the minimum number of knots is eight (since the minimum number of control points is four) and the parameter t varies, in this case, from $t_{k-1} = t_3$ to $t_{n+1} = t_4$. Spline segment $\mathbf{P}_i(t)$ depends on control points $\mathbf{P}_{i-3}$, $\mathbf{P}_{i-2}$, $\mathbf{P}_{i-1}$, and $\mathbf{P}_i$ and its expression is

$$\mathbf{P}_i(t) = N_{i-3,4}(t)\mathbf{P}_{i-3} + N_{i-2,4}(t)\mathbf{P}_{i-2} + N_{i-1,4}(t)\mathbf{P}_{i-1} + N_{i,4}(t)\mathbf{P}_i,$$

where $3 \le i \le n$ and $t_i \le t \le t_{i+1}$. There are $n-2$ segments denoted by $\mathbf{P}_3(t)$ through $\mathbf{P}_n(t)$. When $n = 3$ (four control points), the curve consists of just one segment. When knot t_i has multiplicity 2 (i.e., $t_i = t_{i+1}$), segment $\mathbf{P}_i(t)$ reduces to a point. As has been mentioned earlier, it is this feature that makes the nonuniform B-spline so flexible, powerful, and therefore useful in practical work.

The weight functions are defined recursively by Equations (7.24) and (7.25) but go up to N_{i4} only:

$$\begin{aligned}
N_{i1}(t) &= \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise,} \end{cases} \\
N_{i2}(t) &= \frac{t - t_i}{t_{i+1} - t_i} N_{i,1}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} N_{i+1,1}(t), \\
N_{i3}(t) &= \frac{t - t_i}{t_{i+2} - t_i} N_{i,2}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} N_{i+1,2}(t), \\
N_{i4}(t) &= \frac{t - t_i}{t_{i+3} - t_i} N_{i,3}(t) + \frac{t_{i+4} - t}{t_{i+4} - t_{i+1}} N_{i+1,3}(t).
\end{aligned} \tag{7.28}$$

The first set, $N_{i1}(t)$, are horizontal segments. The second set, $N_{i2}(t)$, are straight lines. The third set are quadratic polynomials and the fourth set, $N_{i4}(t)$, are cubic polynomials. Each cubic segment is defined by four control points and lies in the convex hull defined by the points. Thus, segment $\mathbf{P}_i(t)$ is defined by points $\mathbf{P}_{i-3}$, $\mathbf{P}_{i-2}$, $\mathbf{P}_{i-1}$, and $\mathbf{P}_i$, while segment $\mathbf{P}_{i+1}(t)$ is defined by points $\mathbf{P}_{i-2}$, $\mathbf{P}_{i-1}$, $\mathbf{P}_i$, and $\mathbf{P}_{i+1}$.

Figure 7.19 illustrates the effect of knot multiplicities using $n = 7$ (i.e., eight points) as an example. The knot vector should contain $n + k + 1 = 7 + 4 + 1 = 12$ values and t should vary from $t_{k-1} = t_3$ to $t_{n+1} = t_8$, a total of five subintervals. The four parts of the figure show cubic B-spline curves constructed with the knot vectors

$$(-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8), \quad (-3, -2, -1, 0, 1, 1, 2, 3, 4, 5, 6, 7),$$

```
(* 8-Point Nonuniform Cubic B-Spline Example. Five Segments *)
Clear[g,Q,pts,seg];
P0={0,0}; P1={0,1}; P2={1,1}; P3={1,0}; P4={2,0}; P5={2.75,1}; P6={3,1}; P7={3,0};
pts=Graphics[{PointSize[.01], Point/@{P0,P1,P2,P3,P4,P5,P6,P7}}];
seg={AbsoluteDashing[{2,2}], Line[{P1,P2,P3}], Line[{P4,P5,P6,P7}]};
Q[t_]:={((1-t)^3 P0 +(3t^3-6t^2+4) P1 +(-3t^3+3t^2+3t+1) P2 +t^3 P3)/6,
((2-t)^3 P1 +(3t^3-15t^2+21t-5) P2 +(-3t^3+12t^2-12t+4) P3 +(t-1)^3 P4)/6,
((3-t)^3 P2 +(3t^3-24t^2+60t-44) P3 +(-3t^3+21t^2-45t+31) P4 +(t-2)^3 P5)/6,
((4-t)^3 P3 +(3t^3-33t^2+117t-131) P4 +(-3t^3+30t^2-96t+100) P5 +(t-3)^3 P6)/6,
((5-t)^3 P4 +(3t^3-42t^2+192t-284) P5 +(-3t^3+39t^2-165t+229) P6 +(t-4)^3 P7)/6};
g=Table[ParametricPlot[Q[t][[i]], {t,i-1,0.97i},
 Compiled->False, DisplayFunction->Identity], {i,1,5}];
Show[g, pts, Graphics[seg], PlotRange->All, DefaultFont->{"cmr10", 10},
 DisplayFunction->$DisplayFunction, AspectRatio->Automatic];
```

For the four segments of part (b), the only difference is

```
Q[t_]:={(1-t)^3/6 P0 +(11t^3-15t^2-3t+7)/12 P1+(-5t^3+3t^2+3t+1)/4 P2 +t^3/2 P3,
(2-t)^3/2 P2 +(5t^3-27t^2+45t-21)/4 P3+(-11t^3+51t^2-69t+29)/12 P4 +(t-1)^3/6 P5,
(3-t)^3/4 P3 +(7t^3-57t^2+147t-115)/12 P4+(-3t^3+21t^2-45t+31)/6 P5 +(t-2)^3/6 P6,
((4-t)^3 P4 +(3t^3-33t^2+117t-131)P5+(-3t^3+30t^2-96t+100)P6 +(t-3)^3 P7)/6};
g=Table[ParametricPlot[Q[t][[i]], {t,i-1,0.97i},
 Compiled->False, DisplayFunction->Identity], {i,1,4}];
```

For the three segments of part (c), the only difference is

```
Q[t_]:={(1-t)^3 P0 /6+(11t^3-15t^2-3t+7)P1 /12+(-7t^3+3t^2+3t+1)P2 /4+t^3 P3,
(2-t)^3 P3+(7t^3-39t^2+69t-37)P4 /4+(-11t^3+51t^2-69t+29) P5 /12+(t-1)^3 P6 /6,
(3-t)^3 P4 /4+(7t^3-57t^2+147t-115)P5 /12+(-3t^3+21t^2-45t+31) P6 /6+(t-2)^3 P7 /6};
g=Table[ParametricPlot[Q[t][[i]], {t,i-1,0.97i},
 Compiled->False, DisplayFunction->Identity], {i,1,3}];
```

For the two segments of part (d), the only difference is

```
Q[t_]:={(1-t)^3P0 /6 +(11t^3-15t^2-3t+7)P1 /12+(-7t^3+3t^2+3t+1)P2 /4 +t^3 P3,
(2-t)^3 P4 +(7t^3-39t^2+69t-37)P5 /4+(-11t^3+51t^2-69t+29)P6 /12+(t-1)^3P7 /6};
g=Table[ParametricPlot[Q[t][[i]], {t,i-1,0.97i},
 Compiled->False, DisplayFunction->Identity], {i,1,2}];
```

Figure 7.18: Code for an 8-Point Nonuniform B-Spline Example, Figure 7.19.

$$(-3,-2,-1,0,1,1,1,2,3,4,5,6), \quad (-3,-2,-1,0,1,1,1,1,2,3,4,5),$$

respectively. Notice that only six knots, t_3 through t_8, are really important. The rest are distinct and uniform but less important, since only some of them are used in calculating the blending functions.

In Figure 7.19a, all knots have multiplicity 1, each segment is defined by four points, and adjacent segments share three points. The first segment, $\mathbf{P}_3(t)$, is defined by points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, while the last segment, $\mathbf{P}_7(t)$, is defined by points $\mathbf{P}_4$, $\mathbf{P}_5$, $\mathbf{P}_6$, and $\mathbf{P}_7$. The five segments join with C^2 continuity. In Figure 7.19b, we set $t_4 = t_5$, thereby reducing segment $\mathbf{P}_4(t)$ to zero length, causing segments $\mathbf{P}_3(t)$ and $\mathbf{P}_5(t)$ to meet at join $t_4 = t_5$. However, these segments share just two control points, $\mathbf{P}_2$ and $\mathbf{P}_3$, so they have less "in common" and, consequently, join with only C^1 continuity. In Figure 7.19c, we set $t_4 = t_5 = t_6$, thereby reducing segments $\mathbf{P}_4(t)$ and $\mathbf{P}_5(t)$ to zero length and causing segments $\mathbf{P}_3(t)$ and $\mathbf{P}_6(t)$ to meet. These segments share just one control point, namely $\mathbf{P}_3$, so they meet at this point, with C^0 continuity. In Figure 7.19d, we set

$t_4 = t_5 = t_6 = t_7$, so now we have three zero-length segments, namely $\mathbf{P}_4(t)$, $\mathbf{P}_5(t)$, and $\mathbf{P}_6(t)$. Segments $\mathbf{P}_3(t)$ and $\mathbf{P}_7(t)$ now have to meet, but they don't have any common control points. The result is a *discontinuity* (a break) in the curve between points $\mathbf{P}_3$ and $\mathbf{P}_4$.

Figure 7.18 lists the code for Figure 7.19.

Example: This long example is divided into two parts.

Part a. In this part, we calculate the blending functions and spline segments of the curve of Figure 7.19a, where the knot vector is the uniform sequence

$$(-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8).$$

The calculations are done bearing in mind that t varies from $t_3 = 0$ to $t_8 = 5$. We need to calculate all the functions $N_{i4}(t)$ that are nonzero in the five subintervals $[0,1)$, $[1,2)$, $[2,3)$, $[3,4)$, and $[4,5)$. Four blending functions are used to construct each of the five spline segments, so segment $\mathbf{P}_3(t)$ is defined by functions $N_{04}(t)$ through $N_{34}(t)$, segment $\mathbf{P}_4(t)$ is defined by functions $N_{14}(t)$ through $N_{44}(t)$, and segment $\mathbf{P}_7(t)$ is defined by functions $N_{44}(t)$ through $N_{74}(t)$. The first step is to calculate N_{i1}:

$$\begin{gathered} N_{31} = 1 \quad \text{for} \quad t \in [0,1), \quad N_{41} = 1 \quad \text{for} \quad t \in [1,2), \\ N_{51} = 1 \quad \text{for} \quad t \in [2,3), \quad N_{61} = 1 \quad \text{for} \quad t \in [3,4), \quad N_{71} = 1 \quad \text{for} \quad t \in [4,5), \end{gathered}$$

and N_{01}, N_{11}, N_{21}, N_{81}, N_{91}, $N_{10,1}$, and $N_{11,1}$ are zero in the range $0 \le t < 5$.

Step 2 is to calculate functions N_{i2} that are nonzero for $0 \le t < 5$:

$$\begin{aligned}
N_{02}(t) &= \frac{t-t_0}{t_1-t_0}N_{01} + \frac{t_2-t}{t_2-t_1}N_{11} = 0,\\
N_{12}(t) &= \frac{t-t_1}{t_2-t_1}N_{11} + \frac{t_3-t}{t_3-t_2}N_{21} = 0,\\
N_{22}(t) &= \frac{t-t_2}{t_3-t_2}N_{21} + \frac{t_4-t}{t_4-t_3}N_{31} = (1-t) \quad \text{for} \quad t \in [0,1),\\
N_{32}(t) &= \frac{t-t_3}{t_4-t_3}N_{31} + \frac{t_5-t}{t_5-t_4}N_{41} = \begin{cases} t & \text{for } t \in [0,1),\\ 2-t & \text{for } t \in [1,2),\end{cases}\\
N_{42}(t) &= \frac{t-t_4}{t_5-t_4}N_{41} + \frac{t_6-t}{t_6-t_5}N_{51} = \begin{cases} t-1 & \text{for } t \in [1,2),\\ 3-t & \text{for } t \in [2,3),\end{cases}\\
N_{52}(t) &= \frac{t-t_5}{t_6-t_5}N_{51} + \frac{t_7-t}{t_7-t_6}N_{61} = \begin{cases} t-2 & \text{for } t \in [2,3),\\ 4-t & \text{for } t \in [3,4),\end{cases}\\
N_{62}(t) &= \frac{t-t_6}{t_7-t_6}N_{61} + \frac{t_8-t}{t_8-t_7}N_{71} = \begin{cases} t-3 & \text{for } t \in [3,4),\\ 5-t & \text{for } t \in [4,5),\end{cases}\\
N_{72}(t) &= \frac{t-t_7}{t_8-t_7}N_{71} + \frac{t_9-t}{t_9-t_8}N_{81} = t-4 \qquad \text{for } t \in [4,5).
\end{aligned}$$

This step terminates at $N_{72}(t)$ since $N_{82}(t)$ and its successors are zero for $0 \le t < 5$.

Step 3 requires the calculation of several functions N_{i3}:

$$N_{03}(t) = \frac{t-t_0}{t_2-t_0}N_{02} + \frac{t_3-t}{t_3-t_1}N_{12} = 0,$$

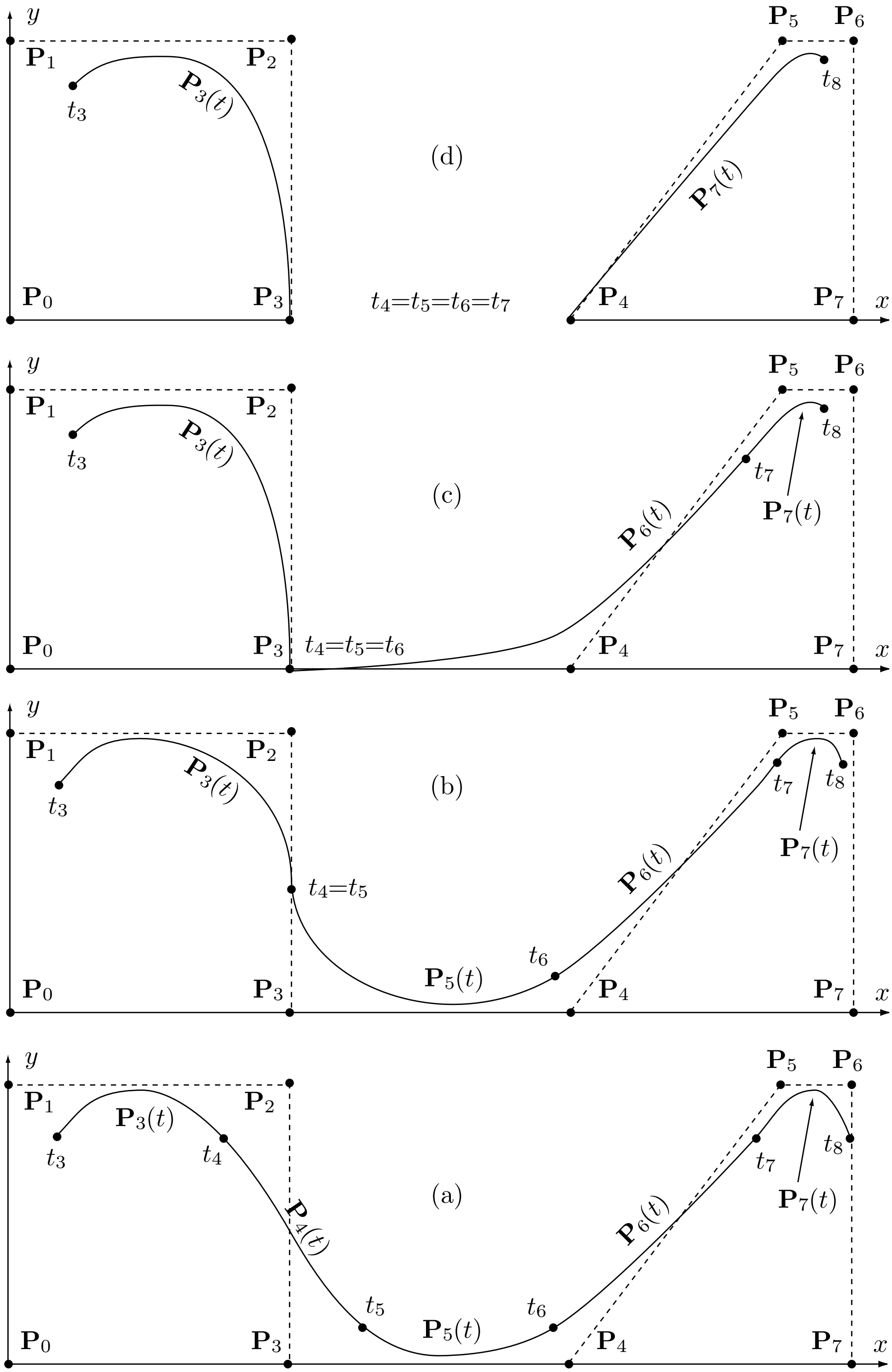


Figure 7.19: An Eight-Point Nonuniform B-Spline Curve with Multiple Knots.

