CARTESIAN TENSORS

NTRODUCTION

Tensor analysis may be regarded as a generalization of vector analysis . It is of great value to a
r an engineer in two ways . First , it allows complex mathematical and physical relationships to
sed in a compact way and simplifies the mechanics of the development of theory . Second , it
reater understanding to vector notations and also establishes certain invariance properties with
and simplicity . It is of great use in mechanics , fluid dynamics , elasticity , differential geometry ,
gnetic theory , general relativity theory , and numerous other fields of science and. engineering .
apter , we shall discuss Cartesian tensors i.e. tensors which are expressed in terms of components
o rectangular Cartesian coordinate systems . At first sight , the notation of Cartesian tensors is
t complicated . The aim of this chapter is to provide a familiarity with the notation which will
e reader to study other texts and applications without difficulty .

e start the actual studies of Cartesian tensors , it will be important to give some basic ideas
s , definitions , transformations , etc.) which are useful in the study of Cartesian tensors .

SUMMATION CONVENTION

Consider an expression $a_1 x_1 + a_2 x_2 + a_3 x_3$ (1)

which can be written using summation sign as $\sum_{j=1}^{3} a_j x_j$ (2)

nit the summation sign and write it simply as $a_j x_j$ (3)

s understood that the repeated index (or suffix) j represents the summation from 1 to 3 .
t the form (3) is much more convenient than the original form (1) . This situation occurs so
y that it is convenient to adopt a convention which avoids the necessity of writing summation
his convention known as the summation convention is as follows :

er a suffix appears twice in the same expression that expression is to be summed over all
the suffix namely , 1 , 2 , 3 .

DUMMY AND FREE INDICES

An index which is repeated in a given expression so that the summation convention applies , is
dummy index , while an index occurring only once in a given expression is called a free index and
imply any summation. For example, in the expression $A_k B_{jk}$, k is dummy index while j is a
x .

XAMPLE (1): Write each of the following using summation convention .

(i) $a_{11} x_1 + a_{12} x_2 + a_{13} x_3$ (ii) $a_{11} b_{11} + a_{12} b_{12} + a_{13} b_{13}$

(iii) $(x_1)^2 + (x_2)^2 + (x_3)^2$ (iv) $\frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3}$

OLUTION: We have

) $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = a_{1i} x_i$

i) $a_{11} b_{11} + a_{12} b_{12} + a_{13} b_{13} = a_{1i} b_{1i}$

ii) $(x_1)^2 + (x_2)^2 + (x_3)^2 = x_1 x_1 + x_2 x_2 + x_3 x_3 = x_i x_i$

v) $\frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 = \frac{\partial \phi}{\partial x_i} dx_i$

XAMPLE (2): Write out explicitly the following summations and compare the results :

(i) $a_i (x_i + y_i)$ (ii) $a_j x_j + a_k y_k$

OLUTION: We have

) $a_i(x_i + y_i) = a_1(x_1 + y_1) + a_2(x_2 + y_2) + a_3(x_3 + y_3)$

$= a_1 x_1 + a_1 y_1 + a_2 x_2 + a_2 y_2 + a_3 x_3 + a_3 y_3$

i) $a_j x_j + a_k y_k = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_1 y_1 + a_2 y_2 + a_3 y_3$

he two summations are identical except for the order in which the terms occur .

OTE: (i) A repeated suffix may be replaced by any other suitable symbol not already in use . xample , $a_j b_j = a_k b_k = a_\alpha b_\alpha$ since in each expression summation over the repeated suffix is implie

i) No suffix may occur more than twice in an expression . For example , $a_{ii} x_i$ is ambigu ecause of the differences in the three quantities :

$a_{ij} x_j = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3$

$a_{ii} x_j = (a_{11} + a_{22} + a_{33}) x_j$

$a_{ij} x_i = a_{1j} x_1 + a_{2j} x_2 + a_{3j} x_3$

n putting $j = i$ in these equations , we obtain entirely different expressions on the R.H.S.

i) An expression of the form $a_i (x_i + y_i)$ is considered well – defined , for it is obtained mposition of the meaningful expressions $a_i z_i$ and $x_i + y_i = z_i$. In other words , the index i garded as occurring once in the term $(x_i + y_i)$.

3 DOUBLE SUMS

An expression can involve more than one summation indices . For example , $a_{ij} x_i x_j$ indicates nmation taking place on both i and j simultaneously . If an expression has two summation (dumm lices , there will be a total of 3^2 terms in the sum ; if there are three indices , there will be 3^3 terms d so on .

EXAMPLE (3): Write the terms in the expression $a_{ij}x_ix_j;\ i,j = 1,2,3$.

SOLUTION: The given expression represents the double sum and has 9 terms. It can be written logically by first summing over i, and then over j. Since i varies from 1 to 3, holding j fixed, the given expression is the sum of three terms. That is

$$a_{ij}x_ix_j = a_{1j}x_1x_j + a_{2j}x_2x_j + a_{3j}x_3x_j$$

Now each term on the R.H.S. has the repeated index j which implies summation. Hence

$$a_{ij}x_ix_j = a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x_2x_2 + a_{23}x_2x_3$$
$$+ a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3x_3$$

The result is the same if one sums over j first, and then over i.

EXAMPLE (4): Write the following expression using summation convention.

$$a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13} + a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23} + a_{13}b_{31} + \ldots$$

SOLUTION: The given expression can be written as

$$(a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13}) + (a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23}) + (a_{13}b_{31} + a_{23}b_{32} + \ldots$$
$$= a_{i1}b_{1i} + a_{i2}b_{2i} + a_{i3}b_{3i} = a_{ij}b_{ji}$$

7.4 SUBSTITUTIONS

Suppose it is required to substitute $y_i = a_{ij}x_j$ in the equation $Q = b_{ij}y_ix_j$ … substitution would lead to an absurd expression like $Q = b_{ij}a_{ij}x_jx_j$.

The correct procedure is first to identify any dummy indices in the expression to be substituted … coincide with indices occurring in the main expression. Changing these dummy indices to … found in the main expression, one may then carry out the substitution in the usual manner …

Step (1) $y_i = a_{ij}x_j,\ Q = b_{ij}y_ix_j$. We see that the dummy index j is duplicated.

Step (2) Change the dummy index from j to r, to get $y_i = a_{ir}x_r$.

Step (3) Substitute and rearrange to get $Q = b_{ij}(a_{ir}x_r)x_j = a_{ir}b_{ij}x_rx_j$

EXAMPLE (5): If $y_i = a_{ij}x_j$, express the quadratic form $Q = g_{ij}y_iy_j$ … x-variables.

SOLUTION: First write $y_i = a_{ir}x_r,\quad y_j = a_{js}x_s$

Then by substitution, $Q = g_{ij}(a_{ir}x_r)(a_{js}x_s) = g_{ij}a_{ir}a_{js}x_rx_s$

7.5 ALGEBRA AND THE SUMMATION CONVENTION

Certain routine algebraic manipulations in tensors can be easily justified by properties of sums. However, some care should be taken. The following are several valid identities … repeatedly from now on.

(1) $a_{ij}(x_j + y_j) \equiv a_{ij}x_j + a_{ij}y_j$ (2) $a_{ij}x_iy_j \equiv a_{ij}y_jx_i$

(3) $a_{ij}x_ix_j \equiv a_{ji}x_ix_j$ (4) $(a_{ij} + a_{ji})x_ix_j \equiv 2a_{ij}x_ix_j$

(5) $(a_{ij} - a$ …

The following non-identities should be noted:

(2) $a_{ij}x_iy_j \not\equiv a_{ij}y_ix_j$

$a_{ij}(x_i+y_j) \not\equiv a_{ij}x_i + a_{ij}y_j$

$(a_{ij}+a_{ji})x_iy_j \not\equiv 2a_{ij}x_iy_j$

Problem. Show that, generally, $a_{ijk}(x_i+y_j)z_k \neq a_{ijk}x_iz_k + a_{ijk}y_jz_k$.

Solution. Simply observe that on the left side there are no free indices, but on the right, j is free for the first term and i is free for the second.

KRONECKER DELTA δ_{ij}

The Kronecker delta or substitution operator written δ_{ij}, is defined as $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\delta_{11}=\delta_{22}=\delta_{33}=1$, and $\delta_{12}=\delta_{21}=\delta_{23}=\delta_{32}=\delta_{31}=\delta_{13}=0$

Problem. Using the definition of Kronecker delta, calculate $\delta_{ij}x_ix_j$.

Solution. We have $\delta_{ij}x_ix_j = \delta_{1j}x_1x_j + \delta_{2j}x_2x_j + \delta_{3j}x_3x_j$

$= \delta_{11}x_1x_1 + \delta_{12}x_1x_2 + \delta_{13}x_1x_3 + \delta_{21}x_2x_1 + \delta_{22}x_2x_2 + \delta_{23}x_2x_3$
$\qquad + \delta_{31}x_3x_1 + \delta_{32}x_3x_2 + \delta_{33}x_3x_3$

$= 1x_1x_1 + 0x_1x_2 + 0x_1x_3 + 0x_2x_1 + 1x_2x_2 + 0x_2x_3 + 0x_3x_1 + 0x_3x_2 + 1x_3x_3$

$= x_1x_1 + x_2x_2 + x_3x_3 = x_ix_i$

Problem. Show that if x_1, x_2, x_3 are independent variables, then $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$

Solution. We have, if $i = j$, $\dfrac{\partial x_i}{\partial x_j} = \dfrac{\partial x_i}{\partial x_i} = 1$

$\dfrac{\partial x_i}{\partial x_j} = 0$ since x_i and x_j are independent variables.

$\therefore \dfrac{\partial x_i}{\partial x_j} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Problem. Prove that $\delta_{ij}A_j = A_i$.

