OSCILLATIONS 1

Resnick, Halliday, Krane

Q.1 Define simple harmonic motion. Write down its equation of motion and solve for displacement, hence calculate its time period, velocity, acceleration and energy? (PU. 2007, 2011,2014, SU. 2014)

SIMPLE HARMONIC MOTION

The to and fro motion of a body in which acceleration is directly proportional to the displacement and always directed towards mean position is called **simple harmonic motion.** The body executing simple harmonic motion is called **simple harmonic oscillator.**

EXAMPLES

1. The motion of simple pendulum is SHM.
2. The motion of spring mass system is SHM.

SIMPLE HARMONIC OSCILLATOR

Consider a block of mass m is attached with one end of a spring. The other end of spring is fixed to a support. The block is free to move to and fro over a frictionless horizontal surface as shown in fig.

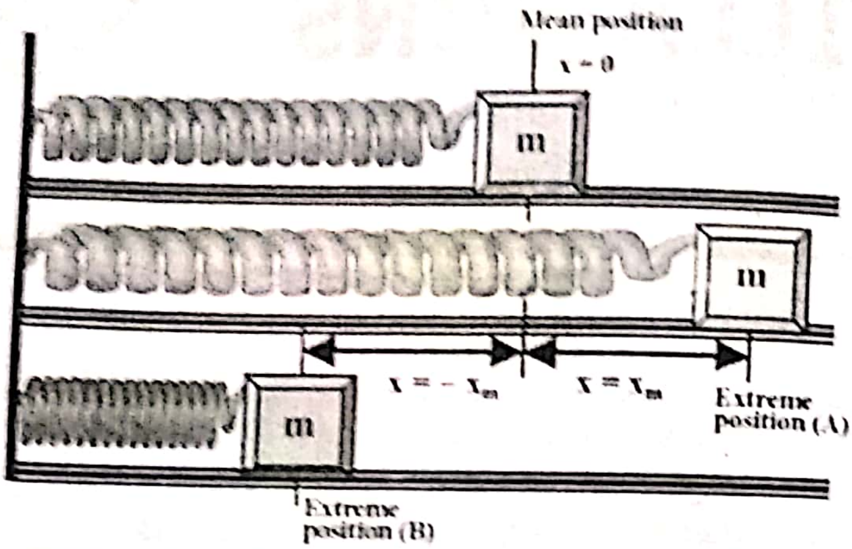
MEAN POSITION

The position $x = 0$ when block is at rest is called mean position because spring is not exerting any no force on the block.

RESTORING FORCE

Now apply external force on the block towards right which extends the spring through distance x_m(extreme position). The spring exerts a force on the block towards mean position when applied force is removed.

This force is equal in magnitude to the applied force but opposite in direction called **spring force or restoring force.**



VIBRATORY MOTION

The block moves towards mean position under spring force. It gets maximum velocity when reaches at mean position and does not stop there due to inertia but continues to move towards extreme position(B). The velocity of block becomes zero at extreme position(B) due to restoring force.

The acceleration is directed from extreme position(B) to mean position because velocity of block is going to decrease when block is moving from mean position to extreme position(B).

Now block moves from extreme position(B) towards mean position due to spring force. The velocity of block increases from B towards mean position and becomes maximum when reaches at mean position. The acceleration is again directed towards mean position. In this way acceleration of block is always directed towards mean position during its to and fro motion.

EQUATION OF MOTION

The block attached with spring having spring constant k takes to and fro motion under restoring force F given as

$$F = -kx \quad \text{------ (1)}$$

The negative sign means restoring force F and displacement x are always in opposite direction. Now apply Newton's second law of motion

$$F = ma$$

$$F = m\frac{d^2x}{dt^2} \qquad \text{------(2)}$$

Compare eq (1) and eq (2)

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This is called **equation of motion of simple harmonic oscillator.**

VELOCITY OF HARMONIC OSCILLATOR

The equation of motion of simple harmonic oscillator is

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\boxed{\omega^2 = \frac{k}{m}}$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) + \omega^2 x = 0$$

$$\frac{dV}{dt} + \omega^2 x = 0$$

$$\frac{dV}{dt}\frac{dx}{dx} + \omega^2 x = 0$$

$$V\frac{dV}{dx} = -\omega^2 x$$

$$V\,dV = -\omega^2 x\,dx$$

Integrating on both sides

$$\int V\,dV = -\omega^2 \int x\,dx$$

$$\frac{V^2}{2} = -\omega^2\left(\frac{x^2}{2}\right) + C$$

$$V^2 = -\omega^2 x^2 + 2C \qquad \text{------(3)}$$

Where C is integration constant. Its value can be determined by initial boundary conditions. The velocity $V = 0$ when $x = x_m$

$$(0)^2 = -\omega^2 x^2_m + 2C$$

$$2C = \omega^2 x^2_m$$

Put this value in eq(3)

$$V^2 = \omega^2(x_m^2 - x^2)$$

The velocity of harmonic oscillator is

$$V = \pm\omega\sqrt{(x_m^2 - x^2)}$$

MAXIMUM VELOCITY

The velocity of harmonic oscillator is maximum at mean position($x = 0$)

$$V_{max} = \pm\omega\sqrt{(x_m^2 - 0)}$$

$$V_{max} = \pm x_m\sqrt{\frac{k}{m}}$$

MINIMUM VELOCITY

The velocity of harmonic oscillator is minimum at extreme position($x = x_m$)

$$V_{min} = \pm\omega\sqrt{(x_m^2 - x_m^2)}$$

$$V_{min} = 0$$

DISPLACEMENT OF HARMONIC OSCILLATOR

The velocity of harmonic oscillator is

$$V = \pm\omega\sqrt{(x_m^2 - x^2)}$$

$$\frac{dx}{dt} = \pm\omega\sqrt{(x_m^2 - x^2)}$$

$\frac{-dx}{\sqrt{(x_m^2 - x^2)}} = \omega dt$ Integrate on both sides $\cos^{-1}\left(\frac{x}{x_m}\right) = \omega t + \phi$ Where ϕ is integration constant $\frac{x}{x_m} = \cos(\omega t + \phi)$ $x = x_m \cos(\omega t + \phi)$	$\frac{dx}{\sqrt{(x_m^2 - x^2)}} = \omega dt$ Integrate on both sides $\sin^{-1}\left(\frac{x}{x_m}\right) = \omega t + \phi$ Where ϕ is integration constant $\frac{x}{x_m} = \sin(\omega t + \phi)$ $x = x_m \sin(\omega t + \phi)$