$$N_{13}(t) = \frac{t-t_1}{t_3-t_1}N_{12} + \frac{t_4-t}{t_4-t_2}N_{22} = \frac{1}{2}(1-t)^2 \qquad \text{for } t\in[0,1),$$

$$N_{23}(t) = \frac{t-t_2}{t_4-t_2}N_{22} + \frac{t_5-t}{t_5-t_3}N_{32} = \frac{1}{2}\begin{cases}(-2t^2+2t+1) & \text{for } t\in[0,1),\\ (2-t)^2 & \text{for } t\in[1,2),\end{cases}$$

$$N_{33}(t) = \frac{t-t_3}{t_5-t_3}N_{32} + \frac{t_6-t}{t_6-t_4}N_{42} = \frac{1}{2}\begin{cases}t^2 & \text{for } t\in[0,1),\\ (-2t^2+6t-3) & \text{for } t\in[1,2),\\ (3-t)^2 & \text{for } t\in[2,3),\end{cases}$$

$$N_{43}(t) = \frac{t-t_4}{t_6-t_4}N_{42} + \frac{t_7-t}{t_7-t_5}N_{52} = \frac{1}{2}\begin{cases}(t-1)^2 & \text{for } t\in[1,2),\\ (-2t^2+10t-11) & \text{for } t\in[2,3),\\ (4-t)^2 & \text{for } t\in[3,4),\end{cases}$$

$$N_{53}(t) = \frac{t-t_5}{t_7-t_5}N_{52} + \frac{t_8-t}{t_8-t_6}N_{62} = \frac{1}{2}\begin{cases}(t-2)^2 & \text{for } t\in[2,3),\\ (-2t^2+14t-23) & \text{for } t\in[3,4),\\ (5-t)^2 & \text{for } t\in[4,5),\end{cases}$$

$$N_{63}(t) = \frac{t-t_6}{t_8-t_6}N_{62} + \frac{t_9-t}{t_9-t_7}N_{72} = \frac{1}{2}\begin{cases}(t-3)^2 & \text{for } t\in[3,4),\\ (-2t^2+18t-39) & \text{for } t\in[4,5),\end{cases}$$

$$N_{73}(t) = \frac{t-t_7}{t_9-t_7}N_{72} + \frac{t_{10}-t}{t_{10}-t_8}N_{82} = \frac{1}{2}(t-4)^2 \qquad \text{for } t\in[4,5).$$

We stop at N_{73} since N_{83} and its successors are zero for $0 \le t < 5$.

The last step involves the calculation of eight functions N_{i4}:

$$N_{04}(t) = \frac{t-t_0}{t_3-t_0}N_{03} + \frac{t_4-t}{t_4-t_1}N_{13} = \frac{1}{6}(1-t)^3 \qquad \text{for } t\in[0,1),$$

$$N_{14}(t) = \frac{t-t_1}{t_4-t_1}N_{13} + \frac{t_5-t}{t_5-t_2}N_{23} = \frac{1}{6}\begin{cases}(3t^3-6t^2+4) & \text{for } t\in[0,1),\\ (2-t)^3 & \text{for } t\in[1,2),\end{cases}$$

$$N_{24}(t) = \frac{t-t_2}{t_5-t_2}N_{23} + \frac{t_6-t}{t_6-t_3}N_{33} = \frac{1}{6}\begin{cases}(-3t^3+3t^2+3t+1) & \text{for } t\in[0,1),\\ (3t^3-15t^2+21t-5) & \text{for } t\in[1,2),\\ (3-t)^3 & \text{for } t\in[2,3),\end{cases}$$

$$N_{34}(t) = \frac{t-t_3}{t_6-t_3}N_{33} + \frac{t_7-t}{t_7-t_4}N_{43} = \frac{1}{6}\begin{cases}t^3 & \text{for } t\in[0,1),\\ (-3t^3+12t^2-12t+4) & \text{for } t\in[1,2),\\ (3t^3-24t^2+60t-44) & \text{for } t\in[2,3),\\ (4-t)^3 & \text{for } t\in[3,4),\end{cases}$$

$$N_{44}(t) = \frac{t-t_4}{t_7-t_4}N_{43} + \frac{t_8-t}{t_8-t_5}N_{53} = \frac{1}{6}\begin{cases}(t-1)^3 & \text{for } t\in[1,2),\\ (-3t^3+21t^2-45t+31) & \text{for } t\in[2,3),\\ (3t^3-33t^2+117t-131) & \text{for } t\in[3,4),\\ (5-t)^3 & \text{for } t\in[4,5),\end{cases}$$

$$N_{54}(t) = \frac{t-t_5}{t_8-t_5}N_{53} + \frac{t_9-t}{t_9-t_6}N_{63} = \frac{1}{6}\begin{cases}(t-2)^3 & \text{for } t\in[2,3),\\ (-3t^3+30t^2-96t+100) & \text{for } t\in[3,4),\\ (3t^3-42t^2+192t-284) & \text{for } t\in[4,5),\end{cases}$$

$$N_{64}(t) = \frac{t-t_6}{t_9-t_6}N_{63} + \frac{t_{10}-t}{t_{10}-t_7}N_{73} = \frac{1}{6}\begin{cases}(t-3)^3 & \text{for } t\in[3,4),\\ (-3t^3+39t^2-165t+229) & \text{for } t\in[4,5),\end{cases}$$

$$N_{74}(t) = \frac{t-t_7}{t_{10}-t_7}N_{73} + \frac{t_{11}-t}{t_{11}-t_8}N_{83} = \frac{1}{6}(t-4)^3 \qquad \text{for } t \in [4,5).$$

A careful study of this last group shows that N_{84} and its successors are zero for $0 \le t < 5$.

The last group of blending functions can now be used to construct the five spline segments:

$$\begin{aligned}
\mathbf{P}_3(t) &= N_{04}(t)\mathbf{P}_0 + N_{14}(t)\mathbf{P}_1 + N_{24}(t)\mathbf{P}_2 + N_{34}(t)\mathbf{P}_3 && t \in [0,1)\\
&= \frac{1}{6}\big[(1-t)^3\mathbf{P}_0 + (3t^3 - 6t^2 + 4)\mathbf{P}_1\\
&\quad + (-3t^3 + 3t^2 + 3t + 1)\mathbf{P}_2 + t^3\mathbf{P}_3\big],\\
\mathbf{P}_4(t) &= N_{14}(t)\mathbf{P}_1 + N_{24}(t)\mathbf{P}_2 + N_{34}(t)\mathbf{P}_3 + N_{44}(t)\mathbf{P}_4 && t \in [1,2)\\
&= \frac{1}{6}\big[(2-t)^3\mathbf{P}_1 + (3t^3 - 15t^2 + 21t - 5)\mathbf{P}_2\\
&\quad + (-3t^3 + 12t^2 - 12t + 4)\mathbf{P}_3 + (t-1)^3\mathbf{P}_4\big],\\
\mathbf{P}_5(t) &= N_{24}(t)\mathbf{P}_2 + N_{34}(t)\mathbf{P}_3 + N_{44}(t)\mathbf{P}_4 + N_{54}(t)\mathbf{P}_5 && t \in [2,3)\\
&= \frac{1}{6}\big[(3-t)^3\mathbf{P}_2 + (3t^3 - 24t^2 + 60t - 44)\mathbf{P}_3\\
&\quad + (-3t^3 + 21t^2 - 45t + 31)\mathbf{P}_4 + (t-2)^3\mathbf{P}_5\big],\\
\mathbf{P}_6(t) &= N_{34}(t)\mathbf{P}_3 + N_{44}(t)\mathbf{P}_4 + N_{54}(t)\mathbf{P}_5 + N_{64}(t)\mathbf{P}_6 && t \in [3,4)\\
&= \frac{1}{6}\big[(4-t)^3\mathbf{P}_3 + (3t^3 - 33t^2 + 117t - 131)\mathbf{P}_4\\
&\quad + (-3t^3 + 30t^2 - 96t + 100)\mathbf{P}_5 + (t-3)^3\mathbf{P}_6\big],\\
\mathbf{P}_7(t) &= N_{44}(t)\mathbf{P}_4 + N_{54}(t)\mathbf{P}_5 + N_{64}(t)\mathbf{P}_6 + N_{74}(t)\mathbf{P}_7 && t \in [4,5)\\
&= \frac{1}{6}\big[(5-t)^3\mathbf{P}_4 + (3t^3 - 42t^2 + 192t - 284)\mathbf{P}_5\\
&\quad + (-3t^3 + 39t^2 - 165t + 229)\mathbf{P}_6 + (t-4)^3\mathbf{P}_7\big].
\end{aligned}$$

A direct check verifies that each segment has barycentric weights. The entire curve starts at $\mathbf{P}_3(0) = (\mathbf{P}_0 + 4\mathbf{P}_1 + \mathbf{P}_2)/6$ and ends at $\mathbf{P}_7(5) = (\mathbf{P}_5 + 4\mathbf{P}_6 + \mathbf{P}_7)/6$. The four joint points between the segments are

$$\begin{aligned}
\mathbf{P}_3(1) &= \mathbf{P}_4(1) = (\mathbf{P}_1 + 4\mathbf{P}_2 + \mathbf{P}_3)/6, & \mathbf{P}_4(2) &= \mathbf{P}_5(2) = (\mathbf{P}_2 + 4\mathbf{P}_3 + \mathbf{P}_4)/6,\\
\mathbf{P}_5(3) &= \mathbf{P}_6(3) = (\mathbf{P}_3 + 4\mathbf{P}_4 + \mathbf{P}_5)/6, & \mathbf{P}_6(4) &= \mathbf{P}_7(4) = (\mathbf{P}_4 + 4\mathbf{P}_5 + \mathbf{P}_6)/6.
\end{aligned}$$

The coordinates of the control points of Figure 7.19a are $\mathbf{P}_0 = (0,0)$, $\mathbf{P}_1 = (0,1)$, $\mathbf{P}_2 = (1,1)$, $\mathbf{P}_3 = (1,0)$, $\mathbf{P}_4 = (2,0)$, $\mathbf{P}_5 = (2.75,1)$, $\mathbf{P}_6 = (3,1)$, and $\mathbf{P}_7 = (3,0)$. The curve therefore starts at $(1/6, 5/6)$, ends at $(2.96, 5/6)$, and passes through the joins $(5/6, 5/6)$, $(7/6, 1/6)$, $(1.96, 1/6)$, and $(2.67, 5/6)$.

Figure 7.20 lists the code that computes the weight functions for this case. This code is general and can also compute B-spline weight functions for the uniform and open uniform cases.

Part b: To continue the example, we now calculate the blending functions and spline segments of the curve of Figure 7.19b where the knot vector is the nonuniform

```
(* Compute the nonuniform weight functions for the 8-point example that follows *)
Clear[bspl,knt]
bspl[i_,k_,t_]:=If[knt[[i+k]]==knt[[i+1]],0, (* 0<=i<=n *)
 bspl[i,k-1,t] (t-knt[[i+1]])/(knt[[i+k]]-knt[[i+1]])] \
 +If[knt[[i+1+k]]==knt[[i+2]],0,
 bspl[i+1,k-1,t] (knt[[i+1+k]]-t)/(knt[[i+1+k]]-knt[[i+2]])];
bspl[i_,1,t_]:=If[knt[[i+1]]<=t<knt[[i+2]], 1, 0];
n=4; k=4; (* Note: 0<=k<=n *)
knt={-3,-2,-1,0,1,2,3,4,5,6,7,8}; (* knots for nonuniform case *)
bspl[i,k,t] (* assign a value to i *)
```

Figure 7.20: Eight-Point Nonuniform B-Spline Example; Code for Blending Functions.

$(-3,-2,-1,0,1,1,2,3,4,5,6,7)$. Notice that we now have $t_4 = t_5 = 1$, resulting in different blending functions and different spline segments.

It is important to realize that t varies in this case from $t_3 = 0$ to $t_8 = 4$. The five intervals of t for the five spline segments are $[0,1)$, $[1,1)$, $[1,2)$, $[2,3)$, and $[3,4)$. The second segment $\mathbf{P}_4(t)$ has now been reduced to a single point.

The first step is to calculate N_{i1}:

$$N_{31} = 1 \text{ for } t \in [0,1), \quad N_{41} = 1 \text{ for } t \in [1,1),$$
$$N_{51} = 1 \text{ for } t \in [1,2), \quad N_{61} = 1 \text{ for } t \in [2,3), \quad N_{71} = 1 \text{ for } t \in [3,4),$$

and N_{01}, N_{11}, N_{21}, N_{81}, N_{91}, $N_{10,1}$, and $N_{11,1}$ are zero in the range $0 \le t < 4$.

Step 2 is to calculate functions N_{i2} that are nonzero for $0 \le t < 4$:

$$\begin{aligned}
N_{02}(t) &= \frac{t-t_0}{t_1-t_0}N_{01} + \frac{t_2-t}{t_2-t_1}N_{11} = 0,\\
N_{12}(t) &= \frac{t-t_1}{t_2-t_1}N_{11} + \frac{t_3-t}{t_3-t_2}N_{21} = 0,\\
N_{22}(t) &= \frac{t-t_2}{t_3-t_2}N_{21} + \frac{t_4-t}{t_4-t_3}N_{31} = (1-t) \quad \text{for } t \in [0,1),\\
N_{32}(t) &= \frac{t-t_3}{t_4-t_3}N_{31} + \frac{t_5-t}{t_5-t_4}N_{41} = t \quad \text{for } t \in [0,1),\\
N_{42}(t) &= \frac{t-t_4}{t_5-t_4}N_{41} + \frac{t_6-t}{t_6-t_5}N_{51} = 2-t \quad \text{for } t \in [1,2),\\
N_{52}(t) &= \frac{t-t_5}{t_6-t_5}N_{51} + \frac{t_7-t}{t_7-t_6}N_{61} = \begin{cases} t-1 & \text{for } t \in [1,2),\\ 3-t & \text{for } t \in [2,3),\end{cases}\\
N_{62}(t) &= \frac{t-t_6}{t_7-t_6}N_{61} + \frac{t_8-t}{t_8-t_7}N_{71} = \begin{cases} t-2 & \text{for } t \in [2,3),\\ 4-t & \text{for } t \in [3,4),\end{cases}\\
N_{72}(t) &= \frac{t-t_7}{t_8-t_7}N_{71} + \frac{t_9-t}{t_9-t_8}N_{81} = t-4 \quad \text{for } t \in [3,4).
\end{aligned}$$

This step terminates at $N_{72}(t)$ since $N_{82}(t)$ and its successors are zero for $0 \le t < 4$.