Solution. We know that the index j represents summation, therefore

$\delta_{ij}A_j = \delta_{i1}A_1 + \delta_{i2}A_2 + \delta_{i3}A_3 \quad (i = 1, 2, 3)$

$\delta_{1j}A_j = \delta_{11}A_1 + \delta_{12}A_2 + \delta_{13}A_3 = A_1$, and

$\delta_{2j}A_j = \delta_{21}A_1 + \delta_{22}A_2 + \delta_{23}A_3 = A_2$, and

$\delta_{3j}A_j = \delta_{31}A_1 + \delta_{32}A_2 + \delta_{33}A_3 = A_3$.

$\delta_{ij}A_j = A_i$.

THEOREM (7.3): Prove that $\delta_{ik}\delta_{jk} = \delta_{ij}$.

PROOF: We have $\delta_{ik}\delta_{jk} = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} + \delta_{i3}\delta_{j3}$ $(i, j = 1, 2, 3)$

when $j = 1$, $\delta_{ik}\delta_{1k} = \delta_{i1}$ and

when $j = 2$, $\delta_{ik}\delta_{2k} = \delta_{i2}$ and

when $j = 3$, $\delta_{ik}\delta_{3k} = \delta_{i3}$

It therefore follows that $\delta_{ik}\delta_{jk} = \delta_{ij}$.

EXAMPLE (8): Show that

(i) $\delta_{ii} = 3$ (ii) $\delta_{ik}\delta_{ik} = 3$

(iii) $\delta_{ij}\delta_{jk}\delta_{ki} = 3$ (iv) $\delta_{ij}\delta_{k\ell}A_{ik} = A_{j\ell}$

(v) $\delta_{ij}\delta_{jk}A_{ik} = A_{ii}$

SOLUTION: We have (i) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

(ii) $\delta_{ik}\delta_{ik} = \delta_{1k}\delta_{1k} + \delta_{2k}\delta_{2k} + \delta_{3k}\delta_{3k}$, $k = 1, 2, 3$

$$= (\delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13}) + (\delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23}) + (\delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33})$$

$$= \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = (1)(1) + (1)(1) + (1)(1) = 1 + 1 + 1 = 3$$

(iii) $\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ik}\delta_{ki} = \delta_{ii} = 3$

(iv) $\delta_{ij}\delta_{k\ell}A_{ik} = \delta_{ij}A_{i\ell} = A_{j\ell}$

(v) $\delta_{ij}\delta_{jk}A_{ik} = \delta_{ik}A_{ik} = A_{ii}$.

7.7 RECTANGULAR COORDINATE SYSTEM

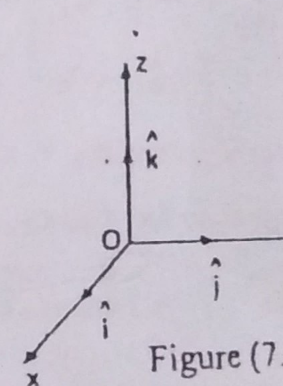


Figure (7.1)

From vector analysis, we are familiar with the rectangular coordinate system in which we take Ox, Oy, Oz as the coordinate axes and $\hat{i}, \hat{j}, \hat{k}$ the unit vectors along these coordinate axes respectively, as shown in figure (7.1).

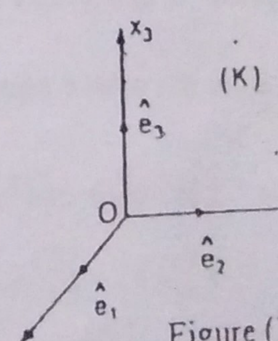


Figure (7.2)

In tensor analysis, in stead of this system we take the system in which we have Ox_1, Ox_2, Ox_3 as the coordinate axes and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ the unit vectors along these axes respectively as shown in figure (7.2). This system of coordinate axes will be denoted by K.

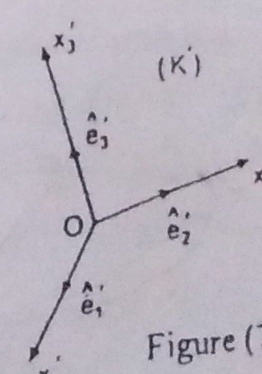


Figure (7.3)

In addition to the system K we need another coordinate system which will be denoted by K'. In the system K', we take Ox'_1, Ox'_2, Ox'_3 as the coordinate axes and $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ the unit vectors along these axes respectively as

THEOREM (7.4): Prove that $\hat{e}_i . \hat{e}_j = \delta_{ij}$.

PROOF: From vector analysis, we know that when $j = i$, $\hat{e}_i . \hat{e}_j = \hat{e}_i . \hat{e}_i = 1$, and when $j \neq i$, $\hat{e}_i . \hat{e}_j = 0$.

Therefore $\hat{e}_i . \hat{e}_j = \delta_{ij}$.

NOTE: Similarly, we can prove $\hat{e}_i' . \hat{e}_j' = \delta_{ij}$.

7.8 DIRECTION COSINES

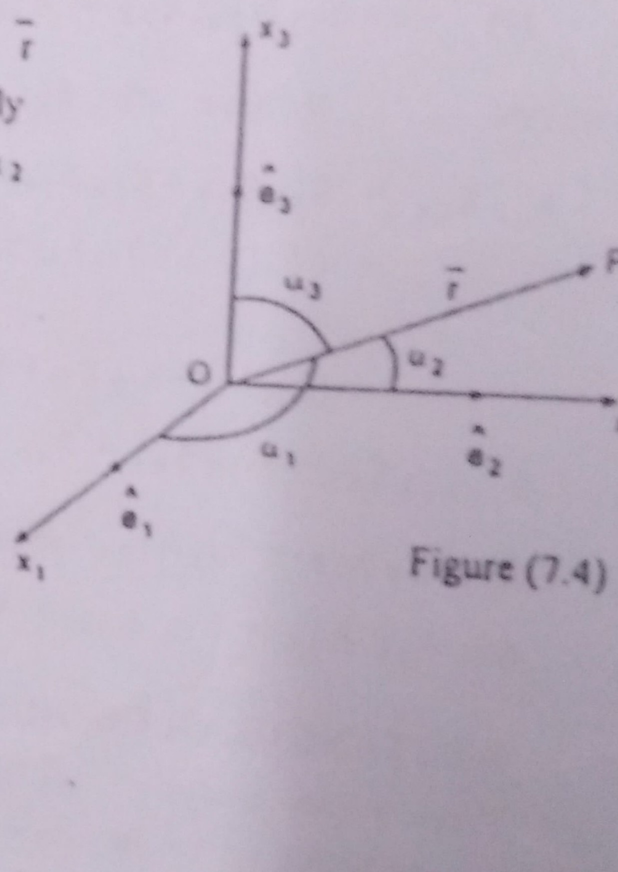


Figure (7.4)

Let $\alpha_1, \alpha_2, \alpha_3$ be the angles which the position vector $\bar{r}$ makes with the positive directions of Ox_1, Ox_2, Ox_3 respectively as shown in figure (7.4). Then the three quantities $\cos\alpha_1, \cos\alpha_2$ $\cos\alpha_3$ are called the direction cosines of the vector $\bar{r}$.

For convenience, we write

$$\ell_1 = \cos\alpha_1, \; \ell_2 = \cos\alpha_2, \; \ell_3 = \cos\alpha_3$$

Now $\bar{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$ (1)

nd $\bar{r} . \hat{e}_1 = x_1$ or $r\cos\alpha_1 = x_1$

imilarly, $r\cos\alpha_2 = x_2$, and $r\cos\alpha_3 = x_3$

ubstitution for x_1, x_2, and x_3 in equation (1), gives

$$\bar{r} = r\cos\alpha_1\hat{e}_1 + r\cos\alpha_2\hat{e}_2 + r\cos\alpha_3\hat{e}_3$$

herefore, a unit vector in the direction of $\bar{r}$ is $\hat{r} = \frac{\bar{r}}{r} = \cos\alpha_1\hat{e}_1 + \cos\alpha_2\hat{e}_2 + \cos\alpha_3\hat{e}_3$

so from the above equations, $\cos\alpha_1 = \frac{x_1}{r}$, $\cos\alpha_2 = \frac{x_2}{r}$, and $\cos\alpha_3 = \frac{x_3}{r}$.

nce $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} = \frac{r^2}{r^2} = 1$ (since $x_1^2 + x_2^2 + x_3^2 = r^2$)

$$\ell_1^2 + \ell_2^2 + \ell_3^2 = 1$$

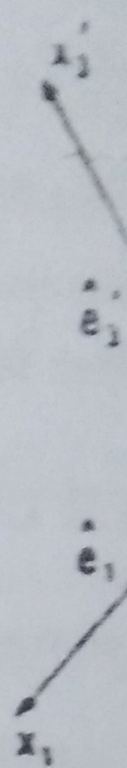
is we have shown that the unit vector in the direction of $\bar{r}$ has its components the direction cosines $\bar{r}$ and that the sum of the squares of the direction cosines is unity.

TE: The direction cosines of the x_1 – axis are $1, 0, 0$. Similarly, the direction cosines of x_2 – axis $0, 1, 0$ and that of x_3 – axes are $0, 0, 1$.

DEFINITION OF ℓ_{ij}

We define ℓ_{ij} to be the cosine of the angle between the ith – axis of the

ORTHOGONAL ROTATION OF AXES



Consider two right–handed rectangular coordinate systems $_1 x_2 x_3$ and $Ox'_1 x'_2 x'_3$ (i.e. systems K and K') having the me origin O and with unit vectors along the coordinate axes , $\hat{e}_2, \hat{e}_3$ and $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ as shown in figure (7.5). Rotate the stem $Ox_1 x_2 x_3$ about O (with Ox_1, Ox_2, Ox_3 always fixed lative to each other) so that it coincides with the system $Ox'_1 x'_2 x'_3$. ch a movement is called a rotation of axes .