The eq $x = x_m \cos(\omega t + \phi)$ is the displacement of simple harmonic oscillator when motion starts from extreme position. The eq $x = x_m \sin(\omega t + \phi)$ is the displacement of simple harmonic oscillator when motion starts from mean position. Here we shall consider motion is started from extreme position.

AMPLITUDE

The displacement of simple harmonic oscillator is given as

$$x = x_m \cos(\omega t + \phi)$$

The maximum displacement of harmonic oscillator from mean position is called **amplitude.**

$$\text{Amplitude} = x_m \cos(\omega t + \phi)_{max} = x_m (\pm 1)$$

$$\text{Amplitude} = \pm x_m$$

PHASE

The term $(\omega t + \phi)$ is called **phase of the motion.** The constant ϕ is called phase constant.

The displacement of simple harmonic oscillator when $\phi = 0$ is

$$x = x_m \cos(\omega t + \phi)$$

$$x = x_m \cos(\omega t + 0) = x_m \cos\omega t$$

The displacement is maximum at time $t = 0$

$$x = x_m \cos(0) = x_m$$

The displacement of simple harmonic oscillator when $\phi = -90^0$ is

$$x = x_m \cos(\omega t + \phi) = x_m \cos(\omega t - 90^0)$$

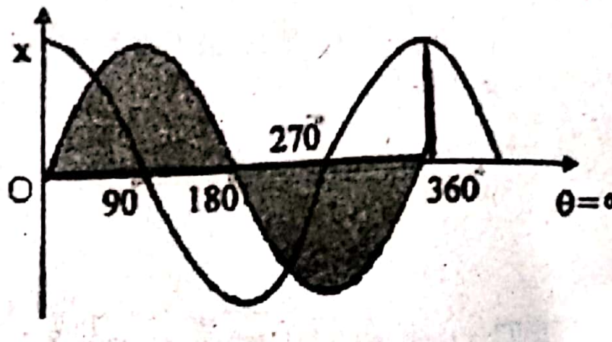
$$x = x_m (\cos\omega t \cos 90^0 + \sin\omega t \sin 90^0)$$

$$x = x_m \sin\omega t$$

The displacement is minimum at time $t = 0$

$$x = x_m \sin(0) = 0$$

It simply shows two motions may have same frequency and same amplitude but differ in phase. The phase behavior is shown in Fig.



LINEAR FREQUENCY AND ANGULAR FREQUENCY

The amount of vibrations completed in unit time is called **linear frequency.** Its SI unit is called hertz(Hz). The amount of rotations completed in unit time is called **angular frequency**. The linear frequency f and angular frequency ω are related as

$$f = \frac{\omega}{2\pi}$$

TIME PERIOD OF MOTION

The time needed to complete one vibration is called time **period of motion**. The reciprocal of linear frequency is equal to time period of motion. The time after which a function repeats itself is called time period of periodic motion. The time period of periodic motion is $\frac{2\pi}{\omega}$. It can be proved as

The displacement of simple harmonic motion is

$$x = x_m \text{Cos}(\omega t + \phi)$$

The displacement of simple harmonic oscillator after time $\left(t + \frac{2\pi}{\omega}\right)$ is

$$x = x_m \text{Cos}\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$$

$$x = x_m \text{Cos}\,(\omega t + 2\pi + \phi)$$

$$x = x_m \text{Cos}\,[(\omega t + \phi) + 2\pi]$$

$$x = x_m [\text{Cos}(\omega t + \phi)\,\text{Cos}2\pi - \text{Sin}(\omega t + \phi)\,\text{Sin}2\pi]$$

$$x = x_m \text{Cos}(\omega t + \phi)$$

It shows function is repeated after time $\frac{2\pi}{\omega}$. Hence time period of periodic motion is $T = \frac{2\pi}{\omega}$. The displacement of simple harmonic motion is $x = x_m \text{Cos}(\omega t + \phi)$. It must satisfy equation of motion of simple harmonic oscillator.

$$\frac{d^2[x_m \text{Cos}(\omega t + \phi)]}{dt^2} + \frac{k}{m}[x_m \text{Cos}(\omega t + \phi)] = 0$$

$$-\omega^2[x_m \text{Cos}(\omega t + \phi)] + \frac{k}{m}[x_m \text{Cos}(\omega t + \phi)] = 0$$