Step 3 requires the calculation of several functions N_{i3}:

$$N_{03}(t) = \frac{t-t_0}{t_2-t_0}N_{02} + \frac{t_3-t}{t_3-t_1}N_{12} = 0,$$

$$N_{13}(t) = \frac{t-t_1}{t_3-t_1}N_{12} + \frac{t_4-t}{t_4-t_2}N_{22} = \frac{1}{2}(1-t)^2 \qquad \text{for } t\in[0,1),$$

$$N_{23}(t) = \frac{t-t_2}{t_4-t_2}N_{22} + \frac{t_5-t}{t_5-t_3}N_{32} = \frac{1}{2}(-3t^2+2t+1) \qquad \text{for } t\in[0,1),$$

$$N_{33}(t) = \frac{t-t_3}{t_5-t_3}N_{32} + \frac{t_6-t}{t_6-t_4}N_{42} = \begin{cases} t^2 & \text{for } t\in[0,1), \\ (2-t)^2 & \text{for } t\in[1,2), \end{cases}$$

$$N_{43}(t) = \frac{t-t_4}{t_6-t_4}N_{42} + \frac{t_7-t}{t_7-t_5}N_{52} = \frac{1}{2}\begin{cases} (-3t^2+10t-7) & \text{for } t\in[1,2), \\ (3-t)^2 & \text{for } t\in[2,3), \end{cases}$$

$$N_{53}(t) = \frac{t-t_5}{t_7-t_5}N_{52} + \frac{t_8-t}{t_8-t_6}N_{62} = \frac{1}{2}\begin{cases} (t-1)^2 & \text{for } t\in[1,2), \\ (-2t^2+10t-11) & \text{for } t\in[2,3), \\ (4-t)^2 & \text{for } t\in[3,4), \end{cases}$$

$$N_{63}(t) = \frac{t-t_6}{t_8-t_6}N_{62} + \frac{t_9-t}{t_9-t_7}N_{72} = \frac{1}{2}\begin{cases} (t-2)^2 & \text{for } t\in[2,3), \\ (-2t^2+14t-23) & \text{for } t\in[3,4), \end{cases}$$

$$N_{73}(t) = \frac{t-t_7}{t_9-t_7}N_{72} + \frac{t_{10}-t}{t_{10}-t_8}N_{82} = \frac{1}{2}(t-3)^2 \qquad \text{for } t\in[3,4).$$

Here we stop at N_{73} since N_{83} and its successors are zero for $0 \le t < 4$.

The last step involves the calculation of eight functions N_{i4}:

$$N_{04}(t) = \frac{t-t_0}{t_3-t_0}N_{03} + \frac{t_4-t}{t_4-t_1}N_{13} = \frac{1}{6}(1-t)^3 \qquad \text{for } t\in[0,1),$$

$$N_{14}(t) = \frac{t-t_1}{t_4-t_1}N_{13} + \frac{t_5-t}{t_5-t_2}N_{23} = \frac{1}{12}(11t^3-15t^2-3t+7) \qquad \text{for } t\in[0,1),$$

$$N_{24}(t) = \frac{t-t_2}{t_5-t_2}N_{23} + \frac{t_6-t}{t_6-t_3}N_{33} = \begin{cases} \frac{1}{4}(-5t^3+3t^2+3t+1) & \text{for } t\in[0,1), \\ \frac{1}{2}(2-t)^3 & \text{for } t\in[1,2), \end{cases}$$

$$N_{34}(t) = \frac{t-t_3}{t_6-t_3}N_{33} + \frac{t_7-t}{t_7-t_4}N_{43} = \begin{cases} \frac{1}{2}t^3 & \text{for } t\in[0,1), \\ \frac{1}{4}(5t^3-27t^2+45t-21) & \text{for } t\in[1,2), \\ \frac{1}{4}(3-t)^3 & \text{for } t\in[2,3), \end{cases}$$

$$N_{44}(t) = \frac{t-t_4}{t_7-t_4}N_{43} + \frac{t_8-t}{t_8-t_5}N_{53} = \begin{cases} \frac{1}{12}(-11t^3+51t^2-69t+29) & \text{for } t\in[1,2), \\ \frac{1}{12}(7t^3-57t^2+147t-115) & \text{for } t\in[2,3), \\ \frac{1}{6}(4-t)^3 & \text{for } t\in[3,4), \end{cases}$$

$$N_{54}(t) = \frac{t-t_5}{t_8-t_5}N_{53} + \frac{t_9-t}{t_9-t_6}N_{63} = \frac{1}{6}\begin{cases} (t-1)^3 & \text{for } t\in[1,2), \\ (-3t^3+21t^2-45t+31) & \text{for } t\in[2,3), \\ (3t^3-33t^2+117t-131) & \text{for } t\in[3,4), \end{cases}$$

$$N_{64}(t) = \frac{t-t_6}{t_9-t_6}N_{63} + \frac{t_{10}-t}{t_{10}-t_7}N_{73} = \frac{1}{6}\begin{cases} (t-2)^3 & \text{for } t\in[2,3), \\ (-3t^3+30t^2-96t+100) & \text{for } t\in[3,4), \end{cases}$$

$$N_{74}(t) = \frac{t-t_7}{t_{10}-t_7}N_{73} + \frac{t_{11}-t}{t_{11}-t_8}N_{83} = \frac{1}{6}(t-3)^3 \qquad \text{for } t\in[3,4).$$

This group of blending functions can now be used to construct the five spline segments

$$\mathbf{P}_3(t) = N_{04}(t)\mathbf{P}_0 + N_{14}(t)\mathbf{P}_1 + N_{24}(t)\mathbf{P}_2 + N_{34}(t)\mathbf{P}_3 \qquad t\in[0,1)$$

$$
\begin{aligned}
&= \frac{1}{6}(1-t)^3\mathbf{P}_0 + \frac{1}{12}(11t^3 - 15t^2 - 3t + 7)\mathbf{P}_1\\
&\quad + \frac{1}{4}(-5t^3 + 3t^2 + 3t + 1)\mathbf{P}_2 + \frac{1}{2}t^3\mathbf{P}_3,\\
\mathbf{P}_4(t) &= N_{14}(1)\mathbf{P}_1 + N_{24}(1)\mathbf{P}_2 + N_{34}(1)\mathbf{P}_3 + N_{44}(1)\mathbf{P}_4 \qquad t \in [1,1)\\
&= 0\mathbf{P}_1 + \frac{1}{2}\mathbf{P}_2 + \frac{1}{2}\mathbf{P}_3 + 0\mathbf{P}_4 = (\mathbf{P}_2 + \mathbf{P}_3)/2 \quad \text{(a point)},\\
\mathbf{P}_5(t) &= N_{24}(t)\mathbf{P}_2 + N_{34}(t)\mathbf{P}_3 + N_{44}(t)\mathbf{P}_4 + N_{54}(t)\mathbf{P}_5 \qquad t \in [1,2)\\
&= \frac{1}{2}(2-t)^3\mathbf{P}_2 + \frac{1}{4}(5t^3 - 27t^2 + 45t - 21)\mathbf{P}_3\\
&\quad + \frac{1}{12}(-11t^3 + 51t^2 - 69t + 29)\mathbf{P}_4 + \frac{1}{6}(t-1)^3\mathbf{P}_5,\\
\mathbf{P}_6(t) &= N_{34}(t)\mathbf{P}_3 + N_{44}(t)\mathbf{P}_4 + N_{54}(t)\mathbf{P}_5 + N_{64}(t)\mathbf{P}_6 \qquad t \in [2,3)\\
&= \frac{1}{4}(3-t)^3\mathbf{P}_3 + \frac{1}{12}(7t^3 - 57t^2 + 147t - 115)\mathbf{P}_4\\
&\quad + \frac{1}{6}(-3t^3 + 21t^2 - 45t + 31)\mathbf{P}_5 + \frac{1}{6}(t-2)^3\mathbf{P}_6,\\
\mathbf{P}_7(t) &= N_{44}(t)\mathbf{P}_4 + N_{54}(t)\mathbf{P}_5 + N_{64}(t)\mathbf{P}_6 + N_{74}(t)\mathbf{P}_7 \qquad t \in [3,4)\\
&= \frac{1}{6}\big[(4-t)^3\mathbf{P}_4 + (3t^3 - 33t^2 + 117t - 131)\mathbf{P}_5\\
&\quad + (-3t^3 + 30t^2 - 96t + 100)\mathbf{P}_6 + (t-3)^3\mathbf{P}_7\big].
\end{aligned}
$$

A direct check verifies that each segment has barycentric weights. The entire curve starts at $\mathbf{P}_3(0) = (2\mathbf{P}_0 + 7\mathbf{P}_1 + 3\mathbf{P}_2)/12$ and ends at $\mathbf{P}_7(4) = (\mathbf{P}_5 + 4\mathbf{P}_6 + \mathbf{P}_7)/6$. The three joint points between the segments are

$$
\mathbf{P}_3(1) = \mathbf{P}_5(1) = (\mathbf{P}_2 + \mathbf{P}_3)/2, \quad \mathbf{P}_5(2) = \mathbf{P}_6(2) = (3\mathbf{P}_3 + 7\mathbf{P}_4 + 2\mathbf{P}_5)/12,
$$
$$
\mathbf{P}_6(3) = \mathbf{P}_7(3) = (\mathbf{P}_4 + 4\mathbf{P}_5 + \mathbf{P}_6)/6.
$$

(End of example.)

◇ **Exercise 7.11:** Calculate the blending functions and spline segments for the curves of Figure 7.19c,d.

This example illustrates the power and flexibility of the nonuniform B-spline. Other curve methods make it possible to control the shape of a curve by moving control points, by subdividing the curve and adding points, and by repeating certain points. The nonuniform B-spline method can employ all these operations but can also fine-tune the curve by changing the values of knots and by using multiple knots.

7.12 Matrix Form of the Nonuniform B-Spline

The Cox–DeBoor recursive formula, Equations (7.24) and (7.25), is general and can be used to calculate the blending functions of the uniform, open, and nonuniform B-splines. However, it is complex and slow to calculate. Explicit, matrix-based expressions for the B-spline are simpler and faster to use. Such expressions have been derived for the uniform quadratic B-spline in Section 7.1 [Equation (7.6)] and for the uniform cubic B-spline in Section 7.2 [Equation (7.11)]. Similar expressions are derived in this section for the linear, quadratic, and cubic *nonuniform* B-splines. We temporarily use the notation u instead of t for the parameter and u_i instead of t_i for the knots.

For the linear case, where $k = 2$, the Cox–DeBoor formula becomes

$$N_{i2} = \frac{u-u_i}{u_{i+1}-u_i}N_{i1}(u) + \frac{u_{i+2}-u}{u_{i+2}-u_{i+2}}N_{i+1,1}(u) = \begin{cases} \dfrac{u-u_i}{u_{i+1}-u_i} & \text{for } u \in [u_i, u_{i+1}), \\ \dfrac{u_{i+2}-u}{u_{i+2}-u_{i+1}} & \text{for } u \in [u_{i+1}, u_{i+2}), \\ 0 & \text{otherwise.} \end{cases} \tag{7.29}$$

For $i = 0$, this becomes

$$N_{02} = \begin{cases} \dfrac{u-u_0}{u_1-u_0} & \text{for } u \in [u_0, u_1), \\ \dfrac{u_2-u}{u_2-u_1} & \text{for } u \in [u_1, u_2), \\ 0 & \text{otherwise.} \end{cases} \tag{7.30}$$

The other blending function N_{12} is easily obtained from Equation (7.30) by incrementing all the indices.

Blending function N_{02} is zero over the subinterval $[u_2, u_3)$ and blending function N_{12} is zero over $[u_0, u_1)$. It is therefore only over the interval $[u_1, u_2)$ that both these functions are nonzero, so the parameter u should vary from u_1 to u_2. Over this interval, we have

$$N_{02}(u) = \frac{u_2-u}{u_2-u_1}, \quad N_{12}(u) = \frac{u_3-u}{u_3-u_2}. \tag{7.31}$$

To derive the expression for the linear spline, we denote $\Delta = u_2 - u_1$ and define the parameter t by

$$t = \frac{u-u_1}{\Delta} = \frac{u-u_1}{u_2-u_1}.$$

Notice that $u = u_1 \to t = 0$ and $u = u_2 \to t = 1$. Also, $u - u_1 = t\Delta$ and $u - u_2 = \Delta(t-1)$. Substituting this in Equation (7.31) yields the matrix expression for the linear nonuniform B-spline

$$\mathbf{P}(t) = (t, 1)\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \end{pmatrix}. \tag{7.32}$$

When t varies from 0 to 1, this becomes the straight line from $\mathbf{P}_0$ to $\mathbf{P}_1$. The nonuniform linear B-spline does not depend on Δ, so it is identical to the uniform linear B-spline.

When you get an 8 on the midterm, there ain't a curve in the world that can save you. —Unknown

Next, we derive the matrix form of the quadratic case. Applying the Cox–DeBoor formula to Equation (7.30), we get the first quadratic blending function N_{03}:

$$N_{03}(u)=\begin{cases}\dfrac{u-u_0}{u_2-u_0}\cdot\dfrac{u-u_0}{u_1-u_0} & \text{for } u\in[u_0,u_1),\\[2mm] \dfrac{u-u_0}{u_2-u_0}\cdot\dfrac{u_2-u}{u_2-u_1}+\dfrac{u_3-u}{u_3-u_1}\cdot\dfrac{u-u_1}{u_2-u_1} & \text{for } u\in[u_1,u_2),\\[2mm] \dfrac{u_3-u}{u_3-u_1}\cdot\dfrac{u_3-u}{u_3-u_2} & \text{for } u\in[u_2,u_3),\\[2mm] 0 & \text{otherwise.}\end{cases}\tag{7.33}$$

Functions N_{13} and N_{23} are obtained from Equation (7.33) by incrementing all the indices. When this is done, we observe that each of the three blending functions N_{i3} is zero over different intervals and it is only over subinterval $[u_2,u_3)$ that all three are nonzero, and their values are

$$\begin{aligned}N_{03}(u)&=\frac{u_3-u}{u_3-u_1}\cdot\frac{u_3-u}{u_3-u_2},\\ N_{13}(u)&=\frac{u-u_1}{u_3-u_1}\cdot\frac{u_3-u}{u_3-u_2}+\frac{u_4-u}{u_4-u_2}\cdot\frac{u-u_2}{u_3-u_2},\\ N_{23}(u)&=\frac{u-u_2}{u_4-u_2}\cdot\frac{u-u_2}{u_3-u_2}.\end{aligned}\tag{7.34}$$

Since the knot vector is nonuniform, the differences between consecutive knots may be different and we denote them

$$\Delta_1=u_2-u_1,\quad \Delta_2=u_3-u_2,\quad \Delta_3=u_4-u_3.$$

We also define $t=(u-u_2)/\Delta_2$, which implies

$$\begin{aligned}u-u_1&=t\Delta_2+\Delta_1,\\ u-u_2&=t\Delta_2,\\ u-u_3&=(t-1)\Delta_2,\\ u-u_4&=t\Delta_2-(\Delta_2+\Delta_3).\end{aligned}\tag{7.35}$$

Equations (7.34) and (7.35) yield the matrix form of the nonuniform quadratic B-spline

$$\mathbf{P}(t)=(t^2,t,1)\begin{pmatrix}a & -a-b & b\\ -2a & 2a & 0\\ a & 1-a & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix},\tag{7.36}$$

where

$$a=\frac{\Delta_2}{\Delta_1+\Delta_2},\qquad b=\frac{\Delta_2}{\Delta_2+\Delta_3},$$

and t varies from 0 to 1 (note that $u=u_2\rightarrow t=0$ and $u=u_3\rightarrow t=1$).

B-splines were known to and studied by Nikolai Lobachevsky whose major contribution to mathematics is perhaps the so-called non-Euclidean (hyperbolic) geometry in the late eighteenth century. The modern version described here was developed, in the late 1970s, by C. DeBoor, M. Cox and L. Mansfield. Note that their algorithm is a generalization of de Casteljau's scaffolding method.