Let the direction cosines of Ox'_1 relative to the axes x_1, Ox_2, Ox_3 be $\ell_{11}, \ell_{12}, \ell_{13}$ respectively . Furthermore , note the direction cosines of Ox'_2 and Ox'_3 by $\ell_{21}, \ell_{22}, \ell_{23}$ and $_1, \ell_{32}, \ell_{33}$, respectively . We may conveniently summarize this in e adjacent table . In this table , the direction cosines of Ox'_1 lative to the axes Ox_1, Ox_2, Ox_3 occur in the first row , the rection cosines of Ox'_2 occur in the second row , and those of x'_3 in the third row . Furthermore , reading down the three columns turn , it is seen that we obtain the direction cosines of the axes x_1, Ox_2, Ox_3 relative to the axes Ox'_1, Ox'_2, Ox'_3 respectively . he table of direction cosines is called the transformation matrix and is written

Ox'_1
Ox'_2
Ox'_3

$$T = [\ell_{ij}] = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}$$

he transpose of T is $T' = [\ell_{ji}] = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix}$

nd $TT' = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ldots \end{bmatrix}$

TT' is the 3 x 3 unit matrix 1, its principle diagonal elements being unity and all … 0. This means that the transposed matrix T' is the inverse of T and so, as in r… is an orthogonal matrix. Since T, T' have the same determinants, the relation TT' … erminants, gives $(\det T)^2 = 1$ so that $\det T = \pm 1$.

…e case of an orthogonal rotation of rectangular coordinate axes as shown in figure (7.5 … $\times \hat{e}_2 = \hat{e}_3$ and also $\hat{e}_1' \times \hat{e}_2' = \hat{e}_3'$ meaning that a right – handed system Ox_1x… – handed when rotated to $Ox_1'x_2'x_3'$. Thus in this case,

$$T = \hat{e}_1' . \hat{e}_2' \times \hat{e}_3' = \hat{e}_1' . \hat{e}_1' = 1$$

…e define an orthogonal transformation to be one whose matrix $T = [\ell_{ij}]$, w… d for which $\det T = \pm 1$. Such a transformation would leave a right – handed system of a… and would likewise preserve left – handedness. However, a right – handed orthogo… coordinate axes specifically requires that $\det T = +1$. This transformation is ca… **…ed orthogonal (or proper)** transformation. A transformation which is not right – hande… **– handed (or an improper)** transformation.

…, when the axes $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$ coincide, i.e. $x_1' = x_1$, $x_2' = x_2$, $x_3' = x$… en that the values of the direction cosines in the transformation matrix T are $\ell_{ij} = 1$ wh… $\ell_{ij} = 0$ when $i \neq j$; and so for this particular case

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \det(T) = 1$$

… one set of axes is right – handed and the other left – handed, it is impossible to bring them in… … by a rotation.

…quation (1) can be written in full as :

$$\begin{bmatrix} \ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 & \ell_{11}\ell_{21} + \ell_{12}\ell_{22} + \ell_{13}\ell_{23} & \ell_{11}\ell_{31} + \ell_{12}\ell_{32} + \ell_{13}\ell_{33} \\ \ell_{21}\ell_{11} + \ell_{22}\ell_{12} + \ell_{23}\ell_{13} & \ell_{21}^2 + \ell_{22}^2 + \ell_{23}^2 & \ell_{21}\ell_{31} + \ell_{22}\ell_{32} + \ell_{23}\ell_{33} \\ \ell_{31}\ell_{11} + \ell_{32}\ell_{12} + \ell_{33}\ell_{13} & \ell_{31}\ell_{21} + \ell_{32}\ell_{22} + \ell_{33}\ell_{23} & \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

…plies the following six equations called the orthonormality conditions :

$$\ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 \ldots \qquad \ell_{11}\ell_{21} + \ell_{12}\ell_{22} + \ell_{13}\ell_{23} = 0 \qquad (3)$$

EXAMPLE (9): Show that the transformation

(i) $$T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$ is orthogonal and right – handed .

On the other hand , the transformation .

(ii) $$T = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$ is orthogonal but left – handed .

SOLUTION: (i) The transpose of the given matrix T is $$T' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}$$

and so $$TT' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

We note that $$\det T = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = 1$$

and so the corresponding transformation is orthogonal and right – handed .

(ii) The transpose of the given matrix is $$T' = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

and so $$TT' = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

We note that $\det T = \begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{vmatrix} = -1$

and so the corresponding transformation is orthogonal but left – handed .

7.10 PROPER AND IMPROPER TRANSFORMATIONS

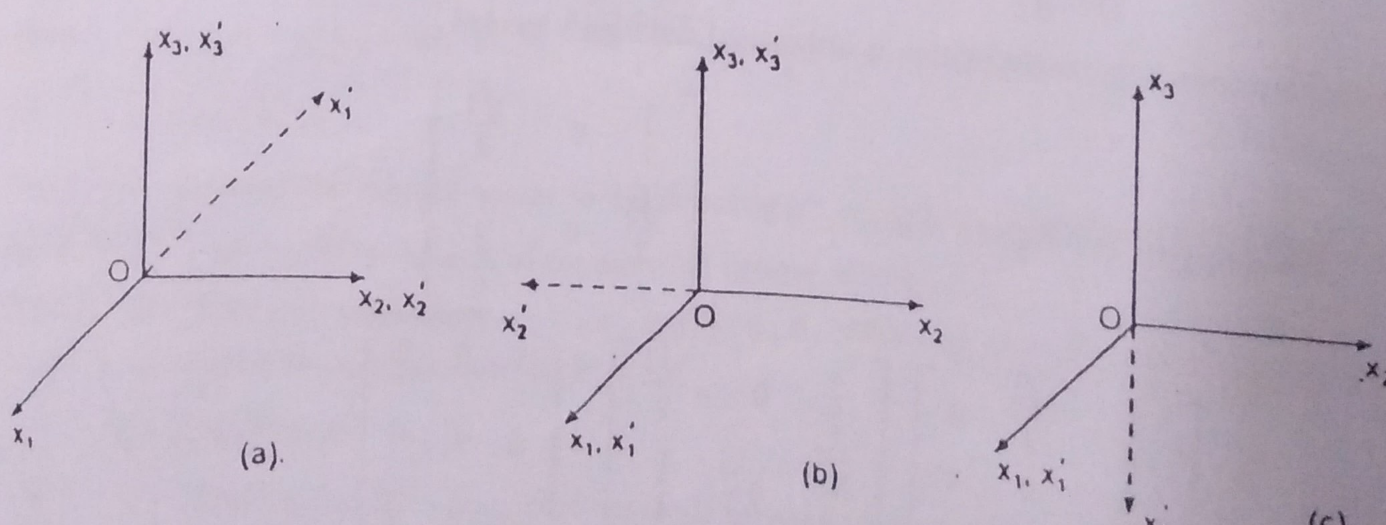
Since for an orthogonal transformation , $\det T = \pm 1$, all coordinate transformations are into two classes . One class consists of those transformations for which $\det T = 1$ and are called transformations ; the other class consists of transformations for which $\det T = -1$ are called im transformations . Under a proper transformation , a right – handed (or left – handed) system right – handed (or left – handed) after rotation . Under an improper transformation , a right – system is changed into a left – handed system and vice–versa . The transformation for which T = unit matrix , is called the identity transformation . Obviously , for the identity transformation

$$x'_1 = x_1 , \quad x'_2 = x_2 , \quad x'_3 = x_3$$

or $$x_i = x_i \quad (i = 1 , 2 , 3)$$

The proper transformation can be obtained from the identity transformation by continuously rotati coordinate axes . On the other hand , the improper transformations cannot be obtained by that pr The improper transformation can be obtained from the identity transformation by two typ discontinuous or discrete operations .

(i) **REFLECTION:** This is the operation in which the new coordinate system $Ox'_1 x'_2$ obtained from the original system $Ox_1 x_2 x_3$ by inverting (reversing) the direction of one of the a the latter , the other two remaining in their original position as shown in figure (7.6) .



(a) (b) (c)

Figure (7.6)

In this case , the transformation will be

$$x'_1 = -x_1 , \quad x'_2 = x_2 , \quad x'_3 = x_3$$

or $$x'_1 = x_1 , \quad x'_2 = -x_2 , \quad x'_3 = x_3$$

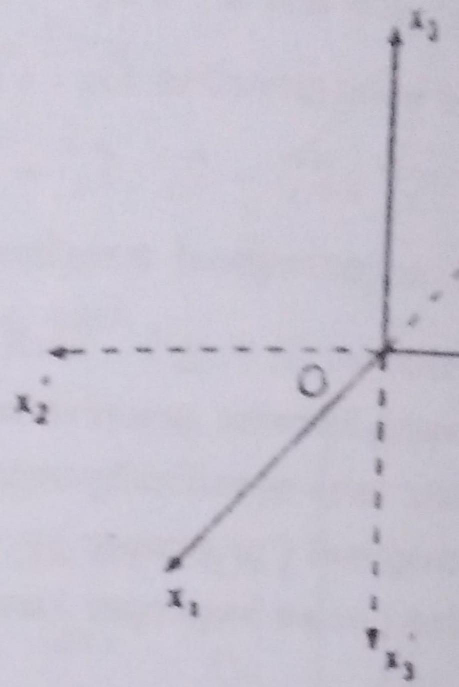
or $$x'_1 = x_1 , \quad x' \quad = x_3$$

The corresponding transformation matrix T will be

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that for each of these matrices $\det T = -1$.