$$\omega^2 = \frac{k}{m}$$

The time period of simple harmonic motion is given as

$$T = \frac{2\pi}{\omega}$$

Put value of ω

$$T = 2\pi\sqrt{\frac{m}{k}}$$

FREQUENCY OF HARMONIC OSCILLATOR

The reciprocal of time period is equal to frequency

$$f = \frac{1}{T}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

GRAPHICAL REPRESENTATION OF DISPLACEMENT, VELOCITY AND ACCELERATION OF SIMPLE HARMONIC OSCILLATOR

The displacement of simple harmonic oscillator is

$$x = x_m \text{Cos}(\omega t + \phi)$$

Differentiate with respect to time

$$V = \frac{dx}{dt}$$

$$V = -\omega x_m \text{Sin}(\omega t + \phi)$$

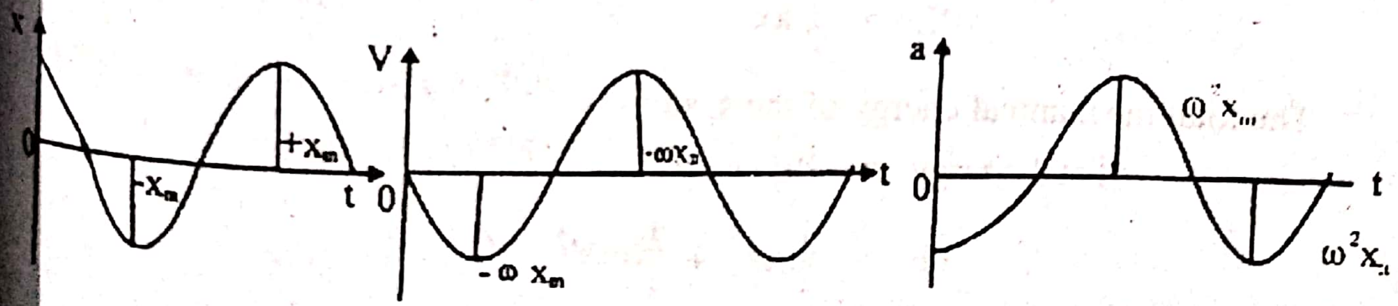
This is the velocity of simple harmonic oscillator. Now differentiate this equation again with respect to time.

$$a = \frac{dV}{dt}$$

$$a = -\omega^2 x_m \text{Cos}(\omega t + \phi)$$

$$a = -\omega^2 x$$

This is the acceleration of a simple harmonic oscillator. The graphical behavior of displacement, velocity and acceleration is shown in Fig.



The graph shows magnitude of maximum displacement of simple harmonic oscillator is x_m. The magnitude of maximum velocity is ωx_m while magnitude of maximum acceleration is $\omega^2 x_m$.

The velocity is zero while magnitude of acceleration is maximum when displacement is maximum. The acceleration is directed in opposite direction to that of displacement.

The acceleration is zero while velocity of simple harmonic oscillator is maximum when displacement is zero.

Q.2 Discuss law of conservation of energy in simple harmonic motion.

LAW OF CONSERVATION OF ENERGY

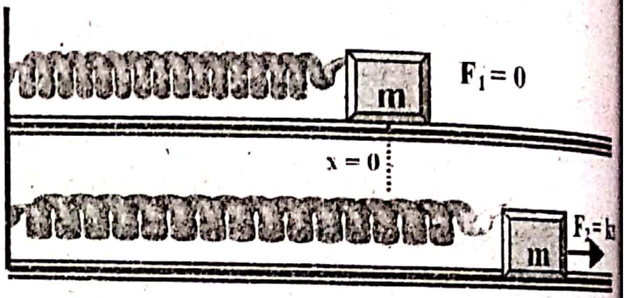
The energy can be transformed from one kind to another kind in an isolated system but it can not be created or destroyed. The total energy of such a system remains constant. This is called **law of conservation of energy.**

The total mechanical energy of simple harmonic oscillator is conserved although it changes between kinetic energy and potential energy.

ENERGY CONSERVATION IN SHM

Consider a block of mass m attached with one end of a spring. The other end of spring is attached with a support. The block is placed on horizontal smooth surface and free to move.

The rest position of block is called mean position ($x = 0$). The force acting on block at mean position is $F_1 = 0$. The block is extended through distance x when force $F_2 = kx$ is applied on block towards right.



The work done is stored as potential energy in spring given as

$$PE = \text{Average force} \times \text{distance}$$

$$PE = \left(\frac{F_1 + F_2}{2}\right) x = \left(\frac{0 + kx}{2}\right) x$$

$$PE = \frac{1}{2} kx^2$$

The total mechanical energy of the system is

$$\text{Total energy} = PE + KE$$

$$TE = \frac{1}{2} kx^2 + \frac{1}{2} mV^2$$

$$TE = \frac{1}{2} kx^2 + \frac{1}{2} m\left(\frac{dx}{dt}\right)^2$$

The displacement of simple harmonic motion is $x = x_m \text{Cos}(\omega t + \phi)$

$$TE = \frac{1}{2} k[x_m \text{Cos}(\omega t + \phi)]^2 + \frac{1}{2} m[-\omega x_m \text{ Sin}(\omega t + \phi)]^2$$

$$TE = \frac{1}{2} kx^2_m \text{ Cos}^2(\omega t + \phi) + \frac{1}{2} m\, \omega^2 x^2_m \text{ Sin}^2(\omega t + \phi)$$

Use $\omega^2 = k/m$

$$TE = \frac{1}{2}kx^2_m \, Cos^2(\omega t + \phi) + \frac{1}{2}kx^2_m \, Sin^2(\omega t + \phi)$$

$$TE = \frac{1}{2}kx^2_m \, [Cos^2(\omega t + \phi) + Sin^2(\omega t + \phi)]$$

$$TE = \frac{1}{2}kx^2_m$$

VELOCITY

It shows total kinetic energy of simple harmonic oscillator is constant. Now put value of total energy.

$$TE = \frac{1}{2}kx^2_m$$

$$TE = \frac{1}{2}k\,x^2 + \frac{1}{2}mV^2$$

On comparing

$$\frac{1}{2}k\,x^2 + \frac{1}{2}mV^2 = \frac{1}{2}kx^2_m$$

$$mV^2 = k\,x^2_m - kx^2$$

$$V = \pm\sqrt{\frac{k}{m}(x^2_m - x^2)}$$

This relation shows velocity is maximum at mean position ($x = 0$) while velocity is zero at extreme position ($x = + x_m$).

MAXIMUM & MINIMUM KE

The kinetic energy of spring mass system is

$$KE = \frac{1}{2}mV^2 = \frac{1}{2}k(x^2_m - x^2)$$

It shows kinetic energy is maximum at mean position($x = 0$) while kinetic energy is zero at extreme position($x = \pm x_m$).