The next example derives the matrix form of the nonuniform cubic B-spline. We apply the Cox–DeBoor formula to Equation (7.33) to obtain the first of the four blending functions N_{i4}:

$$N_{04}(u) = \begin{cases} \frac{u-u_0}{u_3-u_0}\cdot\frac{u-u_0}{u_2-u_0}\cdot\frac{u-u_0}{u_1-u_0} & \text{for } u\in[u_0,u_1), \\ \frac{u-u_0}{u_3-u_0}\cdot\frac{u-u_0}{u_2-u_0}\cdot\frac{u_2-u}{u_2-u_1} & \\ \quad +\frac{u-u_0}{u_3-u_0}\cdot\frac{u_3-u}{u_3-u_1}\cdot\frac{u-u_1}{u_2-u_1} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u-u_1}{u_3-u_1}\cdot\frac{u-u_1}{u_2-u_1} & \text{for } u\in[u_1,u_2), \\ \frac{u-u_0}{u_3-u_0}\cdot\frac{u_3-u}{u_3-u_1}\cdot\frac{u_3-u}{u_3-u_2} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u-u_1}{u_3-u_1}\cdot\frac{u_3-u}{u_3-u_2} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u-u_2}{u_3-u_2} & \text{for } u\in[u_2,u_3), \\ \frac{u_4-u}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3} & \text{for } u\in[u_3,u_4), \\ 0 & \text{otherwise.} \end{cases} \tag{7.37}$$

The remaining three blending functions N_{14}, N_{24}, and N_{34} are obtained from Equation (7.37) by incrementing all the indices. When this is done we observe, as before, that each of the four blending functions N_{i4} is zero over different intervals and it is only over subinterval $[u_3,u_4)$ that all four are nonzero. Their values are

$$\begin{aligned} N_{04}(u) &= \frac{u_4-u}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3}, \\ N_{14}(u) &= \frac{u-u_1}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3}+\frac{u_5-u}{u_5-u_2}\cdot\frac{u-u_2}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3} \\ &\quad +\frac{u_5-u}{u_5-u_2}\cdot\frac{u_5-u}{u_5-u_3}\cdot\frac{u-u_3}{u_4-u_3}, \\ N_{24}(u) &= \frac{u-u_2}{u_5-u_2}\cdot\frac{u-u_2}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3}+\frac{u-u_2}{u_5-u_2}\cdot\frac{u_5-u}{u_5-u_3}\cdot\frac{u-u_3}{u_4-u_3} \\ &\quad +\frac{u_6-u}{u_6-u_3}\cdot\frac{u-u_3}{u_5-u_3}\cdot\frac{u-u_3}{u_4-u_3}, \\ N_{34}(u) &= \frac{u-u_3}{u_6-u_3}\cdot\frac{u-u_3}{u_5-u_3}\cdot\frac{u-u_3}{u_4-u_3}. \end{aligned} \tag{7.38}$$

Since the knot vector is nonuniform, the differences between consecutive knots may be different and we denote them by

$$\Delta_1 = u_2 - u_1, \quad \Delta_2 = u_3 - u_2, \quad \Delta_3 = u_4 - u_3,$$
$$\Delta_4 = u_5 - u_4, \quad \Delta_5 = u_6 - u_5, \quad t = (u - u_3)/\Delta_3.$$

This implies

$$\begin{aligned} u - u_1 &= t\Delta_3 + (\Delta_1 + \Delta_2),\\ u - u_2 &= t\Delta_3 + \Delta_2,\\ u - u_3 &= t\Delta_3,\\ u - u_4 &= (t-1)\Delta_3,\\ u - u_5 &= t\Delta_3 - (\Delta_3 + \Delta_4),\\ u - u_6 &= t\Delta_3 - (\Delta_3 + \Delta_4 + \Delta_5). \end{aligned} \tag{7.39}$$

Equations (7.38) and (7.39) yield the matrix form of the nonuniform cubic B-spline:

$$\mathbf{P}(t) = (t^3, t^2, t, 1)\begin{pmatrix} -a & a+b+c & -b-c-d & d\\ 3a & -3a-3b & 3b & 0\\ -3a & 3a-3e & 3e & 0\\ a & 1-a-f & f & 0 \end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\end{pmatrix}, \tag{7.40}$$

where

$$\begin{aligned} a &= \frac{\Delta_3^2}{(\Delta_1+\Delta_2+\Delta_3)(\Delta_2+\Delta_3)}, & d &= \frac{\Delta_3^2}{(\Delta_3+\Delta_4+\Delta_5)(\Delta_4+\Delta_5)},\\ b &= \frac{\Delta_3^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)}, & e &= \frac{\Delta_2\Delta_3}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)},\\ c &= \frac{\Delta_3^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_3+\Delta_4)}, & f &= \frac{\Delta_2^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)}. \end{aligned}$$

The quantities Δ_i are defined as differences of knot values $u_{i+1} - u_i$ and a good choice for those differences is the chord lengths between points. However, a cubic spline segment requires five Δ_i's, but there are only three chords between the four points defining it. In general, a B-spline curve is defined by $n+1$ points, having n chords between them, but $n+2$ differences Δ_i are required. A standard technique is to select

$$\Delta_1 = \Delta_2 = |\mathbf{P}_1 - \mathbf{P}_0|, \qquad \Delta_{n+1} = \Delta_{n+2} = |\mathbf{P}_n - \mathbf{P}_{n-1}|,$$

and $\Delta_i = |\mathbf{P}_{i-1} - \mathbf{P}_{i-2}|$ for $i = 3, 4, \ldots, n$.

The last topic discussed in this section is the relation between the quadratic uniform and quadratic nonuniform B-splines. Given three control points $\mathbf{Q}_0$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$, the uniform quadratic B-spline $\mathbf{Q}(t)$ defined by them is given by Equation (7.6)

$$\mathbf{Q}(t) = \frac{1}{2}(t^2, t, 1)\begin{pmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix}\mathbf{Q}_0\\ \mathbf{Q}_1\\ \mathbf{Q}_2\end{pmatrix}. \tag{7.6}$$

The nonuniform quadratic B-spline defined by three control points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$ is given by Equation (7.36). If we require the two curves to be identical for any value of the parameter t, we obtain the equation

$$\frac{1}{2}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{Q}_0 \\ \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{pmatrix} = \begin{pmatrix} a & -a-b & b \\ -2a & 2a & 0 \\ a & 1-a & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix}.$$

This is a system of three equations where we assume that the unknowns are the $\mathbf{Q}_i$'s. The solutions are

$$\mathbf{Q}_0 = 2a\mathbf{P}_0 + (1-2a)\mathbf{P}_1, \quad \mathbf{Q}_1 = \mathbf{P}_1, \quad \text{and} \quad \mathbf{Q}_2 = (1-2b)\mathbf{P}_1 + 2b\mathbf{P}_2.$$

To see the geometrical interpretation of these relations, we write

$$\mathbf{Q}_0 = 2a\mathbf{P}_0 + (1-2a)\mathbf{P}_1 = 2a\mathbf{P}_0 + 2(1-a)\mathbf{P}_1 - \mathbf{P}_1 = 2\mathbf{P}(0) - \mathbf{P}_1 = 2\mathbf{Q}(0) - \mathbf{Q}_1,$$

which implies $\mathbf{Q}_0 - \mathbf{Q}(0) = \mathbf{Q}(0) - \mathbf{Q}_1$. The distance between $\mathbf{Q}_0$ and $\mathbf{Q}(0)$ equals the distance between $\mathbf{Q}(0)$ and $\mathbf{Q}_1$, and a similar relation among $\mathbf{Q}_1$, $\mathbf{Q}(1)$, and $\mathbf{Q}_2$.

The conclusion is that a group of three points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$ defining a single quadratic nonuniform B-spline segment $\mathbf{P}(t)$ can be replaced by a group of three points $\mathbf{Q}_0$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ defining a single quadratic *uniform* B-spline segment $\mathbf{Q}(t)$ identical to $\mathbf{P}(t)$. However, given a set of $n+1$ control points $\mathbf{P}_i$ for a nonuniform B-spline curve, they cannot, in general, be replaced by a set of $n+1$ points $\mathbf{Q}_i$ that produce an identical uniform B-spline curve.

7.13 Subdividing the B-spline Curve

The B-spline curve is easy to manipulate by moving the control points and varying the knots. Still, if the curve is based on too few points, it may "refuse" to get the right shape, no matter what. More control points can be added, in such a case, by subdividing the curve, a process similar to subdividing the Bézier curve (Section 6.8). The method described here is called the Oslo algorithm and the discussion follows [Cohen et al. 80] and [Prautzsch 84].

(Control points can also be added by raising the degree of the B-spline curve, similar to the degree elevation of the Bézier curve, Section 6.9. This operation is discussed in [Cohen et al. 85].)

The idea behind subdividing a curve is that there are many (even infinitely many) sets of control points that produce the same B-spline curve. Normally, we are interested in the smallest number of control points that will produce a given curve, but if we cannot get the right shape with the original $n+1$ control points, we need to find a set of $n+2$ points that will produce *the same curve*, then move the new points around, attempting to bring the curve to the desired shape.

Given a set of $n+1$ control points $\mathbf{P}_i$ and a knot vector $(t_0, t_1, \ldots, t_{n+k})$, we start the subdivision process by inserting several new knots, thereby obtaining a new knot vector $(u_0, u_1, \ldots, u_{m+k})$ where $m > n$. The new, subdivided curve is based on the $m+1$ control points $\mathbf{Q}_j$ defined by the Oslo algorithm as

$$\mathbf{Q}_j = \sum_{i=0}^{n} a_{ij}^k \mathbf{P}_i, \quad \text{where} \quad 0 \le i \le n \quad \text{and} \quad 0 \le j \le m,$$

where the coefficients a_{ij}^k are defined recursively by a relation similar to the Cox–DeBoor formula

$$a_{ij}^1 = \begin{cases} 1, & t_i \le u_j < t_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{7.41}$$

$$a_{ij}^k = \frac{u_{j+k-1} - t_i}{t_{i+k-1} - t_i} a_{ij}^{k-1} + \frac{t_{i+k} - u_{j+k-1}}{t_{i+k} - t_{i+1}} a_{i+1,j}^{k-1}. \tag{7.42}$$

This relation guarantees that $\sum_i^n a_{ij}^k = 1$, for $0 \le j \le m$.

If the original knot vector is uniform, inserting a single knot will convert it to a nonuniform vector. However, an open knot vector can sometimes remain open after inserting new knots, as the following example shows. Suppose that we have the open vector $(0,0,0,1,2,2,2)$, where t varies from 0 to 2. This corresponds to a two-segment curve and we want to subdivide both segments. We first multiply each knot by 2, obtaining the vector $(0,0,0,2,4,4,4)$ that produces the same curve when $0 \le t < 4$. Next, we insert knots 1 and 3 to obtain the knot vector $(0,0,0,1,2,3,4,4,4)$. This vector is still open and it corresponds to the four segments $[0,1)$, $[1,2)$, $[2,3)$, and $[3,4)$.

Example: We assume four control points and quadratic segments (i.e., $k = 3$). We already know that each segment is defined by three points, so two segments are needed for this curve. The knot vector is assumed to be uniform and it goes from $t_0 = 0$ to $t_{n+k} = t_6 = 6$. The parameter t varies from $t_{k-1} = t_2 = 2$ to $t_{n+1} = t_4 = 4$; two subintervals. This again shows that the curve consists of two spline segments, the first for the subinterval $[t_2, t_3)$ and the second for $[t_3, t_4)$. We decide to subdivide the first segment. This segment is defined by points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$ (notice that $n = 2$ for this subdivision), so the subdivision process should produce four points, $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ (this implies $m = 3$), such that the two quadratic segments defined by them will have the same shape as the segment being subdivided.

To perform the subdivision, we need to insert a new knot between $t_2 = 2$ and $t_3 = 3$. We (somewhat arbitrarily) select its value to be 2.5. The new knot vector is

$$(u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7) = (0, 1, 2, 2.5, 3, 4, 5, 6),$$

and it is nonuniform. The calculation of the a_{ij}^k coefficients is done by varying i from 0 to $n = 2$ and varying j from 0 to $m = 3$. It requires three steps, for $k = 1, 2, 3$ (notice that this k is not the same as the order of the B-spline).

Step 1: We use Equation (7.41). A direct comparison of the t_i and u_i knots shows that the only nonzero a_{ij}^1 coefficients are a_{00}^1, a_{11}^1, a_{22}^1, and a_{23}^1. Each has a value of 1.

Step 2: We calculate a_{ij}^2 for $j = 0, 1, 2, 3$ from Equation (7.42). For each value of j, we stop when we get coefficients that add up to 1. The nonzero coefficients are

$$
\begin{aligned}
a_{00}^2 &= \frac{u_1 - t_0}{t_1 - t_0} a_{00}^1 + \frac{t_2 - u_1}{t_2 - t_1} a_{10}^1 = \frac{1-0}{1-0} \cdot 1 = 1,\\
a_{11}^2 &= \frac{u_2 - t_1}{t_2 - t_1} a_{11}^1 + \frac{t_3 - u_2}{t_3 - t_2} a_{21}^1 = \frac{2-1}{2-1} \cdot 1 = 1,\\
a_{12}^2 &= \frac{u_3 - t_1}{t_2 - t_1} a_{12}^1 + \frac{t_3 - u_3}{t_3 - t_2} a_{22}^1 = \frac{3-2.5}{3-2} \cdot 1 = 1/2,\\
a_{22}^2 &= \frac{u_3 - t_2}{t_3 - t_2} a_{22}^1 + \frac{t_4 - u_3}{t_4 - t_3} a_{32}^1 = \frac{2.5-2}{3-2} \cdot 1 = 1/2,\\
a_{23}^2 &= \frac{u_4 - t_2}{t_3 - t_2} a_{23}^1 + \frac{t_4 - u_4}{t_4 - t_3} a_{33}^1 = \frac{3-2}{3-2} \cdot 1 = 1.
\end{aligned}
$$

Step 3: The coefficients of step 2 are used to calculate a_{ij}^3:

$$
\begin{aligned}
a_{00}^3 &= \frac{u_2 - t_0}{t_2 - t_0} a_{00}^2 + \frac{t_3 - u_2}{t_3 - t_1} a_{10}^2 = \frac{2-0}{2-0} \cdot 1 = 1,\\
a_{01}^3 &= \frac{u_3 - t_0}{t_2 - t_0} a_{01}^2 + \frac{t_3 - u_3}{t_3 - t_1} a_{11}^2 = \frac{3-2.5}{3-1} \cdot 1 = 1/4,\\
a_{11}^3 &= \frac{u_3 - t_1}{t_3 - t_1} a_{11}^2 + \frac{t_4 - u_3}{t_4 - t_2} a_{21}^2 = \frac{2.5-1}{3-1} \cdot 1 = 3/4,\\
a_{12}^3 &= \frac{u_4 - t_1}{t_3 - t_1} a_{12}^2 + \frac{t_4 - u_4}{t_4 - t_2} a_{22}^2 = \frac{3-1}{3-1} \cdot \frac{1}{2} + \frac{4-3}{4-2} \cdot \frac{1}{2} = 3/4,\\
a_{22}^3 &= \frac{u_4 - t_2}{t_4 - t_2} a_{22}^2 + \frac{t_5 - u_4}{t_5 - t_3} a_{32}^2 = \frac{3-2}{4-2} \cdot \frac{1}{2} = 1/4,\\
a_{23}^3 &= \frac{u_5 - t_2}{t_4 - t_2} a_{23}^2 + \frac{t_5 - u_5}{t_5 - t_3} a_{33}^2 = \frac{4-2}{4-2} \cdot 1 = 1.
\end{aligned}
$$

The four new control points can now be calculated. They are

$$
\begin{aligned}
\mathbf{Q}_0 &= \sum_{i=0}^{3} a_{i0}^3 \mathbf{P}_i = a_{00}^3 \mathbf{P}_0 = \mathbf{P}_0,\\
\mathbf{Q}_1 &= \sum_{i=0}^{3} a_{i1}^3 \mathbf{P}_i = a_{01}^3 \mathbf{P}_0 + a_{11}^3 \mathbf{P}_1 = \frac{1}{4}\mathbf{P}_0 + \frac{3}{4}\mathbf{P}_1,\\
\mathbf{Q}_2 &= \sum_{i=0}^{3} a_{i2}^3 \mathbf{P}_i = a_{12}^3 \mathbf{P}_1 + a_{22}^3 \mathbf{P}_2 = \frac{3}{4}\mathbf{P}_1 + \frac{1}{4}\mathbf{P}_2,\\
\mathbf{Q}_3 &= \sum_{i=0}^{3} a_{i3}^3 \mathbf{P}_i = a_{23}^3 \mathbf{P}_2 = \mathbf{P}_2.
\end{aligned}
$$

The two quadratic B-spline segments defined by $\mathbf{Q}_0\mathbf{Q}_1\mathbf{Q}_2$ and $\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$ have the same shape as the original segment defined by $\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2$, but they are easier to modify since they are based on four points.