(ii) **INVERSION:** This is the operation in which the new coordinate system $Ox_1'x_2'x_3'$ is obtained from the original system $Ox_1x_2x_3$ by inverting the directions of all the coordinate axes of the latter as shown in figure (7.7). The transformation equations in this case will be



$$x_1' = -x_1, \quad x_2' = -x_2, \quad x_3' = -x_3$$

The corresponding transformation matrix T in this case will be

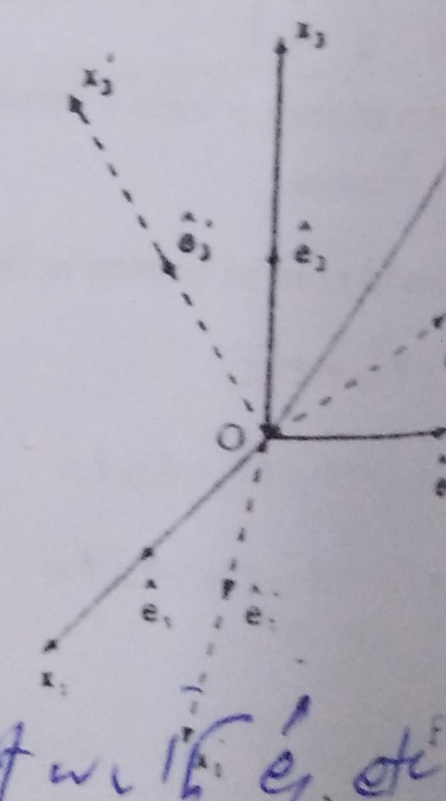
$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that for each of these matrices $\det T = -1$.

7.11 TRANSFORMATION EQUATIONS

(a) TRANSFORMATION EQUATIONS FOR COORDINATES OF A POINT

Consider two rectangular coordinate systems K and K' having the same origin O as shown in figure (7.8). We first find the transformation equations expressing the coordinates x_1', x_2', x_3' of an arbitrary point P in the system K' in terms of its coordinates x_1, x_2, x_3 in the system K, and vice versa. Let $\bar{r}$ and $\bar{r}'$ be the position vectors of any point P in the systems K and K' respectively. Then $\bar{r}' = \bar{r}$



or $\quad x_1'\hat{e}_1' + x_2'\hat{e}_2' + x_3'\hat{e}_3' = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$

or $\quad x_j'\hat{e}_j' = x_i\hat{e}_i, \quad i, j = 1, 2, 3$

For the components x_j', take the dot product of equation (1) with $\hat{e}_j'$, we have

$$x_j'(\hat{e}_j' \cdot \hat{e}_j') = x_i(\hat{e}_i \cdot \hat{e}_j')$$
$$= (\hat{e}_j' \cdot \hat{e}_i)\, x_i$$
$$x_j' = \ell_{ji}\, x_i \qquad (2)$$

where ℓ_{ji} are the direction cosines of the jth - axis of the system K' with the ith - axis of the ...
Equation (2) is called the equation of transformation for the coordinates of the point P from ... to the system K'

... when written in full represents the following three equations:

$$x'_1 = \ell_{11}x_1 + \ell_{12}x_2 + \ell_{13}x_3$$

$$x'_2 = \ell_{21}x_1 + \ell_{22}x_2 + \ell_{23}x_3$$

$$x'_3 = \ell_{31}x_1 + \ell_{32}x_2 + \ell_{33}x_3$$

... be written in matrix form as

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

... be noted that these equations are formed from the transformation matrix.

... inverse transformation law for the coordinates of the point P from the system K' to the syste... (3)

$$x_i = \ell_{ji}x'_j$$

$$\ell_{ji}x'_j = \ell_{ji}(\ell_{jk}x_k) = \ell_{ji}\ell_{jk}x_k = \delta_{ik}x_k = x_i$$

... (3) can also be obtained by taking the dot product of equation (1) with $\hat{e}_i$, since

$$x'_j(\hat{e}'_j \cdot \hat{e}_i) = x_i(\hat{e}_i \cdot \hat{e}_i)$$

$$x_i = \ell_{ji}x'_j$$

... (3) can equivalently be written as

$$x_j = \ell_{ij}x'_i$$

... (3) when written in full represents the following three equations:

$$x_1 = \ell_{11}x'_1 + \ell_{21}x'_2 + \ell_{31}x'_3$$

$$x_2 = \ell_{12}x'_1 + \ell_{22}x'_2 + \ell_{32}x'_3$$

$$x_3 = \ell_{13}x'_1 + \ell_{23}x'_2 + \ell_{33}x'_3$$

... may be written in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

TRANSFORMATION EQUATIONS FOR UNIT VECTORS

We next find the transformation equations expressing the unit vectors $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ of the system K' in terms of the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ of the system K and vice versa.

From vector analysis, we know that

... $(\bar{A}\cdot\hat{e}_3)\hat{e}_3$

Equation (4) when written in full represents the following three equations :

$$\hat{e}_1' = \ell_{11}\hat{e}_1 + \ell_{12}\hat{e}_2 + \ell_{13}\hat{e}_3$$
$$\hat{e}_2' = \ell_{21}\hat{e}_1 + \ell_{22}\hat{e}_2 + \ell_{23}\hat{e}_3$$
$$\hat{e}_3' = \ell_{31}\hat{e}_1 + \ell_{32}\hat{e}_2 + \ell_{33}\hat{e}_3$$

Similarly , from $\bar{A} = (\bar{A}\cdot\hat{e}_1')\hat{e}_1' + (\bar{A}\cdot\hat{e}_2')\hat{e}_2' + (\bar{A}\cdot\hat{e}_3')\hat{e}_3'$

$$= (\bar{A}\cdot\hat{e}_i')\hat{e}_i'$$

Setting $\bar{A} = \hat{e}_j$, we get $\hat{e}_j = (\hat{e}_j\cdot\hat{e}_i')\hat{e}_i' = \ell_{ij}\hat{e}_i'$ (5)

Equation (5) when written in full represents the following three equations:

$$\hat{e}_1 = \ell_{11}\hat{e}_1' + \ell_{21}\hat{e}_2' + \ell_{31}\hat{e}_3'$$
$$\hat{e}_2 = \ell_{12}\hat{e}_1' + \ell_{22}\hat{e}_2' + \ell_{32}\hat{e}_3'$$
$$\hat{e}_3 = \ell_{13}\hat{e}_1' + \ell_{23}\hat{e}_2' + \ell_{33}\hat{e}_3'$$

Equations (4) and (5) are the required transformation equations .

EXAMPLE (10): A set of axes $Ox_1'\,x_2'\,x_3'$ is initially coincident with a set $Ox_1\,x_2\,x_3$. The set $Ox_1'\,x_2'\,x_3'$ is then rotated through an angle θ in the positive sense about the x_3– axis . Show that

$$x_1' = x_1\cos\theta + x_2\sin\theta$$
$$x_2' = -x_1\sin\theta + x_2\cos\theta$$
$$x_3' = x_3$$

SOLUTION: Since the rotation is about x_3 – axis , therefore x_3' – axis coincides with x_3 – axis as shown in the figure (7.9) . If the angle $x_1'Ox_1 = x_2'Ox_2 = \theta$, then

$$\ell_{11} = \cos(x_1'Ox_1) = \cos\theta$$
$$\ell_{12} = \cos(x_1'Ox_2) = \cos(90-\theta) = \sin\theta$$
$$\ell_{13} = \cos(x_1'Ox_3) = \cos 90^0 = 0$$
$$\ell_{21} = \cos(x_2'Ox_1) = \cos(90+\theta) = -\sin\theta$$
$$\ell_{22} = \cos(x_2'Ox_2) = \cos\theta$$
$$\ell_{23} = \cos(x_2'Ox_3) = \cos 90^0 = 0$$
$$\ell_{31} = \cos(x_3'Ox_1) = \cos 90^0 = 0$$
$$\ell_{32} = \cos(x_3'Ox_2) = \cos 90^0 = 0$$
$$\ell_{33} = \cos(x_3'Ox_3) = \cos 0^0 = 1$$

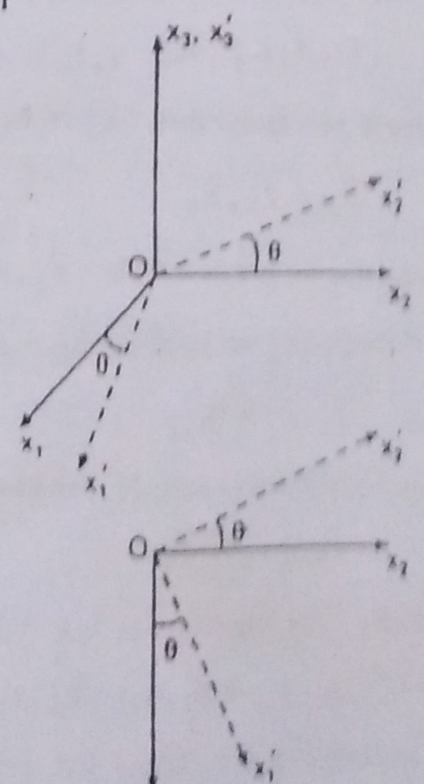


Figure (7.9)

The transformation matrix is given by

$$[\ell_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus the transformation equations for the coordinates become

$$x'_1 = x_1 \cos\theta + x_2 \sin\theta$$

$$x'_2 = -x_1 \sin\theta + x_2 \cos\theta$$

$$x'_3 = x_3$$

Note that effectively we are dealing with a transformation of axes in two–dimensions only and we can write the transformation matrix as

$$[\ell_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

7.12 ORTHONORMALITY CONDITIONS

THEOREM (7.5): Prove that $\ell_{ik}\ell_{jk} = \delta_{ij} = \ell_{ki}\ell_{kj}$.