MAXIMUM & MINIMUM PE

The potential energy of spring mass system is

$$PE = \frac{1}{2}kx^2$$

It shows potential energy is zero at mean position($x = 0$) while potential energy is maximum at extreme position($x = \pm x_m$).

CONCLUSION

The total energy remains constant at extreme position or mean position or at any position during simple harmonic motion although energy changes between kinetic energy and potential energy.

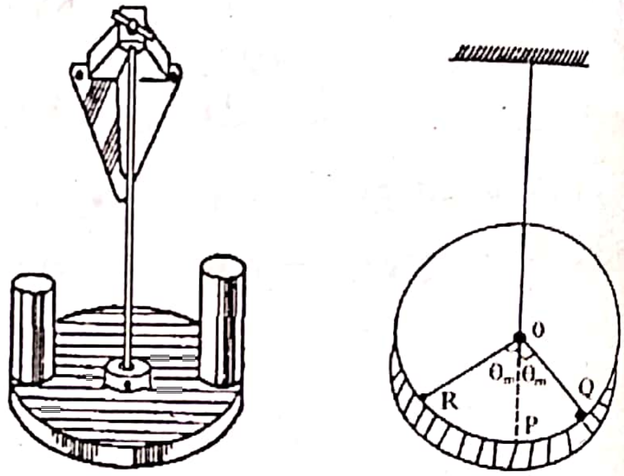
Q.3 What is torsional oscillator. Show that its motion is SHM and find its time period? (PU. 2005, 2008, 2009, 2012, SU 2013)

TORSIONAL OSCILLATOR

A mass or disc suspended from a fixed support by a thin torsion wire when twisted about its axis in horizontal plane is called **torsional oscillator or torsional pendulum.**

EQUATION OF MOTION

Consider a torsion wire whose one end is attached with a center of disc and other end is fixed to a solid support as shown in fig. The rest position of disc is called **mean position.** Now take a point P on the rim of disc and draw a radial line OP. This radial line is called **reference line.**



Rotate the disc in horizontal plane through angle θ_m such that reference line OP reaches at line OQ. The wire gets twisted and torque is produced. The line OQ on disc moves towards reference line OP when wire is allowed to untwist. The twisting of wire creates a restoring torque. The line OP oscillates between OQ and OR under this restoring torque.

The magnitude of applied torque is directly proportional to rotation angle θ.

$$\tau \propto \theta$$

$$\tau = \kappa\theta$$

Where κ is constant of proportionality called **torsional constant.** It depends upon properties of the wire. The twisted wire will exert a restoring torque on the disc which is equal in magnitude but opposite in direction to applied torque

$$\tau = -\kappa\theta \qquad \text{------(1)}$$

The negative sign means direction of torque is opposite to the direction of angular displacement.

The Newton's second law for angular motion is

$$\tau = I\alpha$$

$$\tau = I\frac{d^2\theta}{dt^2} \qquad \text{------ (2)}$$

Compare eq (1) and eq (2)

$$-\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$$

This is called equation of motion of torsional oscillator.

ANGULAR VELOCITY OF ANGULAR HARMONIC OSCILLATOR

The equation of motion of torsional oscillator is

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$$

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = -\frac{\kappa}{I}\theta$$

$$\frac{d\omega}{dt} = -\frac{\kappa}{I}\theta$$

$$\frac{d\omega}{dt}\frac{d\theta}{d\theta} = -\frac{\kappa}{I}\theta$$

$$\omega\, d\omega = -\frac{\kappa}{I}\theta\, d\theta$$

Integrating on both sides

$$\int \omega\, d\omega = -\frac{\kappa}{I}\int \theta\, d\theta$$

$$\frac{\omega^2}{2} = -\left(\frac{\kappa}{I}\right)\frac{\theta^2}{2} + C$$

$$\omega^2 = -\left(\frac{\kappa}{I}\right)\theta^2 + 2C \qquad \text{------ (3)}$$

Where C is integration constant. Its value can be determined by initial boundary conditions. The angular velocity $\omega = 0$ when $\theta = \theta_m$

$$(0)^2 = -\left(\frac{\kappa}{I}\right)\theta^2_m + 2C$$

$$2C = \left(\frac{\kappa}{I}\right)\theta^2_m$$

Put this value in eq(3)

$$\omega^2 = \left(\frac{\kappa}{I}\right)(\theta^2_m - \theta^2)$$

The velocity of torsional oscillator is

$$\omega = \pm\sqrt{\frac{\kappa}{I}}\sqrt{(\theta^2_m - \theta^2)}$$

MAXIMUM ANGULAR VELOCITY

The angular velocity of torsional oscillator is maximum at mean position$(\theta = 0)$

$$\omega_{max} = \pm\sqrt{\frac{\kappa}{I}}\sqrt{(\theta^2_m - 0)}$$

$$\omega_{max} = \pm\,\theta_m\sqrt{\frac{\kappa}{I}}$$

MINIMUM ANGULAR VELOCITY

The angular velocity of torsional oscillator is minimum at extreme position $(\theta = \theta_m)$

$$\omega_{min} = \pm\sqrt{\frac{\kappa}{I}}\sqrt{(\theta^2_m - \theta^2_m)} = 0$$

DISPLACEMENT OF TORSIONAL OSCILLATOR

The angular velocity of torsional oscillator is

$$\omega = \pm\sqrt{\frac{\kappa}{I}}\sqrt{(\theta^2_m - \theta^2)}$$

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{\kappa}{I}}\sqrt{(\theta^2_m - \theta^2)}$$

$$\frac{-d\theta}{\sqrt{(\theta^2_m - \theta^2)}} = \sqrt{\frac{\kappa}{I}}\,dt$$

Here consider only –ve part because motion is stated from extreme position.