7.14 Nonuniform Rational B-Splines (NURBS)

The use of a knot vector is one reason why the B-spline curve is more general than the Bézier and other curve methods. The $n+k+1$ knots can be used as parameters and can be varied by the user/designer to obtain the desired shape of the curve. The rational B-spline, described in this section, employs an additional set of $n+1$ parameters w_i, called *weights*, to add even greater flexibility to the curve. In addition to this feature, the rational B-spline has several more important advantages as follows:

1. It makes it possible to create curves that are true conic sections. It is well known that a polynomial cannot represent a circle. More generally, it cannot represent arbitrary conic sections. It is easy to show that the Bézier and B-spline curves can represent approximate circles (Appendix B). If precise circles or conic sections are needed, then rational curves are the natural choice.

2. It is invariant under perspective projections. We know that curves that are barycentric sums are invariant under affine transformations. If we want to rotate, scale, shear, or translate such a curve, we can apply the transformation to the control points and use the transformed points to draw the transformed curve. There is no need to apply the transformation to every pixel on the curve. However, if we want to project a space (three-dimensional) curve in perspective on a two-dimensional output device, we have to individually project every pixel on the curve. With a rational curve, we can (perspective) project the control points and use the projected, two-dimensional points to calculate the projected curve.

3. It reduces to the nonrational B-spline when all the weights w_i are set to 1. This means that a software package for rational B-splines can be used to generate nonrational B-splines (uniform, open, and nonuniform). This also implies that the nonuniform rational B-spline (NURBS for short) is the most general parametric curve. It can take many shapes and can easily be reduced to simpler forms. Because of this, NURBS is today the defacto standard for curve design. Three excellent references to NURBS are [Farin 99], [Piegl 97], and [Rogers 01].

Perhaps the best way to introduce rational B-splines (and rational curves in general) is by means of homogeneous coordinates. This method starts by adding an extra dimension to points, so a two-dimensional point becomes a triplet (x, y, w) and a three-dimensional point becomes a 4-tuple (x, y, z, w). After transforming or manipulating the point, it is projected back to its original number of dimensions by dividing its coordinates by w. Given four-dimensional control points $\mathbf{Q}_i = (x_i, y_i, z_i, w_i)$, where we assume for convenience that the w_i coordinates are nonnegative, we can define a (nonrational) B-spline curve as

$$\mathbf{P}_{\rm nr}(t) = \sum_{i=0}^{n} \mathbf{Q}_i N_{ik}(t).$$

From this we get the rational B-spline $\mathbf{P}_{\rm r}(t)$ by isolating that part of $\mathbf{P}_{\rm nr}(t)$ that depends on the fourth coordinates w_i and dividing by this part.

$$\mathbf{P}_{\rm r}(t) = \frac{\sum_{i=0}^{n} \mathbf{P}_i w_i N_{ik}(t)}{\sum_{i=0}^{n} w_i N_{ik}(t)} = \sum_{i=0}^{n} \mathbf{P}_i R_{ik}(t), \qquad (7.43)$$

where $\mathbf{P}_i = (x_i, y_i, z_i)$ are three-dimensional control points and $R_{ik}(t)$ are the new, rational blending functions defined by

$$R_{ik}(t) = \frac{w_i N_{ik}(t)}{\sum_{i=0}^{n} w_i N_{ik}(t)}. \tag{7.44}$$

This type of curve has most of the properties of the nonrational B-spline. The following should be mentioned in particular:

1. The new blending functions $R_{ik}(t)$ are nonnegative and barycentric.
2. The curve reduces to the nonrational curve when all the weights w_i equal 1 [this is a direct consequence of Equation (7.44)].
3. Since the rational curve is the four-dimensional generalization of the nonrational B-spline, the algorithms for curve subdivision and degree elevation of the B-spline can be used for the rational version. They simply have to be executed on the four-dimensional control points (x_i, y_i, z_i, w_i).

So much for the definition of the rational B-spline. The main question is how to select values for the weights in order to modify the shape of the curve in a predictable way. In order to isolate the effect of one weight on the curve, we first observe that Equation (7.43) implies that when $w_k = 0$, point $\mathbf{P}_k$ has no effect on the curve. To see how increasing the value of a weight affects the curve, we select an index $0 \le k \le n$ and divide Equation (7.43) by w_k

$$\mathbf{P}_{\mathrm{r}}(t) = \frac{\sum_{i=0, i\neq k}^{n} \mathbf{P}_i \frac{w_i}{w_k} N_{ik}(t) + \mathbf{P}_k N_{kk}(t)}{\sum_{i=0, i\neq k}^{n} \frac{w_i}{w_k} N_{ik}(t) + N_{kk}(t)}.$$

It is easy to see that as w_k grows without limit, the result approaches point $\mathbf{P}_k$. We therefore conclude that those curve segments that are affected by $\mathbf{P}_k$ will approach this point as weight w_k grows.

The rest of this section describes two approaches to understanding the weights and their effects on the curve. The first approach is to set all weights $w_i = 1$, then change the value of one of them and see how it affects the blending functions. The second approach is to derive specific sets of weights that will produce B-spline curves that are conic sections. The first approach is illustrated by a detailed example.

Example: This is an extension of the open B-spline example on page 282. We assume $n = 4$ (five control points), select order $k = 3$ (quadratic polynomial segments), and the knot vector $(0, 0, 0, 1, 2, 3, 3, 3)$. The parameter t varies from $t_{k-1} = t_2 = 0$ to $t_{n+1} = t_5 = 3$, so our curve consists of three segments. The nonrational blending functions $N_{i3}(t)$ are

$$N_{03}(t) = (1-t)^2, \qquad 0 \le t < 1,$$
$$N_{13}(t) = \frac{1}{2}\begin{cases} t(4-3t), & 0 \le t < 1,\\ (2-t)^2, & 1 \le t < 2,\end{cases}$$
$$N_{23}(t) = \frac{1}{2}\begin{cases} t^2, & 0 \le t < 1,\\ (-2t^2 + 6t - 3), & 1 \le t < 2,\\ (3-t)^2, & 2 \le t < 3,\end{cases}$$

$$N_{33}(t)=\frac{1}{2}\begin{cases}(t-1)^2, & 1\le t<2,\\ (-3t^2+14t-15), & 2\le t<3,\end{cases}$$
$$N_{43}(t)=(t-2)^2, \qquad 2\le t<3.$$

Before we can calculate the rational blending functions, we have to select values for the five weights. We choose $(1,1,w_2,1,1)$, where w_2 will later be assigned several different values. The result is

$$R_{03}(t)=\frac{w_0N_{03}(t)}{\sum_{i=0}^4 w_iN_{i3}(t)}=\frac{(1-t)^2}{(1-t)^2+t(4-3t)/2+w_2t^2/2}, \qquad t\in[0,1),$$

$$R_{13}(t)=\frac{w_1N_{13}(t)}{\sum_{i=0}^4 w_iN_{i3}(t)}=\begin{cases}\frac{t(4-3t)/2}{(1-t)^2+t(4-3t)/2+w_2t^2/2}, & t\in[0,1)\\ \frac{(2-t)^2/2}{(2-t)^2/2+w_2(-2t^2+6t-3)/2+(t-1)^2/2}, & t\in[1,2),\end{cases}$$

$$R_{23}(t)=\frac{w_2N_{23}(t)}{\sum_{i=0}^4 w_iN_{i3}(t)}=\begin{cases}\frac{w_2t^2/2}{(1-t)^2+t(4-3t)/2+w_2t^2/2}, & t\in[0,1)\\ \frac{w_2(-2t^2+6t-3)/2}{(2-t)^2/2+w_2(-2t^2+6t-3)/2+(t-1)^2/2}, & t\in[1,2)\\ \frac{w_2(3-t)^2/2}{w_2(3-t)^2/2+(-3t^2+14t-15)/2+(t-2)^2}, & t\in[2,3),\end{cases}$$

$$R_{33}(t)=\frac{w_3N_{33}(t)}{\sum_{i=0}^4 w_iN_{i3}(t)}=\begin{cases}\frac{(t-1)^2/2}{(2-t)^2/2+w_2(-2t^2+6t-3)/2+(t-1)^2/2}, & t\in[1,2)\\ \frac{(-3t^2+14t-15)/2}{w_2(3-t)^2/2+(-3t^2+14t-15)/2+(t-2)^2}, & t\in[2,3),\end{cases}$$

$$R_{43}(t)=\frac{w_4N_{43}(t)}{\sum_{i=0}^4 w_iN_{i3}(t)}=\frac{(t-2)^2}{w_2(3-t)^2/2+(-3t^2+14t-15)/2+(t-2)^2} \qquad t\in[2,3).$$

We next calculate the three spline segments for the four cases $w_2=0, 0.5, 1$, and 5. For $w_2=0$ the three segments are

$$\mathbf{P}_1(t)=\frac{(1-t)^2}{1-t^2/2}\mathbf{P}_0+\frac{(4-3t)t}{2-t^2}\mathbf{P}_1+0\mathbf{P}_2,$$

$$\mathbf{P}_2(t)=\frac{(2-t)^2}{5-6t+2t^2}\mathbf{P}_1+0\mathbf{P}_2+\frac{(t-1)^2}{5-6t+2t^2}\mathbf{P}_3,$$

$$\mathbf{P}_3(t)=0\mathbf{P}_2+\frac{15-14t+3t^2}{7-6t+t^2}\mathbf{P}_3+\frac{2(-2+t)^2}{-7+6t-t^2}\mathbf{P}_4.$$

For $w_2=0.5$ they are

$$\mathbf{P}_1(t)=\frac{(1-t)^2}{1-0.25t^2}\mathbf{P}_0+\frac{(4-3t)t}{2-0.5t^2}\mathbf{P}_1+\frac{0.25t^2}{1-0.25t^2}\mathbf{P}_2,$$
$$\mathbf{P}_2(t)=\frac{(2-t)^2}{3.5-3t+t^2}\mathbf{P}_1+\frac{0.25(-3+6t-2t^2)}{1.75-1.5t+0.5t^2}\mathbf{P}_2+\frac{(t-1)^2}{3.5-0.5t^2}\mathbf{P}_3,$$
$$\mathbf{P}_3(t)=\frac{0.25(3-t)^2}{-1.25+1.5t-0.25t^2}\mathbf{P}_2+\frac{-15+14t-3t^2}{-2.5+3.5t-0.5t^2}\mathbf{P}_3+\frac{(t-2)^2}{-1.25+1.5t-0.25t^2}\mathbf{P}_4.$$

For $w_2 = 1$ we get

$$\begin{aligned}
\mathbf{P}_1(t) &= (1-t)^2\mathbf{P}_0 + \frac{(4-3t)t}{2}\mathbf{P}_1 + \frac{t^2}{2}\mathbf{P}_2,\\
\mathbf{P}_2(t) &= \frac{(2-t)^2}{2}\mathbf{P}_1 + \frac{-3+6t-2t^2}{2}\mathbf{P}_2 + \frac{(t-1)^2}{2+6t-3t^2}\mathbf{P}_3,\\
\mathbf{P}_3(t) &= \frac{(3-t)^2}{2}\mathbf{P}_2 + \frac{-15+14t-3t^2}{2}\mathbf{P}_3 + (t-2)^2\mathbf{P}_4.
\end{aligned}$$

Finally, for $w_2 = 5$ the segments are

$$\begin{aligned}
\mathbf{P}_1(t) &= \frac{(1-t)^2}{1+2t^2}\mathbf{P}_0 + \frac{(4-3t)t}{2+4t^2}\mathbf{P}_1 + \frac{5t^2}{2+4t^2}\mathbf{P}_2,\\
\mathbf{P}_2(t) &= \frac{(2-t)^2}{-10+24t-8t^2}\mathbf{P}_1 + \frac{5(-3+6t-2t^2)}{-10+24t-8t^2}\mathbf{P}_2 + \frac{(t-1)^2}{-10+54t-23t^2}\mathbf{P}_3,\\
\mathbf{P}_3(t) &= \frac{5(3-t)^2}{38-24t+4t^2}\mathbf{P}_2 + \frac{-15+14t-3t^2}{38-24t+4t^2}\mathbf{P}_3 + \frac{(t-2)^2}{19-12t+2t^2}\mathbf{P}_4.
\end{aligned}$$

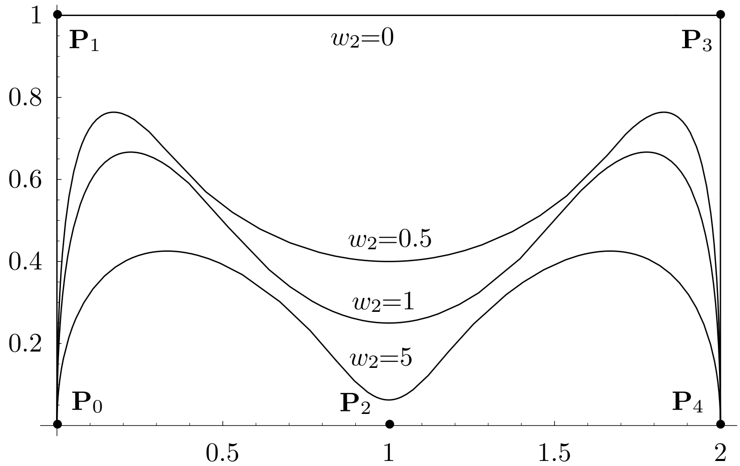
They are plotted in Figure 7.21 for control points $\mathbf{P}_0 = (0,0)$, $\mathbf{P}_1 = (0,1)$, $\mathbf{P}_2 = (1,0)$, $\mathbf{P}_3 = (2,1)$, and $\mathbf{P}_4 = (2,0)$. It is easy to see how weight w_2 affects the shape of the curve by controlling the amount of "pull" that point $\mathbf{P}_2$ exerts on the curve. For $w_2 = 0$, point $\mathbf{P}_2$ has no effect. The curve is defined by the four remaining points and is identical to the control polygon of these points. As w_2 grows toward 5, the curve becomes more and more attracted to $\mathbf{P}_2$.

Now for the second approach. We are looking for specific sets of weights that will generate conic sections. Since the conics are described by quadratic equations and each is fully defined by means of three points, it makes sense to try rational B-splines of order $k = 3$ defined by three points (i.e., $n = 2$). The conic is easier to design if the B-spline curve starts and ends at control points, so it makes sense to use an open B-spline. Since we have selected $k = n + 1$, we know (from Section 7.10) that the open B-spline will be a Bézier curve. The knot vector for our curve is calculated by Equation (7.27) to be $(0,0,0,0,1,1,1,1)$. To simplify our task, we try the simple set of weights $(1, w_1, 1)$. Our problem is to find out for what values, if any, of w_1 we get precise conics.