PROOF: We know that $\hat{e}_j' = \ell_{ji}\hat{e}_i$

or $\hat{e}_j' = \ell_{jk}\hat{e}_k$

Taking the dot product with $\hat{e}_i'$, we get

$$\hat{e}_i' \cdot \hat{e}_j' = \ell_{jk}\hat{e}_i' \cdot \hat{e}_k = \ell_{ik}\ell_{jk} \qquad (1)$$

Also, $\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij}$ (2)

From equations (1) and (2), we have

$$\ell_{ik}\ell_{jk} = \delta_{ij} \qquad (3)$$

Similarly, we know that $\hat{e}_j = \ell_{ij}\hat{e}_i'$

or $\hat{e}_i = \ell_{ki}\hat{e}_k'$

Taking the dot product with $\hat{e}_j$, we get

$$\hat{e}_i \cdot \hat{e}_j = \ell_{ki}\hat{e}_k' \cdot \hat{e}_j = \ell_{ki}\ell_{kj} \qquad (4)$$

Also $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ (5)

From equations (4) and (5), we have

$$\ell_{ki}\ell_{kj} = \delta_{ij} \qquad (6)$$

NOTE: The relation $\ell_{ik}\ell_{jk} = \delta_{ij}$ implies six orthonormality conditions. Write the relation as

$$\ell_{i1}\ell_{j1} + \ell_{i2}\ell_{j2} + \ell_{i3}\ell_{j3} = \delta_{ij} \qquad (A)$$

If we take $i = j = 1$ and $i = j = 2$ and $i = j = 3$ in turn in relation (A), we get

$$\left.\begin{aligned} \ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 &= 1 \\ \ell_{21}^2 + \ell_{22}^2 + \ell_{32}^2 &= 1 \\ \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 &= 1 \end{aligned}\right\} \qquad (I)$$

If we take $i = 1, j = 2$ and $i = 2, j = 3$ and $i = 3, j = 1$ in turn in relation (A), we get

$$\left.\begin{aligned} \ell_{11}\ell_{21}+\ell_{12}\ell_{22}+\ell_{13}\ell_{23} &= 0 \\ \ell_{21}\ell_{31}+\ell_{22}\ell_{32}+\ell_{23}\ell_{33} &= 0 \\ \ell_{31}\ell_{11}+\ell_{32}\ell_{12}+\ell_{33}\ell_{13} &= 0 \end{aligned}\right\} \qquad (2)$$

Similarly, the relation $\ell_{ki}\ell_{kj} = \delta_{ij}$ implies alternative form of orthonormality condition relation as

$$\ell_{1i}\ell_{1j}+\ell_{2i}\ell_{2j}+\ell_{3i}\ell_{3j} = \delta_{ij} \qquad (B)$$

If we take $i = j = 1$ and $i = j = 2$ and $i = j = 3$ in turn in relation (B), we get

$$\left.\begin{aligned} \ell_{11}^2+\ell_{21}^2+\ell_{31}^2 &= 1 \\ \ell_{12}^2+\ell_{22}^2+\ell_{32}^2 &= 1 \\ \ell_{13}^2+\ell_{23}^2+\ell_{33}^2 &= 1 \end{aligned}\right\} \qquad (3)$$

If we take $i = 1, j = 2$ and $i = 2, j = 3$ and $i = 3, j = 1$ in turn in relation (B), we get

$$\left.\begin{aligned} \ell_{11}\ell_{12}+\ell_{21}\ell_{22}+\ell_{31}\ell_{32} &= 0 \\ \ell_{12}\ell_{13}+\ell_{22}\ell_{23}+\ell_{32}\ell_{33} &= 0 \\ \ell_{13}\ell_{11}+\ell_{23}\ell_{21}+\ell_{33}\ell_{31} &= 0 \end{aligned}\right\} \qquad (4)$$

It should be observed how the orthonormality conditions given in equations (1), (2), (3), are formed from the transformation matrix.

7.13 TRANSLATION AND ROTATION

Let the coordinates of a point P be (x_1, x_2, x_3) in the system $Ox_1x_2x_3$ and the coordinates of the point O' in this system be (a_1, a_2, a_3). Shift the system $Ox_1x_2x_3$ to the positions $O'x'_1x'_2x'_3$ through O' as new origin to form a new rectangular coordinate system as shown in figure (7.10). Let the coordinate of P be (x'_1, x'_2, x'_3) in this new system. We know that under the translation only, the axes remain parallel through O', so that the coordinates of P are $(x_i - a_i)$, $i = 1, 2, 3$. Under orthogonal transformation, these become x'_j so that

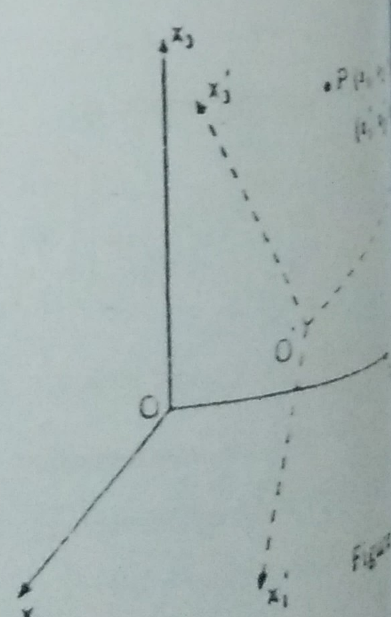


Fig.

$$\begin{aligned} x'_j &= \ell_{ji}(x_i - a_i) \\ &= \ell_{ji}x_i + d_j \end{aligned} \qquad (1)$$

where $d_j = -\ell_{ji}a_i$

Equation (1) illustrates the transformation of coordinates under translation of axes followed by preserving the right-handed rectangular character of the axes.

NVARIANCE WITH RESPECT TO ROTATION OF AXES.

Consider two rectangular coordinate systems K and K$'$ having the same origin O but with axes ...d with respect to each other as shown in figure (7.8). Let (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) be the ...nates of an arbitrary point P in the systems K and K$'$ respectively. Let $\phi(x_1, x_2, x_3)$ be the ... of the scalar point function ϕ at P in the system K and $\phi'(x'_1, x'_2, x'_3)$ be the value of this ...on at the same point in the system K$'$.

...$(x_1, x_2, x_3) = \phi'(x'_1, x'_2, x'_3)$ where x_1, x_2, x_3 and x'_1, x'_2, x'_3 are related by the ...rmation equations (2) or (3) on pages (405) and (406), then $\phi(x_1, x_2, x_3)$ is called an invariant ...spect to the coordinate transformation or rotation of axes.

...rly, a vector point function $\vec{A}(x_1, x_2, x_3)$ is called invariant with respect to rotation of axes if ...$(x_1, x_2, x_3) = \vec{A}'(x'_1, x'_2, x'_3)$. This will be true if

$$A_1(x_1, x_2, x_3)\hat{e}_1 + A_2(x_1, x_2, x_3)\hat{e}_2 + A_3(x_1, x_2, x_3)\hat{e}_3$$

$$= A'_1(x'_1, x'_2, x'_3)\hat{e}'_1 + A'_2(x'_1, x'_2, x'_3)\hat{e}'_2 + A'_3(x'_1, x'_2, x'_3)\hat{e}'_3$$

...MPLE (11): **Show that the quantity $x_1^2 + x_2^2 + x_3^2 = x_i x_i$ is invariant under a rotation of axes.**

...UTION: Let (x_1, x_2, x_3) be the coordinates of a point P in the system K and ...$x'_2, x'_3)$ be the coordinates of the same point in the system K$'$. Then we know that the ...formation equations are :

$$x'_j = \ell_{ji} x_i \tag{1}$$

...

$$x'_j = \ell_{jk} x_k \tag{2}$$

...tiplying equations (1) and (2), we get

$$x'_j x'_j = (\ell_{ji} x_i)(\ell_{jk} x_k) = \ell_{ji}\ell_{jk} x_i x_k$$

$$= \delta_{ik} x_i x_k = (\delta_{ik} x_k) x_i = x_i x_i$$

$$x'_1 x'_1 + x'_2 x'_2 + x'_3 x'_3 = x_1 x_1 + x_2 x_2 + x_3 x_3$$

$$x'^2_1 + x'^2_2 + x'^2_3 = x_1^2 + x_2^2 + x_3^2$$

...hich shows that the quantity $x_1^2 + x_2^2 + x_3^2 = x_i x_i$ is invariant with respect to rotation of axes.

...OTE: This result expresses the fact that the distance between the origin O and a point P does not ...epend upon the system of coordinates used in calculating the distance.

...15 SCALAR INVARIANT OPERATORS

Let $\mathcal{D}$ denote a linear partial differential operator which involves only the rectangular Cartesian ...ordinates x_1, x_2, x_3 as independent variables. For example, we might have

$$\mathcal{D} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} \quad \text{or} \quad \mathcal{D} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_3}$$

The operator $\mathcal{D}$ is called a scalar invariant operator if its form is unchanged under a rotatio coordinate axes . Thus , for example , if the first of the operators above is invariant (it is not , i then upon changing to new axes $Ox'_1 x'_2 x'_3$ it would become

$$\mathcal{D}' = \frac{\partial}{\partial x'_1} + \frac{\partial}{\partial x'_2} + 2\frac{\partial}{\partial x'_3}$$

The following theorem concerning scalar invariant operators will be required .

THEOREM (7.6): Let $\mathcal{D}$ be a scalar invariant operator , and define its operation vector field $\overline{A}$ by $\mathcal{D}\overline{A} = \mathcal{D}(A_1, A_2, A_3) = (\mathcal{D}A_1, \mathcal{D}A_2, \mathcal{D}A_3)$

Then prove that $\mathcal{D}\overline{A}$ is a vector field .

PROOF: Let A_i and A'_j be the components of $\overline{A}$ in the system $Ox_1 x_2 x$ $Ox'_1 x'_2 x'_3$ respectively , then

$$A'_j = \ell_{ji} A_i$$

Using the property of invariance of the form of $\mathcal{D}$ and also the linearity property of $\mathcal{D}$, we have

$$\mathcal{D}' A'_j = \mathcal{D}(\ell_{ji} A_i) = \ell_{ji} \mathcal{D} A_i$$

showing that the components of $\mathcal{D}\overline{A}$ transform according to the vector law under the rotation of th

Thus it follows that $\mathcal{D}\overline{A}$ is a vector field .