Integrate on both sides

$$\cos^{-1}\left(\frac{\theta}{\theta_m}\right) = \sqrt{\frac{\kappa}{I}}\,t + \phi$$

Where ϕ is integration constant. Put $\omega = \sqrt{\frac{\kappa}{I}}$ called angular frequency. The term ω here is not angular velocity.

$$\frac{\theta}{\theta_m} = \cos(\omega t + \phi)$$

$$\theta = \theta_m \cos(\omega t + \phi)$$

This equation is called angular displacement of torsional oscillator. Here θ_m is maximum angular displacement called **amplitude of angular oscillation.**

TIME PERIOD OF MOTION

The disc oscillates about mean position. The total angular range is $2\theta_m$(fromOQ to OR). The time taken to complete one oscillation is called time period of torsional oscillator

$$T = \frac{2\pi}{\omega}$$

Put value of ω

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

FREQUENCY

The reciprocal of time period is equal to linear frequency

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}$$

ANGULAR VELOCITY

The angular displacement of torsional oscillator is

$$\theta = \theta_m \cos(\omega t + \phi)$$

$$\frac{d\theta}{dt} = -\omega\,\theta_m \sin(\omega t + \phi)$$

This is angular velocity of torsional oscillator at any time.

ANGULAR ACCELERATION

The angular displacement of torsional oscillator is

$$\theta = \theta_m \cos(\omega t + \phi)$$

$$\frac{d\theta}{dt} = -\omega\,\theta_m \sin(\omega t + \phi)$$

$$\frac{d^2\theta}{dt^2} = -\omega^2\,\theta_m \cos(\omega t + \phi)$$

$$\alpha = -\omega^2\,[\theta_m \cos(\omega t + \phi)]$$

$$\alpha = -\omega^2\,\theta$$

This is angular acceleration of torsional oscillator at any time during oscillation

APPLICATIONS

The torsional pendulum is used to find value of G in Cavendish balance. The torsional pendulum is used for time keeping such as balance wheel of mechanical watch.

Q.4 What is simple pendulum. Write its equation of motion and show that motion of simple pendulum is SHM. Calculate its time period and frequency? (GCUF. 2013)

SIMPLE PENDULUM

The metallic bob suspended by a weightless inextensible string is called **simple pendulum.**

LENGTH OF SIMPLE PENDULUM

The distance between point of suspension and center of bob is called **length of simple pendulum.**

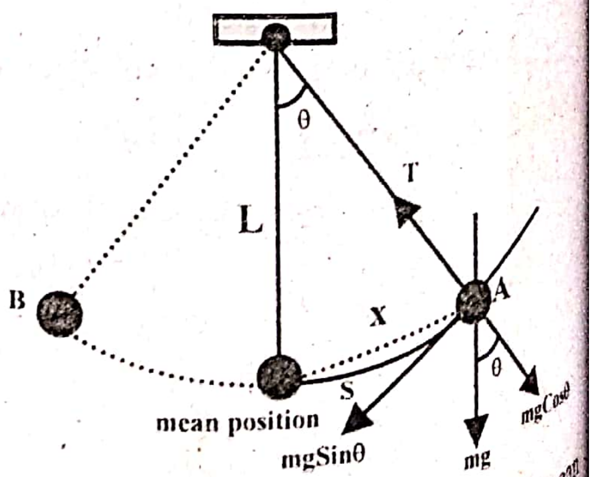
MEAN POSITION

The bob at rest when no resultant force acts on it is called **mean position or equilibrium position.**

MOTION OF BOB

Consider a bob of mass m attached with a string. The string is hanged vertically from a support as shown in fig.

Pull the pendulum from mean position to position A such that string makes a small angle θ with vertical. The bob starts moving towards mean position under restoring force when released. It gets maximum velocity at mean position and does not stop due to inertia but continues to move towards extreme position B. The velocity of bob becomes zero at position B due to restoring force.



It means velocity of bob go on decreasing from mean position to position B so acceleration is directed from position B to mean position.

Now bob moves from position B towards mean position. The velocity go on increasing from B towards mean position and becomes maximum at mean position. The acceleration is again directed towards mean position. In this way bob continues its to and fro motion about mean position. Hence motion of simple pendulum is SHM.

EQUATION OF MOTION OF SIMPLE PENDULUM

The path followed by bob when it moves from mean position to position A is called an arc of circle having radius L. The arc length S and chord length x are approximately equal for small angle.

The forces acting on bob when it is at position A are

1. Weight of bob acting vertically downward
2. Tension acting along the string

RESOLUTION OF WEIGHT FORCE

Resolve weight force into components $mg\cos\theta$ and $mg\sin\theta$

The component $mg\cos\theta$ is called **radial component** and component $mg\sin\theta$ is called **tangential component**.

The radial component of weight is balanced by the tension force

$$T = mg\cos\theta$$

The tangential component provides restoring force to the bob for oscillations

$$F = -mg\sin\theta$$

The negative sign means direction of F is opposite to direction of increasing θ.

For small amplitudes $\sin\theta \approx \theta$

$$F = -mg\theta \qquad \text{------ (1)}$$

The Newton's second law of motion is

$$F = ma \qquad \text{------ (2)}$$

Compare eq (1) and eq (2)

$$ma = -mg\theta$$

$$a = -g\,\theta$$

The relation $S = r\,\theta$ for circular path gives $S = L\,\theta$

$$a = -g\left(\frac{S}{L}\right)$$

For small angles, arc length = chord length i.e. $S = x$

$$a = -g\left(\frac{x}{L}\right)$$

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{L}\right)x$$

$$\frac{d^2x}{dt^2} + g\frac{x}{L} = 0$$

It is called equation of motion of simple pendulum. The solution of eq of motion simple pendulum or simple harmonic oscillator is

$$x = x_m\cos(\omega t + \phi)$$

This is called displacement of bob measured from mean position at any time during motion. Here x_m is amplitude and ϕ is phase constant. This equation shows the motion of simple pendulum is SHM.