There is no need to use the Cox–DeBoor recursive formula [Equation (7.25)] to calculate the blending functions because they are the quadratic Bernstein polynomials. The curve itself can easily be written

$$\begin{aligned}
\mathbf{P}(t) &= \frac{N_{03}(t)\mathbf{P}_0 + w_1N_{13}(t)\mathbf{P}_1 + N_{23}(t)\mathbf{P}_2}{N_{03}(t) + w_1N_{13}(t) + N_{23}(t)}\\
&= \frac{(1-t)^2\mathbf{P}_0 + 2w_1t(1-t)\mathbf{P}_1 + t^2\mathbf{P}_2}{(1-t)^2 + 2w_1t(1-t) + t^2}.
\end{aligned} \tag{7.45}$$

◇ **Exercise 7.12:** Show that in the special case where $w_1 = 0$, the curve of Equation (7.45) reduces to the straight line between $\mathbf{P}_0$ and $\mathbf{P}_2$.




```
(* Rational B-spline example. w_2=0, .5, 1, 5 (Slow!) *)
Clear[bspl,knt,w,pnts,cur1,cur2,cur3,cur4,R] (* weight functions *)
bspl[i_,k_,t_]:=If[knt[[i+k]]==knt[[i+1]],0, (* 0<=i<=n *)
 bspl[i,k-1,t] (t-knt[[i+1]])/(knt[[i+k]]-knt[[i+1]])] \
 +If[knt[[i+1+k]]==knt[[i+2]],0,
 bspl[i+1,k-1,t] (knt[[i+1+k]]-t)/(knt[[i+1+k]]-knt[[i+2]])];
bspl[i_,1,t_]:=If[knt[[i+1]]<=t<knt[[i+2]], 1, 0];
R[i_,t_]:=(w[[i+1]] bspl[i,k,t])/Sum[w[[j+1]] bspl[j,k,t], {j,0,n}];
n=4; k=3; w={1,1,0,1,1}; (* weights *)
knt={0,0,0,1,2,3,3,3}; (* knots *)
pnts={{0,0}, {0,1}, {1,0}, {2,1}, {2,0}};
cur1=ParametricPlot[Sum[(R[i,t] pnts[[i+1]]), {i,0,n}], {t,0,3},
 PlotRange->All, DisplayFunction->Identity, Compiled->False];
w[[3]]=0.5;
cur2=ParametricPlot[Sum[(R[i,t] pnts[[i+1]]), {i,0,n}], {t,0,3},
 PlotRange->All, DisplayFunction->Identity, Compiled->False];
w[[3]]=1;
cur3=ParametricPlot[Sum[(R[i,t] pnts[[i+1]]), {i,0,n}], {t,0,3},
 PlotRange->All, DisplayFunction->Identity, Compiled->False];
w[[3]]=5;
cur4=ParametricPlot[Sum[(R[i,t] pnts[[i+1]]), {i,0,n}], {t,0,3},
 PlotRange->All, DisplayFunction->Identity, Compiled->False];
Show[cur1,cur2,cur3,cur4, PlotRange->All, DefaultFont->{"cmr10", 10},
 DisplayFunction->$DisplayFunction];
```

Figure 7.21: Effects of Varying Weight w_2.

The midpoint $\mathbf{S}$ of the curve of Equation (7.45) is given by

$$\mathbf{S} = \mathbf{P}(0.5) = \frac{(\mathbf{P}_0 + \mathbf{P}_2)/2}{1 + w_1} + \frac{w_1\mathbf{P}_1}{1 + w_1} = \frac{1}{1 + w_1}\mathbf{M} + \frac{w_1}{1 + w_1}\mathbf{P}_1 = (1-u)\mathbf{M} + u\mathbf{P}_1, \quad (7.46)$$

where $\mathbf{M} = (\mathbf{P}_0 + \mathbf{P}_2)/2$ is the midpoint of $\mathbf{P}_0$ and $\mathbf{P}_2$ and $u \stackrel{\text{def}}{=} w_1/(1 + w_1)$. Thus, point $\mathbf{S}$, which is called the *shoulder point* of the curve moves along a straight line from $\mathbf{M}$ to $\mathbf{P}_1$ when w_1 varies from 0 to ∞ (or, equivalently, when u varies from 0 to 1).

Equation (7.46) also yields the relation

$$w_1 = \frac{\mathbf{M} - \mathbf{S}}{\mathbf{S} - \mathbf{P}_1}, \tag{7.47}$$

which shows that w_1 is the ratio of two distances.

It can be shown (see, e.g., [Lee 86]) that the single weight w_1 determines the type of conic generated by Equation (7.45). Values in the range $(0, 1)$ generate an elliptic curve (with a circle as a special case). The value $w_1 = 1$ produces a parabolic curve, and values $w_1 > 1$ result in a hyperbolic curve. Figure 7.22 shows examples of these types of conics (notice that **S** is not necessarily the maximum point on these curves).

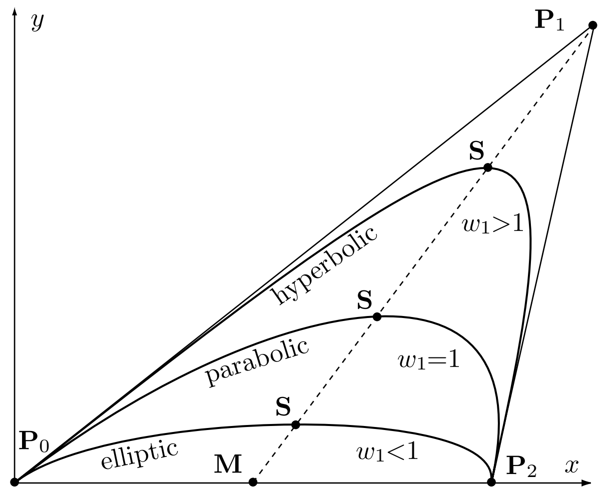
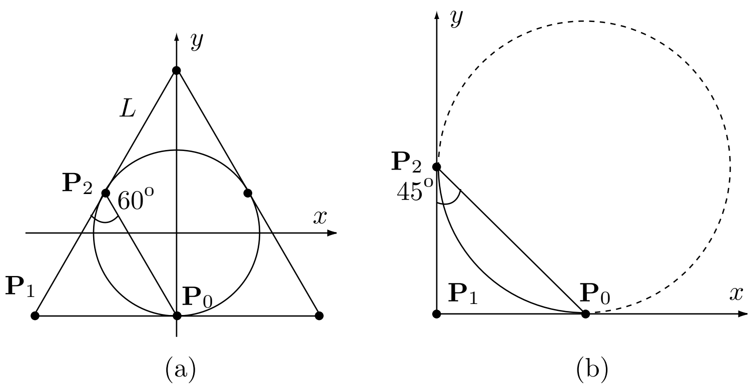


Figure 7.22: Conics Generated by Varying w_1.

A circle is formed when the three control points form an isosceles triangle. If we denote the base angle of this triangle by θ, it can be shown that a circular arc spanning 2θ degrees is obtained when $w_1 = \cos\theta$. The most common cases are $\theta = 60°$ and $\theta = 90°$. In the latter case (Figure 7.23b), a complete circle can easily be formed by using the symmetry of a circle and duplicating every point four times. In the former case (Figure 7.23a), a complete circle can be obtained by specifying six control points and calculating three spline segments.

Example: We are given the three points $\mathbf{P}_0 = (0, -1)R$, $\mathbf{P}_1 = (-1.732, -1)R$, and $\mathbf{P}_2 = (-0.866, 0.5)R$ of Figure 7.23a. Substituting these points in Equation (7.45) and setting w_1 to $\cos 60° = 0.5$ yields the $60°$ circular arc that goes from $\mathbf{P}_0$ to $\mathbf{P}_2$:

$$\begin{aligned}\mathbf{P}(t) &= \frac{(1-t)^2\mathbf{P}_0 + 2w_1t(1-t)\mathbf{P}_1 + t^2\mathbf{P}_2}{(1-t)^2 + 2w_1t(1-t) + t^2}\\ &= R\frac{(1-t)^2(0,-1) + t(1-t)(-1.732,-1) + t^2(-0.866, 0.5)}{(1-t)^2 + t(1-t) + t^2}\end{aligned}$$




```
(* One third of a circle done by rational B-spline *)
P0={0,-1}; P1={-1.732,-1}; P2={-0.866,0.5}; w1=0.5;
pnts=ListPlot[{P0,P1,P2}, Prolog->PointSize[.04], DisplayFunction->Identity];
axs={AbsoluteThickness[1], Line[{P0,P1,P2}]};
th=ParametricPlot[((1-t)^2 P0+2w1 t(1-t)P1+t^2 P2)/((1-t)^2+2w1 t(1-t)+t^2),
 {t,0,1}, PlotRange->All, DisplayFunction->Identity, Compiled->False];
Show[Graphics[axs],th,pnts, PlotRange->All, DisplayFunction->$DisplayFunction];
```

Figure 7.23: Control Points for Circles.

$$= R\frac{(0.866t^2 - 1.732t, 0.5t^2 + t - 1)}{(1-t)^2 + t(1-t) + t^2}.$$

◇ **Exercise 7.13:** Show how to figure out the coordinates of the three points from Figure 7.23a.

◇ **Exercise 7.14:** Given the three points $\mathbf{P}_0 = (1,0)R$, $\mathbf{P}_1 = (0,0)$, and $\mathbf{P}_2 = (0,1)R$ of Figure 7.23b, calculate the quadratic rational B-spline segment defined by the points whose shape is a circular arc spanning 90°.

7.15 Uniform B-Spline Surfaces

The uniform B-Spline surface patch is constructed as a Cartesian product of two uniform B-spline curves. The biquadratic B-spline surface patch, for example, is fully defined by nine control points and is constructed as the Cartesian product of Equation (7.6) with itself

$$\mathbf{P}(u,w) = \left(\frac{1}{2}\right)^2 (u^2,u,1)\begin{pmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02}\\ \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12}\\ \mathbf{P}_{20} & \mathbf{P}_{21} & \mathbf{P}_{22}\end{pmatrix} \times \begin{pmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0\end{pmatrix}^T \begin{pmatrix} w^2\\ w\\ 1\end{pmatrix}. \tag{7.48}$$

Its four corner points are not the four extreme control points, but

$$\begin{aligned}
\mathbf{K}_{00} &= \mathbf{P}(0,0) = \frac{1}{4}(\mathbf{P}_{00}+\mathbf{P}_{01}+\mathbf{P}_{10}+\mathbf{P}_{11}),\\
\mathbf{K}_{01} &= \mathbf{P}(0,1) = \frac{1}{4}(\mathbf{P}_{01}+\mathbf{P}_{02}+\mathbf{P}_{11}+\mathbf{P}_{12}),\\
\mathbf{K}_{10} &= \mathbf{P}(1,0) = \frac{1}{4}(\mathbf{P}_{10}+\mathbf{P}_{11}+\mathbf{P}_{20}+\mathbf{P}_{21}),\\
\mathbf{K}_{11} &= \mathbf{P}(1,1) = \frac{1}{4}(\mathbf{P}_{11}+\mathbf{P}_{12}+\mathbf{P}_{21}+\mathbf{P}_{22}).
\end{aligned} \tag{7.49}$$

Notice that corner point $\mathbf{K}_{00}$ can be written

$$\mathbf{K}_{00} = \frac{1}{2}\left(\frac{\mathbf{P}_{00}+\mathbf{P}_{01}}{2} + \frac{\mathbf{P}_{10}+\mathbf{P}_{11}}{2}\right).$$

This point is therefore located midway between points $(\mathbf{P}_{00}+\mathbf{P}_{01})/2$ and $(\mathbf{P}_{10}+\mathbf{P}_{11})/2$. Figure 7.24a shows this location, as well as the locations of the other three corner points, for the case where the control points are equally spaced.

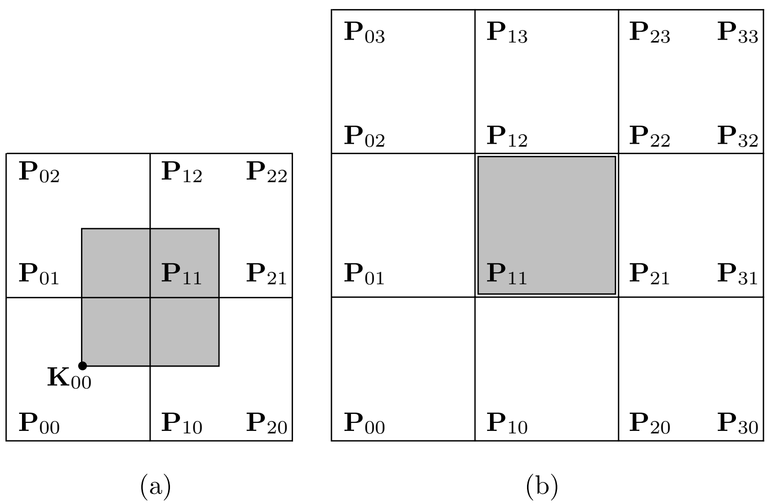


Figure 7.24: Idealized B-Spline Surface Patches.

Example: Given the nine points

$$\begin{aligned}
&\mathbf{P}_{00}=(0,0,0), \quad \mathbf{P}_{01}=(0,1,0), \quad \mathbf{P}_{02}=(0,2,0),\\
&\mathbf{P}_{10}=(1,0,0), \quad \mathbf{P}_{11}=(1,1,1), \quad \mathbf{P}_{12}=(1,2,0),\\
&\mathbf{P}_{20}=(2,0,0), \quad \mathbf{P}_{21}=(2,1,0), \quad \mathbf{P}_{22}=(2,2,0),
\end{aligned}$$

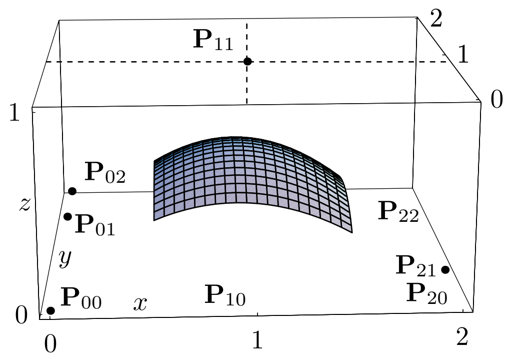
the biquadratic B-spline surface patch defined by them is given by the simple expression

$$\mathbf{P}(u,w) = (u+1/2, w+1/2, (-1-2u+2u^2)(-1-2w+2w^2)/4).$$

Its four corner points are

$$\mathbf{K}_{00} = \mathbf{P}(0,0) = \left(\frac{1}{2},\frac{1}{2},\frac{1}{4}\right), \quad \mathbf{K}_{01} = \mathbf{P}(0,1) = \left(\frac{1}{2},\frac{3}{2},\frac{1}{4}\right),$$
$$\mathbf{K}_{10} = \mathbf{P}(1,0) = \left(\frac{3}{2},\frac{1}{2},\frac{1}{4}\right), \quad \mathbf{K}_{11} = \mathbf{P}(1,1) = \left(\frac{3}{2},\frac{3}{2},\frac{1}{4}\right).$$

Figure 7.25 shows the relation between this surface and its control points.




```
(* BiQuadratic B-spline Patch Example *)
<<:Graphics:ParametricPlot3D.m
Clear[T,Pnts,Q,comb,g1,g2];
T[t_]:={t^2,t,1};
Pnts={{{0,0,0},{0,1.5,0},{0,2,0}},{{1,0,0},{1,1,1},{1,2,0}},
 {{2,0,0},{2,0.5,0},{2,2,0}}};
Q={{1,-2,1},{-2,2,0},{1,1,0}};
g1=Graphics3D[{AbsolutePointSize[3], Table[Point[Pnts[[i,j]]],{i,1,3},{j,1,3}]}];
comb[i_]:=((1/4)T[u].Q.Pnts)[[i]] (Transpose[Q].T[w])[[i]]
g2=ParametricPlot3D[comb[1]+comb[2]+comb[3], {u,0,1},{w,0,1}, AspectRatio->Automatic,
 Ticks->{{0,1,2},{0,1,2},{0,1}}, Compiled->False, DisplayFunction->Identity];
Show[g2,g1, DisplayFunction->$DisplayFunction, ViewPoint->{-0.196, -4.177, 1.160},
 PlotRange->All, DefaultFont->{"cmr10", 10}];
```

Figure 7.25: A Biquadratic B-Spline Surface Patch.

◇ **Exercise 7.15:** Calculate the midpoint $\mathbf{P}(1/2,1/2)$ of this patch.

From the dictionary

A line segment is a part of a line that is bounded by two end points. The midpoint of a segment is the unique point located at an equal distance from the two end points.