THE LAPLACIAN OPERATOR ∇^2

The most important of the scalar invariant operators is the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$$

Formally , the Laplacian operator is the square of the del operator .

THEOREM (7.7): Prove that the Laplacian operator $\nabla^2 = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$ is invariant under a rotation of the axes .

PROOF: The invariance of the Laplacian operator follows from the fact that the com of del-operator transforms as a vector . Under a rotation of coordinate axes $Ox'_1 x'_2 x'_3$, we have

$$\frac{\partial}{\partial x'_j} = \ell_{ji}\frac{\partial}{\partial x_i} \quad \text{and} \quad \frac{\partial}{\partial x'_j} = \ell_{jk}\frac{\partial}{\partial x_k}$$

Thus $$\frac{\partial}{\partial x'_j}\frac{\partial}{\partial x'_j} = \left(\ell_{ji}\frac{\partial}{\partial x_i}\right)\left(\ell_{jk}\frac{\partial}{\partial x_k}\right)$$

$$= \ell_{ji}\ell_{jk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} = \delta_{ik}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$$

which shows that the Laplacian operator ∇^2 is invariant under a rotation of the axes .

7.16 THE ALTERNATING SYMBOL ϵ_{ijk}

The alternating symbol written ϵ_{ijk} is defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is a cyclic permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is an anticyclic permutation of } 1, 2, 3 \\ 0 & \text{if any two of } i, j, k \text{ are equal} \end{cases}$$

Thus $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$

$\epsilon_{223} = \epsilon_{131} = \epsilon_{313} = \ldots\ldots = 0$

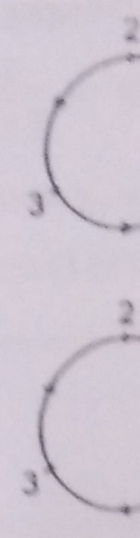


Figure (7.11)

RELATIONSHIP BETWEEN ALTERNATING SYMBOL AND KRONECKER DELTA

THEOREM (7.8): Prove that $\epsilon_{ijk}\,\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$

PROOF: We know that

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2 \qquad (1)$$

Using the definition of the alternating symbol we can write equation (1) as

$$\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k = \epsilon_{ijk}$$

Thus
$$\begin{aligned} \epsilon_{ijk}\,\epsilon_{\ell mk} &= (\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k)(\hat{e}_\ell \times \hat{e}_m \cdot \hat{e}_k) \\ &= (\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k)\hat{e}_k \cdot (\hat{e}_\ell \times \hat{e}_m) \\ &= (\hat{e}_i \times \hat{e}_j) \cdot (\hat{e}_\ell \times \hat{e}_m) \quad [\text{since } (\bar{A} \cdot \hat{e}_k)\hat{e}_k = \bar{A}\,] \\ &= (\hat{e}_i \times \hat{e}_j) \times \hat{e}_\ell \cdot \hat{e}_m \\ &= [(\hat{e}_i \cdot \hat{e}_\ell)\hat{e}_j - (\hat{e}_j \cdot \hat{e}_\ell)\hat{e}_i] \cdot \hat{e}_m \\ &= (\hat{e}_i \cdot \hat{e}_\ell)(\hat{e}_j \cdot \hat{e}_m) - (\hat{e}_j \cdot \hat{e}_\ell)(\hat{e}_i \cdot \hat{e}_m) \\ &= \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \end{aligned}$$

ALTERNATIVE METHOD

We have to prove

$$\epsilon_{ij1}\,\epsilon_{\ell m1} + \epsilon_{ij2}\,\epsilon_{\ell m2} + \epsilon_{ij3}\,\epsilon_{\ell m3} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \qquad (A)$$

Case (1) (a) when $i = j$ but $\ell \neq m$ (b) when $i \neq j$ but $\ell = m$

For these cases we may easily verify from equation (A) that L.H.S. = 0 = R.H.S.

Case (2) When $i \neq j$, $\ell \neq m$ and the pairs (i, j) and (ℓ, m) are different from each other

For example, $(i, j) = (1, 2)$ and $(\ell, m) = (1, 3)$, then from equation (A)

Case (3) When $i \neq j$, $\ell \neq m$ but the pairs (i, j) and $(\ell$ [illegible] can have the following pairs of values in any order :

$$1,2;\ 1,3;\ 2,3;\ 2,1;\ 3,1;\ 3,2$$

The first pair gives rise to the following possibilities :

$$i = 1, j = 2, \quad \ell = 1, m = 2$$

$$i = 1, j = 2, \quad \ell = 2, m = 1$$

$$i = 2, j = 1, \quad \ell = 1, m = 2$$

$$i = 2, j = 1, \quad \ell = 2, m = 1$$

For these possibilities , equation (A) gives

L.H.S. = 1 = R.H.S.

L.H.S. = − 1 = R.H.S.

L.H.S. = − 1 = R.H.S.

L.H.S. = 1 = R.H.S.

The result may easily be seen to be true for the other five pairs also .

Hence equation (A) holds for all possible values of i, j, ℓ and m .

NOTE: Each side of the theorem is a tensor of order 4 , therefore equation (A) is equivalent [illegible] $3^4 = 81$ scalar equations .

EXAMPLE (12): Prove that

(i) $\epsilon_{ijk}\delta_{jk} = 0$

(ii) $\epsilon_{ijk}\epsilon_{ijk} = 6$ [illegible]

(iii) $\epsilon_{ijk}\epsilon_{\ell jk} = 2\delta_{i\ell}$

(iv) $\epsilon_{ijk}\epsilon_{\ell mk}\delta_{jm}$ [illegible]

(v) $\epsilon_{iks}\epsilon_{mps} = \epsilon_{sik}\epsilon_{smp} = \epsilon_{ksi}\epsilon_{psm}$

(vi) $\frac{1}{2}\epsilon_{ijk}\epsilon_{ijr}A_r$ [illegible]

SOLUTION: We know that

(i) $\epsilon_{ijk}\delta_{jk} = \epsilon_{ijj} = \epsilon_{i11} + \epsilon_{i22} + \epsilon_{i33} = 0 + 0 + 0 = 0$

(ii) We know that

$$\epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \quad \text{(A)}$$

Setting $\ell = i$ and $m = j$ in the above relation

$$\epsilon_{ijk}\epsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} = (3)(3) - \delta_{ii} = 9 - 3 = 6$$

(iii) Put $m = j$ in relation (A) we get

$$\epsilon_{ijk}\epsilon_{\ell jk} = \delta_{i\ell}\delta_{jj} - \delta_{ij}\delta_{j\ell} = 3\delta_{i\ell} - \delta_{i\ell} = 2\delta_{i\ell}$$

(iv) $\epsilon_{ijk}\epsilon_{\ell mk}\delta_{jm} = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\delta_{jm}$

$$= \delta_{i\ell}\delta_{jm}\delta_{jm} - \delta_{im}\delta_{j\ell}\delta_{jm}$$

$$= \delta_{i\ell}\delta_{mm} - \delta_{im}\delta_{m\ell} = 3\delta_{i\ell} - \delta_{i\ell} = 2\delta_{i\ell}$$

$\epsilon_{mps} = \epsilon_{sik}\epsilon_{smp} = \epsilon_{ksi}\epsilon_{psm}$

clic permutation of the suffixes in ϵ_{ijk} does not change its value,

$\epsilon_{ijk}\epsilon_{ijl}A_l = \frac{1}{2}(2\delta_{kl})A_l = A_k$ (since $\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il}$ from equation (3))

SORS

now that a scalar is a quantity whose specification (in any coordinate system) requires just On the other hand , a vector is a quantity whose specification in any coordinate system numbers called its **components** . Scalars and vectors are both special cases of a more called a tensor of order n , whose specification in any given coordinate system requires 3^n called the components of the tensor . In fact , scalars are tensors of order 0 , with $3^0 = 1$ nd vectors are tensors of order 1 with $3^1 = 3$ components . The order or rank of a tensor he number of suffixes used in it .

OTH – ORDER TENSORS (OR SCALARS)

scalar (or zeroth order tensor) is meant a quantity uniquely specified in any coordinate ngle real number (the component or value of the scalar) which is invariant under changes of system i.e. which does not change when the coordinate system is changed . Thus if ϕ is the r in the coordinate system K and ϕ' its value in another coordinate system K', then

ϕ'

$$(x_1, x_2, x_3) = \phi'(x'_1, x'_2, x'_3)$$

of order zero is called a **scalar invariant** .