This solution must satisfy the eq of motion of simple harmonic oscillator

$$\frac{d^2[x_m \text{Cos}(\omega t + \phi)]}{dt^2} + \frac{g}{L}[x_m \text{Cos}(\omega t + \phi) = 0$$

$$-\omega^2 [x_m \text{Cos}(\omega t + \phi)] + \frac{g}{L}[x_m \text{Cos}(\omega t + \phi)] = 0$$

$$\omega^2 = \frac{g}{L}$$

$$\omega = \sqrt{\frac{g}{L}}$$

TIME PERIOD OF MOTION

The time taken by bob to complete one vibration is called **time period of simple pendulum.**

$$T = \frac{2\pi}{\omega}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

The time period of simple pendulum is independent of mass of bob and amplitude. The time period depends upon length of simple pendulum and value of g. The pendulum having time period 2s is called **second pendulum.**

The formula for time period of motion of simple pendulum when amplitude of oscillation is not small is

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{2^2}\text{Sin}^2\frac{\theta_m}{2} + \frac{1}{2^2}\frac{3^2}{4^2}\text{Sin}^4\frac{\theta_m}{2} + \dots\dots\dots\right]$$

This shows time period increases with increase in amplitude.

FREQUENCY

The reciprocal of time period is called frequency. Its SI unit is called hertz(Hz)

$$f = \frac{1}{T}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

The frequency of second pendulum is 0.5 Hz.

APPLICATIONS

It is used as timekeeper. It is used to find value of g.

Q.5 What is physical pendulum. Derive its equation of motion and find its time period. Also find the length of the equivalent simple pendulum?

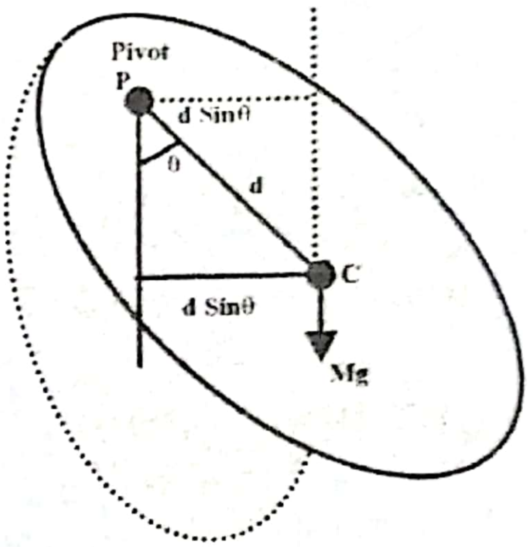
(PU. 2002, 2009, 2011, 2013, Su 2011, 2012, 2013, GCUF. 2012, 2013).

PHYSICAL PENDULUM

A rigid body which swings freely about some pivot point is called **physical pendulum**. All real pendulums are called physical pendulums.

EQUATION OF MOTION

Consider a rigid body of mass M having arbitrary shape is pivoted about an horizontal axis at point P as shown in fig. The position of body when center of mass C is below point P is called equilibrium position or mean position. The distance between pivot point P and center of mass C is d.



B Now pull the physical pendulum from mean position to new position such that line CP makes a small angle θ with vertical. The bob starts moving towards mean position under restoring torque when released. It gets maximum velocity at mean position and does not stop due to inertia but continues to move to and fro about mean position

The restoring torque is given as

$$\tau = -F\,d\,\text{Sin}\theta$$

$$\tau = -Mg\,d\,\text{Sin}\theta$$

For small angular displacements $\theta \approx \text{Sin}\theta$ ------(1)

$$\tau = -(Mgd)\,\theta$$

The general formula of torque is ------(2)

$$\tau = I\alpha$$

Compare eq (1) and eq (2)

$$I\alpha = -(Mgd)\,\theta$$

$$I\frac{d^2\theta}{dt^2} = -(Mgd)\,\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{Mg\,d}{I}\theta = 0$$

Put $Mgd = \kappa$

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$$

This is called **equation of motion of physical pendulum.** The solution of equation of motion of physical pendulum is

$$\theta = \theta_m \cos(\omega t + \phi)$$

It is angular displacement of physical pendulum at any time t from mean position. Where θ_m is maximum angular displacement called amplitude of angular oscillation while ω is called angular frequency.

This solution must satisfy the eq of motion of physical pendulum.

$$\frac{d^2[\theta_m \cos(\omega t + \phi)]}{dt^2} + \frac{\kappa}{I}[\theta_m \cos(\omega t + \phi)] = 0$$

$$-\omega^2[\theta_m \cos(\omega t + \phi)] + \frac{\kappa}{I}[\theta_m \cos(\omega t + \phi)] = 0$$

$$\omega^2 = \frac{\kappa}{I}$$

$$\omega = \sqrt{\frac{\kappa}{I}}$$

TIME PERIOD OF MOTION

The time taken by physical pendulum to complete one oscillation about mean position is called **time period of motion of physical pendulum**

$$T = \frac{2\pi}{\omega}$$

Put value of ω

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

$$T = 2\pi\sqrt{\frac{I}{Mgd}} \quad \text{------(3)}$$

ROTATIONAL INERTIA

The time period of motion of a physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

Squaring on both sides

$$T^2 = 4\pi^2\left(\frac{I}{Mgd}\right)$$

$$I = \frac{MgdT^2}{4\pi^2}$$

This is rotational inertia of a physical pendulum. The rotational inertia of a physical pendulum about an axis of rotation can be determined by above formula because all parameters on right side are measurable.

SIMPLE PENDULUM AS SPECIAL CASE

The simple pendulum is a special case of physical pendulum. Assume that pivot point of physical pendulum is a point of suspension for simple pendulum and center of mass of physical pendulum is mass of the bob of simple pendulum.