The bicubic B-spline patch is defined by a grid of 4×4 control points and is constructed as the Cartesian product of Equation (7.11) with itself

$$\mathbf{P}(u,w)=\left(\frac{1}{6}\right)^2(u^3,u^2,u,1)\begin{pmatrix}-1&3&-3&1\\3&-6&3&0\\-3&0&3&0\\1&4&1&0\end{pmatrix}$$
$$\times\begin{pmatrix}\mathbf{P}_{00}&\mathbf{P}_{01}&\mathbf{P}_{02}&\mathbf{P}_{03}\\\mathbf{P}_{10}&\mathbf{P}_{11}&\mathbf{P}_{12}&\mathbf{P}_{13}\\\mathbf{P}_{20}&\mathbf{P}_{21}&\mathbf{P}_{22}&\mathbf{P}_{23}\\\mathbf{P}_{30}&\mathbf{P}_{31}&\mathbf{P}_{32}&\mathbf{P}_{33}\end{pmatrix}\begin{pmatrix}-1&3&-3&1\\3&-6&3&0\\-3&0&3&0\\1&4&1&0\end{pmatrix}^T\begin{pmatrix}w^3\\w^2\\w\\1\end{pmatrix}. \tag{7.50}$$

Its four corner points are

$$\begin{aligned}\mathbf{K}_{00}&=\mathbf{P}(0,0)\\&=\frac{1}{36}(\mathbf{P}_{00}+\mathbf{P}_{02}+4\mathbf{P}_{10}+4\mathbf{P}_{12}+\mathbf{P}_{20}+4\mathbf{P}_{01}+16\mathbf{P}_{11}+4\mathbf{P}_{21}+\mathbf{P}_{22}),\\\mathbf{K}_{01}&=\mathbf{P}(0,1)\\&=\frac{1}{36}(\mathbf{P}_{01}+4\mathbf{P}_{02}+\mathbf{P}_{03}+4\mathbf{P}_{11}+16\mathbf{P}_{12}+4\mathbf{P}_{13}+\mathbf{P}_{21}+4\mathbf{P}_{22}+\mathbf{P}_{23}),\\\mathbf{K}_{10}&=\mathbf{P}(1,0)\\&=\frac{1}{36}(\mathbf{P}_{10}+\mathbf{P}_{12}+4\mathbf{P}_{20}+4\mathbf{P}_{22}+\mathbf{P}_{30}+4\mathbf{P}_{11}+16\mathbf{P}_{21}+4\mathbf{P}_{31}+\mathbf{P}_{32}),\\\mathbf{K}_{11}&=\mathbf{P}(1,1)\\&=\frac{1}{36}(\mathbf{P}_{11}+4\mathbf{P}_{12}+\mathbf{P}_{13}+4\mathbf{P}_{21}+16\mathbf{P}_{22}+4\mathbf{P}_{23}+\mathbf{P}_{31}+4\mathbf{P}_{32}+\mathbf{P}_{33}).\end{aligned} \tag{7.51}$$

Each is a barycentric sum of nine control points. Notice that the first corner point can be rewritten in the form

$$\mathbf{K}_{00}=\frac{1}{6}\left[\frac{1}{6}(\mathbf{P}_{00}+4\mathbf{P}_{10}+\mathbf{P}_{20})+\frac{4}{6}(\mathbf{P}_{01}+4\mathbf{P}_{11}+\mathbf{P}_{21})+\frac{1}{6}(\mathbf{P}_{02}+4\mathbf{P}_{12}+\mathbf{P}_{22})\right]. \tag{7.52}$$

This point is therefore the weighted sum of three points, each the weighted sum of three control points. Its precise location depends on the positions of the nine points involved.

◇ **Exercise 7.16:** What is the value of $\mathbf{K}_{00}$ for the special case where the control points are equally spaced?

The other three corner points can be expressed similarly. If all 16 points are equally spaced, the bicubic surface patch has its corners at the four control points $\mathbf{P}_{11}$, $\mathbf{P}_{21}$, $\mathbf{P}_{12}$, and $\mathbf{P}_{22}$ (Figure 7.24b shows an idealized diagram).

Large B-spline surfaces can be constructed from these bicubic patches by starting with a mesh of $(m+1)\times(n+1)$ control points $\mathbf{P}_{00}$ through $\mathbf{P}_{mn}$, dividing it into $(m-2)\times(n-2)$ overlapping groups of 4×4 points each, as in Figure 5.9 and applying Equation (7.50) to calculate a cubic patch for each group. The individual patches will not only connect at their joint points but will have C^2 continuity along their boundaries.

To show that the bicubic patches connect at the joints, we note how joint point $\mathbf{K}_{01}$ can be obtained from joint $\mathbf{K}_{00}$ by incrementing the second indices of the nine control points involved in their expressions [Equation (7.51)]. The same is true for joints $\mathbf{K}_{10}$ and $\mathbf{K}_{11}$. Similarly, joint point $\mathbf{K}_{10}$ can be obtained from $\mathbf{K}_{00}$ by incrementing the first index of each control point, and the same is true for joints $\mathbf{K}_{01}$ and $\mathbf{K}_{11}$.

To show first-order continuity we calculate, for example, the two tangent vectors $\mathbf{P}^u(u,0)$ and $\mathbf{P}^u(u,1)$ of boundary curves $\mathbf{P}(u,0)$ and $\mathbf{P}(u,1)$

$$\begin{aligned}
\mathbf{P}^u(u,0) = \big(&-\mathbf{P}_{02}+\mathbf{P}_{20}+4\mathbf{P}_{21}+\mathbf{P}_{22}-\mathbf{P}_{00}(u-1)^2-4\mathbf{P}_{01}(u-1)^2\\
&+2\mathbf{P}_{02}u-4\mathbf{P}_{10}u-16\mathbf{P}_{11}u-4\mathbf{P}_{12}u+2\mathbf{P}_{20}u+8\mathbf{P}_{21}u+2\mathbf{P}_{22}u\\
&-\mathbf{P}_{02}u^2+3\mathbf{P}_{10}u^2+12\mathbf{P}_{11}u^2+3\mathbf{P}_{12}u^2-3\mathbf{P}_{20}u^2-12\mathbf{P}_{21}u^2\\
&-3\mathbf{P}_{22}u^2+\mathbf{P}_{30}u^2+4\mathbf{P}_{31}u^2+\mathbf{P}_{32}u^2\big)/12\\
\mathbf{P}^u(u,1) = \big(&-\mathbf{P}_{03}+\mathbf{P}_{21}+4\mathbf{P}_{22}+\mathbf{P}_{23}-\mathbf{P}_{01}(u-1)^2-4\mathbf{P}_{02}(u-1)^2\\
&+2\mathbf{P}_{03}u-4\mathbf{P}_{11}u-16\mathbf{P}_{12}u-4\mathbf{P}_{13}u+2\mathbf{P}_{21}u+8\mathbf{P}_{22}u+2\mathbf{P}_{23}u\\
&-\mathbf{P}_{03}u^2+3\mathbf{P}_{11}u^2+12\mathbf{P}_{12}u^2+3\mathbf{P}_{13}u^2-3\mathbf{P}_{21}u^2-12\mathbf{P}_{22}u^2\\
&-3\mathbf{P}_{23}u^2+\mathbf{P}_{31}u^2+4\mathbf{P}_{32}u^2+\mathbf{P}_{33}u^2\big)/12
\end{aligned} \tag{7.53}$$

Equation (7.53) shows that tangent vector $\mathbf{P}^u(u,1)$ can be obtained from $\mathbf{P}^u(u,0)$ by incrementing the second index of every control point involved. Equation (7.54) illustrates the same property for the second derivatives, thereby showing second-order continuity:

$$\begin{aligned}
\mathbf{P}^{uu}(u,0) = \big(&\mathbf{P}_{00}+\mathbf{P}_{02}-2\mathbf{P}_{10}-8\mathbf{P}_{11}-2\mathbf{P}_{12}+\mathbf{P}_{20}+4\mathbf{P}_{21}+\mathbf{P}_{22}\\
&-4\mathbf{P}_{01}(u-1)-\mathbf{P}_{00}u-\mathbf{P}_{02}u+3\mathbf{P}_{10}u+12\mathbf{P}_{11}u+3\mathbf{P}_{12}u\\
&-3\mathbf{P}_{20}u-12\mathbf{P}_{21}u-3\mathbf{P}_{22}u+\mathbf{P}_{30}u+4\mathbf{P}_{31}u+\mathbf{P}_{32}u\big)/6\\
\mathbf{P}^{uu}(u,1) = \big(&\mathbf{P}_{01}+\mathbf{P}_{03}-2\mathbf{P}_{11}-8\mathbf{P}_{12}-2\mathbf{P}_{13}+\mathbf{P}_{21}+4\mathbf{P}_{22}+\mathbf{P}_{23}\\
&-4\mathbf{P}_{02}(u-1)-\mathbf{P}_{01}u-\mathbf{P}_{03}u+3\mathbf{P}_{11}u+12\mathbf{P}_{12}u+3\mathbf{P}_{13}u\\
&-3\mathbf{P}_{21}u-12\mathbf{P}_{22}u-3\mathbf{P}_{23}u+\mathbf{P}_{31}u+4\mathbf{P}_{32}u+\mathbf{P}_{33}u\big)/6.
\end{aligned} \tag{7.54}$$

7.16 Relation to Other Surfaces

This short section shows how the uniform bicubic B-spline surface patch can be expressed as either a bicubic Coons or a bicubic Bézier patch.

Bicubic Coons and B-Spline Patches. A bicubic B-spline surface patch can be written as a bicubic Coons patch. That patch [Equation (4.35), duplicated here] is defined in terms of four corner points, eight tangent vectors, and four twist vectors. These 16 quantities (the elements of matrix $\mathbf{C}$ below) can be expressed in terms of the 16 control points $\mathbf{P}_{ij}$ that define the B-spline patch. The idea is to equate the expression for the Coons surface

$$\mathbf{Q}(u,w) = (u^3, u^2, u, 1)\mathbf{H}\begin{pmatrix} \mathbf{Q}_{00} & \mathbf{Q}_{01} & \mathbf{Q}^w_{00} & \mathbf{Q}^w_{01} \\ \mathbf{Q}_{10} & \mathbf{Q}_{11} & \mathbf{Q}^w_{10} & \mathbf{Q}^w_{11} \\ \mathbf{Q}^u_{00} & \mathbf{Q}^u_{01} & \mathbf{Q}^{uw}_{00} & \mathbf{Q}^{uw}_{01} \\ \mathbf{Q}^u_{10} & \mathbf{Q}^u_{11} & \mathbf{Q}^{uw}_{10} & \mathbf{Q}^{uw}_{11} \end{pmatrix}\mathbf{H}^T\begin{pmatrix} w^3 \\ w^2 \\ w \\ 1 \end{pmatrix} = \mathbf{UHCH}^T\mathbf{W}^T, \tag{4.35}$$

with that of the B-spline surface, Equation (7.50), and solve for the 16 elements of matrix $\mathbf{C}$. This process is straightforward and the solutions are

$$\mathbf{Q}_{00} = \frac{1}{6}\left(\frac{\mathbf{P}_{00}}{6} + \frac{4\mathbf{P}_{10}}{6} + \frac{\mathbf{P}_{20}}{6}\right) + \frac{4}{6}\left(\frac{\mathbf{P}_{01}}{6} + \frac{4\mathbf{P}_{11}}{6} + \frac{\mathbf{P}_{21}}{6}\right) + \frac{1}{6}\left(\frac{\mathbf{P}_{02}}{6} + \frac{4\mathbf{P}_{12}}{6} + \frac{\mathbf{P}_{22}}{6}\right),$$

$$\mathbf{Q}_{01} = \frac{1}{6}\left(\frac{\mathbf{P}_{01}}{6} + \frac{4\mathbf{P}_{11}}{6} + \frac{\mathbf{P}_{21}}{6}\right) + \frac{4}{6}\left(\frac{\mathbf{P}_{02}}{6} + \frac{4\mathbf{P}_{12}}{6} + \frac{\mathbf{P}_{22}}{6}\right) + \frac{1}{6}\left(\frac{\mathbf{P}_{03}}{6} + \frac{4\mathbf{P}_{13}}{6} + \frac{\mathbf{P}_{23}}{6}\right),$$

$$\mathbf{Q}_{10} = \frac{1}{6}\left(\frac{\mathbf{P}_{10}}{6} + \frac{4\mathbf{P}_{20}}{6} + \frac{\mathbf{P}_{30}}{6}\right) + \frac{4}{6}\left(\frac{\mathbf{P}_{11}}{6} + \frac{4\mathbf{P}_{21}}{6} + \frac{\mathbf{P}_{31}}{6}\right) + \frac{1}{6}\left(\frac{\mathbf{P}_{12}}{6} + \frac{4\mathbf{P}_{22}}{6} + \frac{\mathbf{P}_{32}}{6}\right),$$

$$\mathbf{Q}_{11} = \frac{1}{6}\left(\frac{\mathbf{P}_{11}}{6} + \frac{4\mathbf{P}_{21}}{6} + \frac{\mathbf{P}_{31}}{6}\right) + \frac{4}{6}\left(\frac{\mathbf{P}_{12}}{6} + \frac{4\mathbf{P}_{22}}{6} + \frac{\mathbf{P}_{32}}{6}\right) + \frac{1}{6}\left(\frac{\mathbf{P}_{13}}{6} + \frac{4\mathbf{P}_{23}}{6} + \frac{\mathbf{P}_{33}}{6}\right),$$

$$\mathbf{Q}^u_{00} = \frac{1}{6}\left(\frac{\mathbf{P}_{20} - \mathbf{P}_{00}}{2}\right) + \frac{4}{6}\left(\frac{\mathbf{P}_{21} - \mathbf{P}_{01}}{2}\right) + \frac{1}{6}\left(\frac{\mathbf{P}_{22} - \mathbf{P}_{02}}{2}\right),$$

$$\mathbf{Q}^u_{01} = \frac{1}{6}\left(\frac{\mathbf{P}_{21} - \mathbf{P}_{01}}{2}\right) + \frac{4}{6}\left(\frac{\mathbf{P}_{22} - \mathbf{P}_{02}}{2}\right) + \frac{1}{6}\left(\frac{\mathbf{P}_{23} - \mathbf{P}_{03}}{2}\right),$$

$$\mathbf{Q}^u_{10} = \frac{1}{6}\left(\frac{\mathbf{P}_{30} - \mathbf{P}_{10}}{2}\right) + \frac{4}{6}\left(\frac{\mathbf{P}_{31} - \mathbf{P}_{11}}{2}\right) + \frac{1}{6}\left(\frac{\mathbf{P}_{32} - \mathbf{P}_{12}}{2}\right),$$

$$\mathbf{Q}^u_{11} = \frac{1}{6}\left(\frac{\mathbf{P}_{31} - \mathbf{P}_{11}}{2}\right) + \frac{4}{6}\left(\frac{\mathbf{P}_{32} - \mathbf{P}_{12}}{2}\right) + \frac{1}{6}\left(\frac{\mathbf{P}_{33} - \mathbf{P}_{13}}{2}\right),$$

$$\mathbf{Q}^w_{00} = \frac{1}{2}\left(\frac{\mathbf{P}_{02}}{6} + \frac{4\mathbf{P}_{12}}{6} + \frac{\mathbf{P}_{22}}{6}\right) - \frac{1}{2}\left(\frac{\mathbf{P}_{00}}{6} + \frac{4\mathbf{P}_{10}}{6} + \frac{\mathbf{P}_{20}}{6}\right),$$

$$\mathbf{Q}^w_{01} = \frac{1}{2}\left(\frac{\mathbf{P}_{03}}{6} + \frac{4\mathbf{P}_{13}}{6} + \frac{\mathbf{P}_{23}}{6}\right) - \frac{1}{2}\left(\frac{\mathbf{P}_{01}}{6} + \frac{4\mathbf{P}_{11}}{6} + \frac{\mathbf{P}_{21}}{6}\right),$$

$$\mathbf{Q}^w_{10} = \frac{1}{2}\left(\frac{\mathbf{P}_{12}}{6} + \frac{4\mathbf{P}_{22}}{6} + \frac{\mathbf{P}_{32}}{6}\right) - \frac{1}{2}\left(\frac{\mathbf{P}_{10}}{6} + \frac{4\mathbf{P}_{20}}{6} + \frac{\mathbf{P}_{30}}{6}\right),$$

$$\mathbf{Q}^w_{11} = \frac{1}{2}\left(\frac{\mathbf{P}_{13}}{6} + \frac{4\mathbf{P}_{23}}{6} + \frac{\mathbf{P}_{33}}{6}\right) - \frac{1}{2}\left(\frac{\mathbf{P}_{11}}{6} + \frac{4\mathbf{P}_{21}}{6} + \frac{\mathbf{P}_{31}}{6}\right),$$

$$\mathbf{Q}^{uw}_{00} = \frac{1}{2}\left(\frac{\mathbf{P}_{22} - \mathbf{P}_{02}}{2}\right) - \frac{1}{2}\left(\frac{\mathbf{P}_{20} - \mathbf{P}_{00}}{2}\right),$$

$$\mathbf{Q}^{uw}_{01} = \frac{1}{2}\left(\frac{\mathbf{P}_{23} - \mathbf{P}_{03}}{2}\right) - \frac{1}{2}\left(\frac{\mathbf{P}_{21} - \mathbf{P}_{01}}{2}\right),$$

$$\mathbf{Q}^{uw}_{10} = \frac{1}{2}\left(\frac{\mathbf{P}_{32} - \mathbf{P}_{12}}{2}\right) - \frac{1}{2}\left(\frac{\mathbf{P}_{30} - \mathbf{P}_{10}}{2}\right),$$

$$\mathbf{Q}^{uw}_{11} = \frac{1}{2}\left(\frac{\mathbf{P}_{33} - \mathbf{P}_{13}}{2}\right) - \frac{1}{2}\left(\frac{\mathbf{P}_{31} - \mathbf{P}_{11}}{2}\right).$$