ST ORDER TENSORS (OR VECTORS)

entity (quantity) representable by a set A_i of three (i.e. 3^1) numbers (called components) a coordinate system K is called a first order tensor , if its components transform under he coordinate system according to the law

$$A'_j = l_{ji}A_i \qquad (1)$$

are the components of the quantity in the coordinate system K' and l_{ji} is the cosine of the een the jth – axis of K' and the ith – axis of K . Equation (1) is called the transformation law mponents of first order tensor i.e. vector . Equation (1) represents the following three equations :

$$A'_1 = l_{11}A_1 + l_{12}A_2 + l_{13}A_3 \qquad (2)$$

$$A'_2 = l_{21}A_1 + l_{22}A_2 + l_{23}A_3$$

$$A'_3 = l_{31}A_1 + l_{32}A_2 + l_{33}A_3$$

ay be

The inverse transformation law for the components of the first order tensor in the system K in terms of its components in the system K' is

$$A_i = \ell_{ji} A'_j \tag{4}$$

since $\ell_{ji} A'_j = \ell_{ji}(\ell_{jk} A_k) = \ell_{ji}\ell_{jk} A_k = \delta_{ik} A_k = A_i$

Equation (4) may equivalently be written as

$$A_j = \ell_{ij} A'_i \tag{5}$$

Equation (4) or (5) represents the following three equations :

$$\begin{aligned} A_1 &= \ell_{11} A'_1 + \ell_{21} A'_2 + \ell_{31} A'_3 \\ A_2 &= \ell_{12} A'_1 + \ell_{22} A'_2 + \ell_{32} A'_3 \\ A_3 &= \ell_{13} A'_1 + \ell_{23} A'_2 + \ell_{33} A'_3 \end{aligned} \tag{6}$$

which may be written in matrix form as

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} \tag{7}$$

SECOND ORDER TENSORS

A quantity representable by a two suffix set A_{ij} of nine (i.e. 3^2) numbers (called co… relatively to a coordinate system K is called a second order tensor , if its components trans… changes of the coordinate system according to the law

$$A'_{mn} = \ell_{mi}\ell_{nj} A_{ij} \tag{1}$$

where A'_{mn} are the components of the quantity in the coordinate system K' and ℓ_{mi} is the c… angle between the mth – axis of K' and the ith – axis of K . (Similarly for ℓ_{nj})

The inverse transformation law expressing the components of the second order tensor in the sy… terms of its components in the system K' is :

$$A_{ij} = \ell_{mi}\ell_{nj} A'_{mn} \tag{3}$$

since
$$\begin{aligned} \ell_{mi}\ell_{nj} A'_{mn} = \ell_{mi}\ell_{nj}(\ell_{mr}\ell_{ns} A_{rs}) &= (\ell_{mi}\ell_{mr})(\ell_{nj}\ell_{ns}) A_{rs} \\ &= \delta_{ir}\delta_{js} A_{rs} = (\delta_{ir} A_{rs})\delta_{js} \\ &= A_{is}\delta_{js} = A_{ij} \end{aligned}$$

Equation (3) can be written equivalently as $A_{mn} = \ell_{im}\ell_{jn} A'_{ij}$

NOTE: (i) The nine components of a second order tensor can be written in the form of a 3×3 …

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

or briefly $[A_{ij}]$, where A_{ij} is the element in the ith – row and jth – column of the above matr…

(ii) Equation (1) when written in matrix form becomes

$$\begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \quad ($$

or $$[A'] = [T][A][T']$$

Equation (2) is easier to use than equation (1) itself

(iii) Given the components of a second order tensor in the coordinate system K, we c equation (1) to determine its components in another coordinate system K'. In particular, if components of a tensor vanish in one coordinate system, they also vanish in any other coordinate sys

EXAMPLE (13): Prove that if A_i and B_j are two first order tensors i.e. vectors, the product $A_i B_j$ $(i, j = 1, 2, 3)$ is a second order tensor.

SOLUTION: Let $C_{ij} = A_i B_j$ (1)

then we have to prove that C_{ij} $(i, j = 1, 2, 3)$ are the components of a second order tensor. Sin and B_j are the first order tensors, their equations of transformation from the system K to K' are

$$A'_m = \ell_{mi} A_i \qquad (2)$$

$$B'_n = \ell_{nj} B_j \qquad (3)$$

Multiplying equations (2) and (3), we obtain

$$A'_m B'_n = \ell_{mi} \ell_{nj} A_i B_j$$

or $$C'_{mn} = \ell_{mi} \ell_{nj} C_{ij} \qquad (4)$$

where $C'_{mn} = A'_m B'_n$ (5)

Equation (5) shows that $C_{ij} = A_i B_j$ are the components of a second order tensor.

THEOREM (7.9): Prove that the Kronecker delta δ_{ij} is a Cartesian tensor of rank 2.

PROOF: Let δ_{ij} and δ'_{mn} be the components of the Kronecker delta in the systems K K' respectively. Then $\delta_{ij} = \hat{e}_i \cdot \hat{e}_j$ and $\delta'_{mn} = \hat{e}'_m \cdot \hat{e}'_n$

Also we know that $\hat{e}'_m = \ell_{mi} \hat{e}_i$ and $\hat{e}'_n = \ell_{nj} \hat{e}_j$

so
$$\begin{aligned} \delta'_{mn} &= \hat{e}'_m \cdot \hat{e}'_n \\ &= (\ell_{mi} \cdot \hat{e}_i) \cdot (\ell_{nj} \hat{e}_j) \\ &= \ell_{mi} \ell_{nj} (\hat{e}_i \cdot \hat{e}_j) \\ &= \ell_{mi} \ell_{nj} \delta_{ij} \end{aligned}$$

which shows that δ_{ij} is a second order Cartesian tensor.

NOTE: (i) The nine components of the Kronecker delta tensor δ_{ij} can be written in the form of 3 x 3 matrix as

$$[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ii) Since x_i are the components of a first order tensor (i.e. ...

...hat $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$ is a tensor of order 2.

THIRD ORDER TENSORS

A quantity representable by a set of three suffixes A_{ijk} of 27 (i.e. 3^3) numbers (c... components) relatively to a coordinate system K is called a third order tensor , if its compo... transform under changes of the coordinate system according to the law

$$A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}A_{ijk} \tag{1}$$

where A'_{mnp} are the components of a quantity in the coordinate system K' and ℓ_{mi} is the cosine o... angle between the mth – axis of K' and the ith – axis of K . (Similarly , for ℓ_{nj} and ℓ_{pk})

The inverse transformation law of equation (1) expressing the components of the third order tensor ... system K in terms of its components in the system K' is

$$A_{ijk} = \ell_{mi}\ell_{nj}\ell_{pk}A'_{mnp} \tag{2}$$

since

$$\begin{aligned}
\ell_{mi}\ell_{nj}\ell_{pk}A'_{mnp} &= \ell_{mi}\ell_{nj}\ell_{pk}(\ell_{mr}\ell_{ns}\ell_{pt}A_{rst}) \\
&= (\ell_{mi}\ell_{mr})(\ell_{nj}\ell_{ns})(\ell_{pk}\ell_{pt})A_{rst} \\
&= (\delta_{ir}\delta_{js}\delta_{kt})A_{rst} \\
&= \delta_{ir}\delta_{js}(\delta_{kt}A_{rst}) \\
&= \delta_{ir}(\delta_{js}A_{rsk}) \\
&= \delta_{ir}A_{rjk} = A_{ijk}
\end{aligned}$$

Equation (2) can be written equivalently as

$$A_{mnp} = \ell_{im}\ell_{jn}\ell_{kp}A'_{ijk} \tag{3}$$

Note that equation (2) represents components of the form

A_{111} , A_{112} , A_{113} , A_{123} , etc.

EXAMPLE (14): Prove that if A_i , B_j , and C_k are three first order tensors, the... product $A_iB_jC_k$ ($i , j , k = 1 , 2 , 3$) is a tensor of order 3. ... $A_iB_jC_j$ ($i , j = 1 , 2 , 3$) form a first order tensor

SOLUTION: Let $D_{ijk} = A_iB_jC_k$ (1)

then we have to prove that D_{ijk} ($i , j , k = 1 , 2 , 3$) are the components of a third order tensor ... D_{ijj} are the components of a first order tensor

Since A_i , B_j , C_k are first order tensors , their equations of transformations from the system K to K'...

$$A'_m = \ell_{mi}A_i \tag{2}$$

$$B'_n = \ell_{nj}B_j \tag{3}$$

$$C'_p = \ell_{pk}C_k \tag{4}$$

...ing (2), (3), and (4), we obtain

$$A'_m B'_n C'_p = \ell_{mi}\ell_{nj}\ell_{pk} A_i B_j C_k \qquad (5)$$

$$D'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk} D_{ijk} \qquad (6)$$

$$D'_{mnp} = A'_m B'_n C'_p$$

...(6) shows that $D_{ijk} = A_i B_j C_k$ are the components of a third order tensor.

...$n = p$ in equation (6), we have

$$D'_{mnn} = \ell_{mi}\ell_{nj}\ell_{nk} D_{ijk}$$

$$D'_{mnn} = \ell_{mi}\delta_{jk} D_{ijk} = \ell_{mi} D_{ijj}$$

...ows that $D_{ijj} = A_i B_j C_j$ are the components of a first order tensor.

...EM (7.10): **Prove that the alternating symbol ϵ_{ijk} is a Cartesian tensor of rank 3.**

...: Let ϵ_{ijk} and ϵ'_{pqr} be the components of the alternating symbol in the systems K ... respectively. Then

$$\left.\begin{array}{l} \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2, \quad \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \\ \hat{e}'_2 \times \hat{e}'_3 = \hat{e}'_1, \quad \hat{e}'_3 \times \hat{e}'_1 = \hat{e}'_2, \quad \hat{e}'_1 \times \hat{e}'_2 = \hat{e}'_3 \end{array}\right\} \qquad (1)$$

...definition of the alternating symbol we can write equations (1) as

$$\left.\begin{array}{l} \hat{e}_i \times \hat{e}_j \cdot \hat{e}_k = \epsilon_{ijk} \\ \hat{e}'_p \times \hat{e}'_q \cdot \hat{e}'_r = \epsilon'_{pqr} \end{array}\right\} \qquad (2)$$

...(i, j, k, p, q, r = 1, 2, 3).

$\hat{e}'_p = \ell_{pi}\hat{e}_i$, $\hat{e}'_q = \ell_{qj}\hat{e}_j$, and $\hat{e}'_r = \ell_{rk}\hat{e}_k$ therefore,

$$\epsilon'_{pqr} = \hat{e}'_p \times \hat{e}'_q \cdot \hat{e}'_r$$

$$= (\ell_{pi}\hat{e}_i) \times (\ell_{qj}\hat{e}_j) \cdot (\ell_{rk}\hat{e}_k) = \ell_{pi}\ell_{qj}\ell_{rk}\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k \qquad (3)$$

$$\epsilon'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}\epsilon_{ijk}$$

...m equation (3) it is clear that ϵ_{ijk} is a Cartesian tensor of rank 3.