The distance between pivot point and center of mass becomes length of weightless string of simple pendulum given as

$$d = L$$

The moment of inertia of physical pendulum takes the form

$$I = Md^2$$

$$I = ML^2$$

The time period of physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

Under above stated conditions

$$T = 2\pi\sqrt{\frac{ML^2}{MgL}}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

This is the time period of simple pendulum. Hence simple pendulum is a special case of physical pendulum.

CENTER OF OSCILLATION

The resulting simple pendulum will have the same time period as the physical pendulum when mass of physical pendulum is concentrated at proper length L from the pivot point.

Time period of physical pendulum = Time period of simple pendulum

$$2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$\frac{L}{g} = \frac{I}{Mgd}$$

$$L = \frac{I}{Md}$$

The point lying at distance L = I/Md from pivot point is called center of oscillation of the physical pendulum.

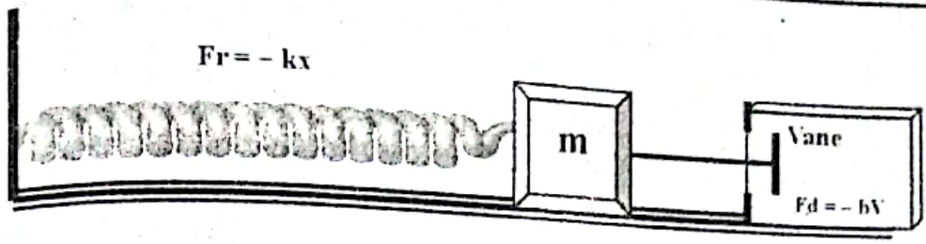
Q.7 What is damped harmonic oscillator? Write its equation of motion and find its solution, also calculate its amplitude and frequency. (GCUF. 2013)

DAMPED HARMONIC OSCILLATOR

The oscillator which moves in a resistive medium under restoring force is called **damped harmonic oscillator.** The amplitude of damped harmonic oscillator gradually goes on decreasing and finally becomes zero when it moves to and fro in a resistive medium. The energy is dissipated during damped harmonic motion.

EQUATION OF MOTION

Consider a block of mass m whose one end is connected with a spring and other end is connected with a massless vane. The free end of spring is attached to a fixed support while vane is immersed in a resistive fluid. The block is placed on frictionless horizontal surface as shown in fig.



Now displace the block towards right through some displacement and release. The block along with vane moves to and fro under spring force or restoring force.

The restoring force or spring force acting on block is

$$F_r = -k\,x$$

The damping force experienced by vane when it moves in resistive medium is given as

$$F_d = -bV$$

Where b is constant ofproportionality. It depends upon properties of resistive medium such as density and shape and dimensions of vane.

The net force acting on the system is

$$\text{Net force} = F_r + F_d$$

$$\text{Net force} = -k\,x - bV$$

The acceleration produced by net force acting on the system is given by Newton's second law of motion as

$$\text{Net force} = \text{mass (acceleration)}$$

$$-k\,x - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

This is called **equation of motion of damped harmonic oscillator.**

SOLUTION OF EQUATION OF MOTION

The equation of motion of damped harmonic oscillator is

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Put $\frac{b}{m} = 2\beta$ and $\frac{k}{m} = \omega^2$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = 0 \qquad \text{------ (1)}$$

Assume that solution of this equation is $x = Ae^{\mu t}$. It must satisfy the equation of motion.

$$\frac{d^2(Ae^{\mu t})}{dt^2} + 2\beta\frac{d\,(Ae^{\mu t})}{dt} + \omega^2\,(Ae^{\mu t}) = 0$$

$$\mu^2\,(Ae^{\mu t}) + 2\beta\mu(Ae^{\mu t}) + \omega^2\,(Ae^{\mu t}) = 0$$

$$\mu^2 + 2\beta\mu + \omega^2 = 0$$

The roots of this quadratic equation are

$$\mu = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega^2}}{2}$$

$$\mu = -\beta \pm \sqrt{\beta^2 - \omega^2}$$

$$\mu_1 = -\beta + \sqrt{\beta^2 - \omega^2} \quad \text{and} \quad \mu_2 = -\beta - \sqrt{\beta^2 - \omega^2}$$

The damped oscillatory motion is only possible when $\omega^2 > \beta^2$. It means spri force is greater than damping force. Under this condition

$$\sqrt{\beta^2 - \omega^2} = \sqrt{(-1)(\omega^2 - \beta^2)} = i\sqrt{\omega^2 - \beta^2} = i\,\omega'$$

Where

$$\omega' = \sqrt{\omega^2 - \beta^2}$$

Now values of μ_1 and μ_2 are

$$\mu_1 = -\beta + i\omega' \quad \text{and} \quad \mu_2 = -\beta - i\omega'$$

The corresponding solution of equation of motion of damped harmonic oscillat is

$$x = x_1 + x_2$$

$$x = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} \qquad \text{------(2)}$$

$$x = A_1 e^{(-\beta + i\omega')t} + A_2 e^{(-\beta - i\omega')t}$$

$$x = e^{-\beta t}[A_1 e^{i\omega' t} + A_2 e^{-i\omega' t}]$$

$$x = e^{-\beta t}[A_1(\text{Cos}\omega' t + i\,\text{Sin}\omega' t) + A_2\,(\text{Cos}\omega' t - i\,\text{Sin}\omega' t)]$$

$$x = e^{-\beta t}[(A_1 + A_2)\,\text{Cos}\omega' t + (A_1 - A_2)\,i\,\text{Sin}\omega' t]$$

The displacement x of damped harmonic oscillator must be real. Therefore, righ side of above eq must be real. This is only possible when A_1 and A_2 are comple conjugate of each other. i.e. $A_1 = a + ib$ & $A_2 = a - ib$