Bézier and B-Spline Bicubic Patches. A bicubic B-spline surface patch can also be written in the form of a bicubic Bézier patch. The bicubic Bézier patch is fully defined by 16 control points $\mathbf{Q}_{ij}$ [the elements of matrix $\mathbf{P}$ of Equation (6.32)]. They can be expressed in terms of the 16 control points $\mathbf{P}_{ij}$ defining the B-spline patch. The idea is to equate the expressions for the bicubic Bézier and B-spline surface patches and solve for the elements of matrix $\mathbf{P}$. The solutions are

$$
\begin{aligned}
\mathbf{Q}_{00} &= \frac{1}{6}\left(\frac{\mathbf{P}_{00}}{6}+\frac{4\mathbf{P}_{10}}{6}+\frac{\mathbf{P}_{20}}{6}\right)+\frac{4}{6}\left(\frac{\mathbf{P}_{01}}{6}+\frac{4\mathbf{P}_{11}}{6}+\frac{\mathbf{P}_{21}}{6}\right)+\frac{1}{6}\left(\frac{\mathbf{P}_{02}}{6}+\frac{4\mathbf{P}_{12}}{6}+\frac{\mathbf{P}_{22}}{6}\right),\\
\mathbf{Q}_{01} &= \frac{4}{6}\left(\frac{\mathbf{P}_{01}}{6}+\frac{4\mathbf{P}_{11}}{6}+\frac{\mathbf{P}_{21}}{6}\right)+\frac{2}{6}\left(\frac{\mathbf{P}_{02}}{6}+\frac{4\mathbf{P}_{12}}{6}+\frac{\mathbf{P}_{32}}{6}\right),\\
\mathbf{Q}_{02} &= \frac{2}{6}\left(\frac{\mathbf{P}_{01}}{6}+\frac{4\mathbf{P}_{11}}{6}+\frac{\mathbf{P}_{21}}{6}\right)+\frac{4}{6}\left(\frac{\mathbf{P}_{02}}{6}+\frac{4\mathbf{P}_{12}}{6}+\frac{\mathbf{P}_{32}}{6}\right),\\
\mathbf{Q}_{03} &= \frac{1}{6}\left(\frac{\mathbf{P}_{01}}{6}+\frac{4\mathbf{P}_{11}}{6}+\frac{\mathbf{P}_{21}}{6}\right)+\frac{4}{6}\left(\frac{\mathbf{P}_{02}}{6}+\frac{4\mathbf{P}_{12}}{6}+\frac{\mathbf{P}_{22}}{6}\right)+\frac{1}{6}\left(\frac{\mathbf{P}_{03}}{6}+\frac{4\mathbf{P}_{13}}{6}+\frac{\mathbf{P}_{23}}{6}\right),\\
\mathbf{Q}_{10} &= \frac{1}{6}\left(\frac{4\mathbf{P}_{10}+2\mathbf{P}_{20}}{6}\right)+\frac{4}{6}\left(\frac{4\mathbf{P}_{11}+2\mathbf{P}_{21}}{6}\right)+\frac{1}{6}\left(\frac{4\mathbf{P}_{12}+2\mathbf{P}_{22}}{6}\right),\\
\mathbf{Q}_{11} &= \frac{4}{6}\left(\frac{4\mathbf{P}_{11}+2\mathbf{P}_{21}}{6}\right)+\frac{2}{6}\left(\frac{4\mathbf{P}_{12}+2\mathbf{P}_{22}}{6}\right),\\
\mathbf{Q}_{12} &= \frac{2}{6}\left(\frac{4\mathbf{P}_{11}+2\mathbf{P}_{21}}{6}\right)+\frac{4}{6}\left(\frac{4\mathbf{P}_{12}+2\mathbf{P}_{22}}{6}\right),\\
\mathbf{Q}_{13} &= \frac{1}{6}\left(\frac{4\mathbf{P}_{11}+2\mathbf{P}_{21}}{6}\right)+\frac{4}{6}\left(\frac{4\mathbf{P}_{12}+2\mathbf{P}_{22}}{6}\right)+\frac{1}{6}\left(\frac{4\mathbf{P}_{13}+2\mathbf{P}_{23}}{6}\right),\\
\mathbf{Q}_{20} &= \frac{1}{6}\left(\frac{4\mathbf{P}_{10}+2\mathbf{P}_{20}}{6}\right)+\frac{4}{6}\left(\frac{4\mathbf{P}_{11}+2\mathbf{P}_{21}}{6}\right)+\frac{1}{6}\left(\frac{4\mathbf{P}_{12}+2\mathbf{P}_{22}}{6}\right),\\
\mathbf{Q}_{21} &= \frac{4}{6}\left(\frac{2\mathbf{P}_{11}+4\mathbf{P}_{21}}{6}\right)+\frac{2}{6}\left(\frac{2\mathbf{P}_{12}+4\mathbf{P}_{22}}{6}\right),\\
\mathbf{Q}_{22} &= \frac{2}{6}\left(\frac{2\mathbf{P}_{11}+4\mathbf{P}_{21}}{6}\right)+\frac{4}{6}\left(\frac{2\mathbf{P}_{12}+4\mathbf{P}_{22}}{6}\right),\\
\mathbf{Q}_{23} &= \frac{1}{6}\left(\frac{2\mathbf{P}_{11}+4\mathbf{P}_{21}}{6}\right)+\frac{4}{6}\left(\frac{2\mathbf{P}_{12}+4\mathbf{P}_{22}}{6}\right)+\frac{1}{6}\left(\frac{2\mathbf{P}_{13}+4\mathbf{P}_{23}}{6}\right),\\
\mathbf{Q}_{30} &= \frac{1}{6}\left(\frac{\mathbf{P}_{10}}{6}+\frac{4\mathbf{P}_{20}}{6}+\frac{\mathbf{P}_{30}}{6}\right)+\frac{4}{6}\left(\frac{\mathbf{P}_{11}}{6}+\frac{4\mathbf{P}_{21}}{6}+\frac{\mathbf{P}_{31}}{6}\right)+\frac{1}{6}\left(\frac{\mathbf{P}_{12}}{6}+\frac{4\mathbf{P}_{22}}{6}+\frac{\mathbf{P}_{32}}{6}\right),\\
\mathbf{Q}_{31} &= \frac{4}{6}\left(\frac{\mathbf{P}_{11}+4\mathbf{P}_{21}+\mathbf{P}_{31}}{6}\right)+\frac{2}{6}\left(\frac{\mathbf{P}_{12}+4\mathbf{P}_{22}+\mathbf{P}_{32}}{6}\right),\\
\mathbf{Q}_{32} &= \frac{2}{6}\left(\frac{\mathbf{P}_{11}+4\mathbf{P}_{21}+\mathbf{P}_{31}}{6}\right)+\frac{4}{6}\left(\frac{\mathbf{P}_{12}+4\mathbf{P}_{22}+\mathbf{P}_{32}}{6}\right),\\
\mathbf{Q}_{33} &= \frac{1}{6}\left(\frac{\mathbf{P}_{11}}{6}+\frac{4\mathbf{P}_{21}}{6}+\frac{\mathbf{P}_{31}}{6}\right)+\frac{4}{6}\left(\frac{\mathbf{P}_{12}}{6}+\frac{4\mathbf{P}_{22}}{6}+\frac{\mathbf{P}_{32}}{6}\right)+\frac{1}{6}\left(\frac{\mathbf{P}_{13}}{6}+\frac{4\mathbf{P}_{23}}{6}+\frac{\mathbf{P}_{33}}{6}\right).
\end{aligned}
$$

7.17 An Interpolating Bicubic Patch

The uniform bicubic B-spline surface patch is defined by 16 control points. A mesh of $(m+1)\times(n+1)$ control points can be used to calculate $(m-2)\times(n-2)$ such patches. Each patch has four corner points, but since the patches are connected, the total number of joint points is $(m-1)\times(n-1)$. This section shows how to solve the opposite problem, namely given a mesh of $(m-1)\times(n-1)$ data points $\mathbf{Q}_{1,1}$ through $\mathbf{Q}_{m-1,n-1}$, how to calculate the bicubic B-spline surface that passes through them.

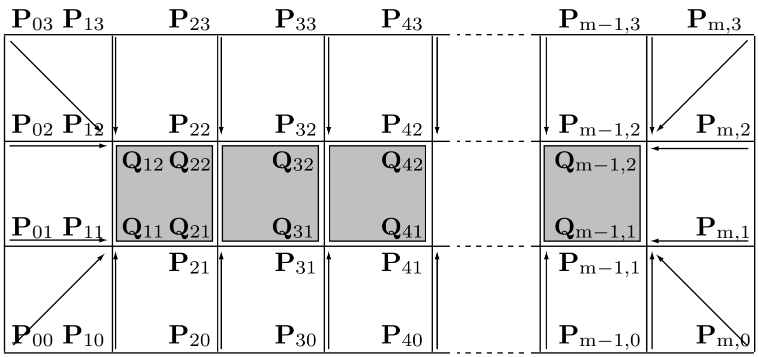


Figure 7.26: An Interpolating B-Spline Surface.

The given data points $\mathbf{Q}_{ij}$ are considered the joint points of the unknown surface and Equation (7.52) shows how they are related to the (yet unknown) control points $\mathbf{P}_{00}$ through $\mathbf{P}_{mn}$:

$$\begin{aligned}\mathbf{Q}_{ij} =& \frac{1}{6}\Big[\frac{1}{6}(\mathbf{P}_{i-1,j-1}+4\mathbf{P}_{i,j-1}+\mathbf{P}_{i+1,j-1})\\ &+\frac{4}{6}(\mathbf{P}_{i-1,j}+4\mathbf{P}_{i,j}+\mathbf{P}_{i+1,j})\\ &+\frac{1}{6}(\mathbf{P}_{i-1,j+1}+4\mathbf{P}_{i,j+1}+\mathbf{P}_{i+1,j+1})\Big].\end{aligned} \tag{7.55}$$

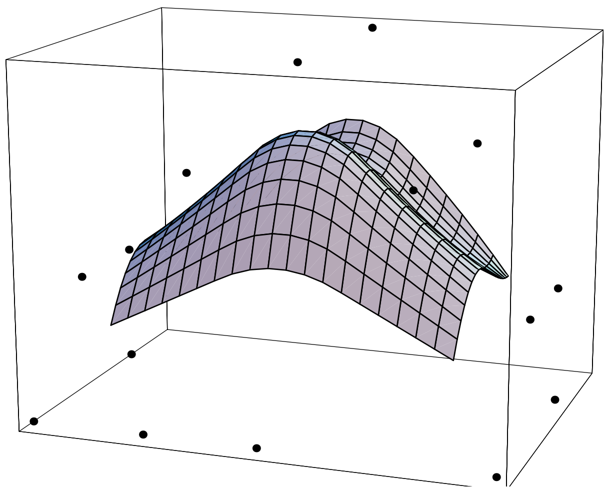
Equation (7.55) can be written $(m-1)\times(n-1)$ times, once for each given data point $\mathbf{Q}_{ij}$. The number of equations needed, however, is $(m+1)\times(n+1)$. We use the relation

$$(m+1)\times(n+1)=(m-1)\times(n-1)+2m+2n,$$

to figure out how many more equations are needed. The extra equations are obtained by the user specifying the vectors shown in Figure 7.26. There are $m-1$ vectors going from boundary control points $\mathbf{P}_{i,0}$ to the "bottom" data points $\mathbf{Q}_{i1}$. There are $m-1$ more such vectors going from the boundary control points $\mathbf{P}_{i,n+1}$ to the "top" data points $\mathbf{Q}_{i,n}$. In addition, there are $2(n-1)$ vectors going from the "left" and "right" boundary control points to the extreme data points $\mathbf{Q}_{1,j}$ and $\mathbf{Q}_{m-1,j}$. Finally, there are four vectors going from the four corner control points to the four corner data points.

Once all $2(n-1)+2(m-1)+4$ vectors have been specified, a system of $(m+1)\times(n+1)$ linear equations can be set and solved, to yield the control points.

If the surface should be closed along one dimension, some of the vectors don't have to be specified. For example, if the surface of Figure 7.26 should be closed in the vertical direction (i.e., if it should resemble a horizontal cylinder), then the bottom row of control points $\mathbf{P}_{i,0}$ should be duplicated and renamed $\mathbf{P}_{i,4}$, and the top row $\mathbf{P}_{i,3}$ should be duplicated and renamed $\mathbf{P}_{i,-1}$. Two extra rows of surface patches should be calculated, but every patch now has control points above and below it, so the $2(m-1)$ vertical vectors need not be specified by the user.



```
0 0 0  0 1 1  0 2 1  0 2 0
1 0 0  1 1 2  1 2 1  1 3 2
2 0 0  2 1 3  2 2 2  2 3 3
3 0 0  3 1 2  3 2 1  3 3 2
4 0 0  4 1 1  4 2 1  4 2 0
```

```
(* a general uniform B-spline surface patch *)
Clear[bsplSurf,surpnts,bspl,g1,g2,knt,i,j,km,kn,m,n,u,w]
bspl[i_,k_,t_]:=bspl[i,k-1,t] (t-knt[[i+1]])/(knt[[i+k]]-knt[[i+1]]) \
 +bspl[i+1,k-1,t] (knt[[i+1+k]]-t)/(knt[[i+1+k]]-knt[[i+2]]) (* 0<=i<=n *)
bspl[i_,1,t_]:=If[knt[[i+1]]<=t<knt[[i+2]], 1, 0];
n=3; kn=3; m=4; km=3; (* Note: 0<=kn<=n 0<=km<=m *)
knt=Table[i, {i,0,m+km}]; (* uniform knots *)
(* Input triplets from data file *)
surpnts=ReadList["surf.pnts", {Number,Number,Number}, RecordLists->True];
bsplSurf[u_,w_]:=Sum[Sum[surpnts[[i+1,j+1]]bspl[i,km,u],{i,0,m}]bspl[j,kn,w],{j,0,n}]
g1=Graphics3D[{AbsolutePointSize[3], Table[Point[surpnts[[i,j]]],{i,1,5},{j,1,4}]}];
g2=ParametricPlot3D[bsplSurf[u,w], {u,km-1,m+1},{w,kn-1,n+1},
 DisplayFunction->Identity,
 AspectRatio->Automatic, Compiled->False];
Show[g1,g2, PlotRange->All, DisplayFunction->$DisplayFunction,
 DefaultFont->{"cmr10", 10}, ViewPoint->{1.389, -3.977, 1.042}];
```

Figure 7.27: A Quadratic-Cubic B-Spline Surface Patch.

7.18 The Quadratic-Cubic B-Spline Surface

This type of surface patch is defined by a 3×4 mesh of control points and its expression is a Cartesian product of the quadratic and cubic B-spline curves:

$$\mathbf{P}(u,w) = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)(u^2,u,1)\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} & \mathbf{P}_{03} \\ \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} w^3 \\ w^2 \\ w \\ 1 \end{pmatrix}.$$

Figure 7.27 is an example.

The excellent mathematical and algorithmic properties, combined with successful industrial applications, have contributed to the enormous popularity of NURBS. NURBS play a role in the CAD/CAM/CAE world similar to that of the English language in science and business: "Want to talk business? Learn to talk NURBS".

Les Piegl and Wayne Tiller, *The NURBS Book* (1996)