Hence

$$A_1 + A_2 = 2a \quad \text{and} \quad A_1 - A_2 = 2\,ib$$

The above eq under this condition becomes

$$x = e^{-\beta t}\,[2a\,\text{Cos}\omega' t + (2ib)i\,\text{Sin}\omega' t]$$

$$x = e^{-\beta t}\,[2a\,\text{Cos}\omega' t - 2\,b\,\text{Sin}\omega' t]$$

Put

$$2a = x_m \text{Cos}\phi \quad \text{------(A)}$$

$$2b = x_m \text{Sin}\phi \quad \text{------(B)}$$

$$x = e^{-\beta t}[x_m \text{Cos}\phi \text{Cos}\omega' t - x_m \text{Sin}\phi \text{Sin}\omega' t]$$

$$x = x_m e^{-\beta t} \text{Cos}(\omega' t + \phi)$$

$$x = x_m e^{-\frac{bt}{2m}} \text{Cos}(\omega' t + \phi)$$

This is solution of equation of motion of damped harmonic oscillator and called **displacement of damped harmonic oscillator.** Where x_m is called amplitude of motion without damping. Square eq (A) and eq (B) and add to find value x_m.

$$x_m = 2\sqrt{a^2 + b^2}$$

The term ϕ is called phase constant. Divide eq(B) and eq(A) to find its value

$$\frac{x_m \text{Sin}\phi}{x_m \text{Cos}\phi} = \frac{2b}{2a}$$

$$\text{Tan}\phi = \frac{b}{a}$$

$$\phi = \text{Tan}^{-1}\left(\frac{b}{a}\right)$$

The term ω` is called angular frequency of damped harmonic oscillator. Its value is given as

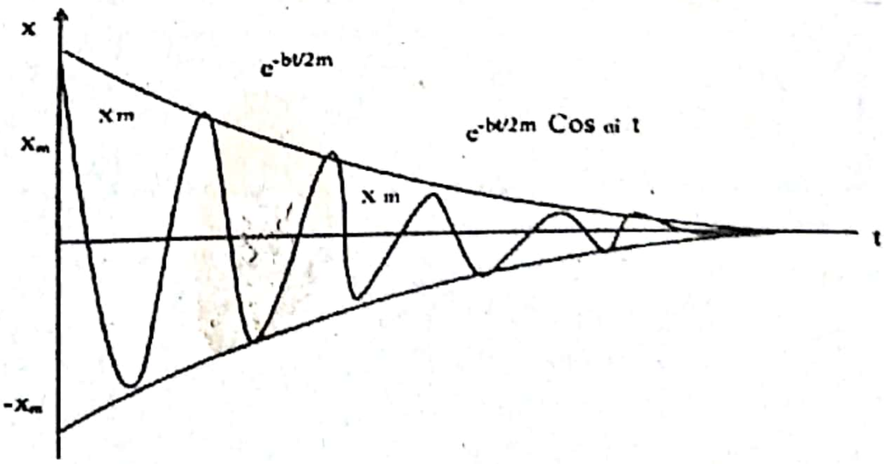
$$\omega' = \sqrt{\omega^2 - \beta^2} = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

DISPLACEMENT OF DAMPED HARMINC OSCILLATOR

The displacement of damped harmonic oscillator is

$$x = x_m e^{-\frac{bt}{2m}} \text{Cos}(\omega' t + \phi)$$

The graphical behavior of displacement of damped harmonic oscillator is shown in diagram.



FEATURES OF DAMPED OSCILLATIONS

The damped oscillations have two major features called frequency of oscillation and amplitude of motion.

FREQUENCY OF OSCILLATION

The angular frequency of damped harmonic oscillator is

$$\omega' = \sqrt{\omega^2 - \beta^2} = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

The frequency of damped oscillations ω` is smaller than frequency of undamped oscillations when resistive friction is present. It means resistive friction slows down oscillations.

The term b = 0 when resistive friction is absent. In this case frequency of damped oscillations is equal to frequency ω of undamped oscillations.

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\frac{k}{m} - \left(\frac{0}{2m}\right)^2}$$

$$\omega' = \omega$$

AMPLITUDE OF OSCILLATIONS

The amplitude of damped harmonic oscillator is

$$\text{Amplitude} = x_m e^{-\frac{bt}{2m}}$$

It is exponentially decreasing function of time and becomes zero after infinite time. The amplitude of damped oscillations is equal to amplitude of undamped oscillations when resistive friction is absent(b = 0)

$$\text{Amplitude} = x_m e^0$$

$$\text{Amplitude} = x_m$$

MEAN LIFE TIME OF OSCILLATIONS

The time interval(τ) in which amplitude of damped harmonic oscillator drops to $\frac{1}{e}$ times of its initial value is called mean life time of damped oscillations.

$$\text{Amplitude of damped oscillation} = \frac{1}{e}\text{(Initial amplitude)}$$

$$x_m e^{-\frac{b\tau}{2m}} = \frac{1}{e} x_m$$

$$e = e^{\frac{b\tau}{2m}}$$

$$\text{Ln}(e) = \frac{b\tau}{2m}\text{Ln}(e)$$

$$\tau = \frac{2m}{b}$$

The mean life time of damped oscillations depends upon mass of block and properties of resistive medium.

TICAL DAMPING

The condition when mean life time of damped oscillations has its smallest value is d critical **damping.**

The damped oscillatory motion is only possible when spring force is greater than ping force.

$$\omega^2 \geq \beta^2$$

$$\frac{k}{m} \geq \left(\frac{b}{2m}\right)^2$$

$$\frac{k}{m} \geq \frac{b^2}{4m^2}$$

$$b \geq 2\sqrt{km}$$

The frequency ω` of damped oscillations reduces to zero when $b = 2\sqrt{km}$.

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{2\sqrt{km}}{2m}\right)^2} = 0$$

Hence displacement of damped oscillations becomes zero. The value of minimum time corresponding to critical damping is given as

$$\tau = \frac{2m}{b} = \frac{2m}{2\sqrt{km}}$$

$$\tau = \sqrt{\frac{m}{k}} = \omega^{-1}$$

The critical damping is important for mechanical engineers in designing such stems having shortest possible time.