HIGHER ORDER TENSORS

A quantity [illegible] by a set of n suffixes $A_{i_1 i_2 \cdots i_n}$ of 3^n numbers (called ...ponents) relatively to a coordinate system K is said to be a tensor of order (rank) n, if its ...ponents [illegible] changes of the [illegible]inate system according to the law

$$[illegible] \cdots \ell_{j_n i_n} A_{i_1 i_2 \cdots i_n} \qquad (1)$$

...where [illegible] ...nents of the quantity in the coordinate system K′ and $\ell_{j_1 i_1}$, etc., ha...

[illegible]

NOTE: Given the components of a tensor of order n in the coordinate system K, we can use equati to determine its components in another rectangular coordinate system K'. In particular, if a components of a tensor vanish in one coordinate system, they also vanish in any other coordinate syste

7.18 ALGEBRA OF TENSORS

ADDITION OF TENSORS

The sum of two or more tensors of the same rank is the tensor whose components are equal to sum of the corresponding components of the individual tensors. Note that the tensors of different ra cannot be added. Also addition of tensors is commutative and associative.

SUBTRACTION OF TENSORS

The difference of two tensors of the same rank is also a tensor whose components are equal to difference of the corresponding components of the two tensors. Note that subtraction of tensors different orders is not defined.

EXAMPLE (15): If A_{jk} and B_{jk} are tensors of rank 2, prove that

(i) $C_{jk} = A_{jk} + B_{jk}$

(ii) $D_{jk} = A_{jk} - B_{jk}$

are also tensors of rank 2.

SOLUTION: (i) Since A_{jk} and B_{jk} are the second order tensors, their equations o transformation from the system K to K' are

$$A'_{mn} = \ell_{mj}\ell_{nk}A_{jk} \qquad (1)$$

$$B'_{mn} = \ell_{mj}\ell_{nk}B_{jk} \qquad (2)$$

Adding equations (1) and (2) we get

$$A'_{mn} + B'_{mn} = \ell_{mj}\ell_{nk}(A_{jk} + B_{jk})$$

or $$C'_{mn} = \ell_{mj}\ell_{nk}C_{jk} \qquad (3)$$

where $C'_{mn} = A'_{mn} + B'_{mn}$

Equation (3) shows that $C_{jk} = A_{jk} + B_{jk}$ is also a tensor of rank 2.

(ii) Subtracting equation (1) from equation (2), we obtain

$$A'_{mn} - B'_{mn} = \ell_{mj}\ell_{nk}(A_{jk} - B_{jk})$$

or $$D'_{mn} = \ell_{mj}\ell_{nk}D_{jk} \qquad (4)$$

where $D''_{mn} = A''_{mn} - B''_{mn}$

From equation (4) it is clear that $D_{jk} = A_{jk} - B_{jk}$ is a tensor of rank 2.
This shows that the sum (or difference) of two second order tensors is a second order tensor. In general the sum (or difference) of two tensors of order n is another tensor of order n.

MULTIPLICATION OF A TENSOR BY A SCALAR

The multiplication of a tensor of any rank by a scalar yields another tensor of the same

EXAMPLE (16): **Prove that if ϕ is a scalar and A_{ij} is a second order tensor, then $C_{ij} = \phi A_{ij}$ is also a second order tensor.**

SOLUTION: Since A_{ij} is a second order tensor, its equation of transformation from t K to K' is

$$A'_{mn} = \ell_{mi}\ell_{nj}A_{ij} \tag{1}$$

Multiplying equation (1) by the scalar ϕ, we get

$$\phi A'_{mn} = \ell_{mi}\ell_{nj}(\phi A_{ij})$$

or $$C'_{mn} = \ell_{mi}\ell_{nj}C_{ij} \tag{2}$$

where $$C'_{mn} = \phi A'_{mn}$$

Equation (2) shows that $C_{ij} = \phi A_{ij}$ is also a second order tensor.

In general, multiplication of a tensor of order n by a scalar gives another tensor of order n.

(OUTER) MULTIPLICATION OF TENSORS

The product of two or more tensors is the tensor whose components are the produ components of the given tensors. The order of a tensor product is clearly the sum of the ord given tensors.

EXAMPLE (17): **If A_{ijk} and B_{mn} are two Cartesian tensors of rank 3 and 2 resp prove that $C_{ijkmn} = A_{ijk}B_{mn}$ is also a tensor of rank 5.**

SOLUTION: Since A_{ijk} and B_{mn} are tensors, their equations of transformation systems K to K' are

$$A'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}A_{ijk}$$

$$B'_{st} = \ell_{sm}\ell_{tn}B_{mn} \tag{1}$$

Multiplying equations (1) and (2), we get (2)

$$A'_{pqr}B'_{st} = \ell_{pi}\ell_{qj}\ell_{rk}\ell_{sm}\ell_{tn}A_{ijk}B_{mn}$$

or $$C'_{pqrst} = \ell_{pi}\ell_{qj}\ell_{rk}\ell_{sm}\ell_{tn}C_{ijkmn}$$

where $$C'_{pqrst} = A'_{pqr}B'_{st} \tag{3}$$

which shows that $C_{ijkmn} = A_{ijk}B_{mn}$ called the outer product of A_{ijk} and B_{mn} is a rank $3 + 2 = 5$.

NOTE: (i) The tensor multiplication is non – commutative. For example,

$$C_{ijkmn} = A_{ijk}B_{mn} \neq B_{mn}A_{ijk} = C\ldots$$

7.19 CONTRACTION OF TENSORS

The process of setting two indices in a tensor equal and [illegible] contraction . For example , the three possible contractions of a third order [illegible] and A_{ijj} .

CONTRACTION THEOREM

THEOREM (7.11): The contraction of a tensor of order n $(n \geq 2)$ leads to a tensor of order [illegible]

PROOF: We prove this theorem for $n = 3$ i.e. for the tensor A_{ijk}.

By hypotheses , A_{ijk} is a tensor of order 3 . Therefore ,

$$A'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}A_{ijk} \qquad (1)$$

Let us contract w.r.t. j and k . Place the corresponding indices q and r equal to each other and [illegible] this index . Then

$$A'_{pqq} = \ell_{pi}\ell_{qj}\ell_{qk}A_{ijk}$$

$$= \ell_{pi}\delta_{jk}A_{ijk} = \ell_{pi}A_{ijj}$$

which shows that $B_i = A_{ijj}$ is a tensor of rank 1 i.e. a vector .

NOTE: (i) We have seen that contraction can be applied to a tensor of rank 2 or higher .

(ii) We know that contraction of a tensor of order n $(n \geq 2)$ leads to a tensor of order $n-2$ [illegible] tensor of order $(n-2)$ can then be contracted again (provided that $n \geq 4$) , giving a tensor of order [illegible] and so on ; until we obtain a tensor of order less than 2 . In fact , repeated contraction of a tensor of order n eventually gives a scalar if n is even and a vector if n is odd .

7.20 (INNER) MULTIPLICATION OF TENSORS

The process of multiplying tensors (outer multiplication) and then contracting the product [illegible] indices belonging to different factors is called inner multiplication and the result is called an inner [illegible] of the given tensors . For example the expression $A_i B_{ik}$ is the inner product of the tensors [illegible] B_{jk} . Similarly $A_i B_i$ is the inner product of two vectors A_i and B_j (i.e. $\bar{A}$ and $\bar{B}$)

EXAMPLE (18): If A_{ijk} and B_{mn} are two tensors of rank 3 and 2 respectively , then their inner product $A_{ijk}B_{in}$ is a tensor of rank $3+2-2=3$.

SOLUTION: Since A_{ijk} and B_{mn} are tensors , their equations of transformation [illegible] system K to K' are

(1)

(letting $p = s$ in equation (3) and summing), we get

$$B'_{qrt} = l_{pi} l_{qj} l_{rk} l_{pm} l_{tn} A_{ijk} B_{mn}$$
$$= l_{qj} l_{rk} l_{tn} (l_{pi} l_{pm}) A_{ijk} B_{mn}$$
$$= l_{qj} l_{rk} l_{tn} \delta_{im} A_{ijk} B_{mn}$$
$$= l_{qj} l_{rk} l_{tn} A_{ijk} B_{in}$$

that $C_{jkn} = A_{ijk} B_{in}$ called the inner product of A_{ijk} and B_{mn} is a tensor of rank 3. ... w.r.t. j and n or k and m in the product $A_{ijk} B_{mn}$, we can similarly show that any ... is a tensor of rank 3.

...: The inner product $A_i B_i$ of two vectors A_i and B_j (i.e. $\overline{A}$ and $\overline{B}$) is a tensor of ... a scalar. For this reason $A_i B_i$ is called the scalar or dot product of $\overline{A}$ and $\overline{B}$.

GENERALIZATION

If $A_{i_1 i_2 \cdots i_m}$ and $B_{j_1 j_2 \cdots j_n}$ are two tensors of rank m and n respectively, then any of ... products is a tensor of rank $m + n - 2$.

QUOTIENT THEOREM

With the help of this theorem we can decide whether a quantity representable by a multi – suffix ... or not.

THEOREM (7.1?): If an inner product of a quantity X with an arbitrary tensor is itself a tensor, then X is also a tensor.

To illustrate this theorem we consider the following example:

EXAMPLE (1?): If $A_{ij} B_j$ is a vector where B_j is an arbitrary vector, then prove that the 2-suffix set A_{ij} is also a tensor of rank 2.

SOLUTION: Let $C_i = A_{ij} B_j$, $C'_p = A'_{pq} B'_q$ where A_{ij}, B_j, C_i and A'_{pq}, B'_q, C'_p are components of the 2 – suffix set and the two vectors in the systems K and K′ respectively.

Since C_i is a vector, therefore

$$C'_p = l_{pi} C_i$$

$$A'_{pq} B'_q = l_{pi} A_{ij} B_j \qquad (1)$$

B_j being arbitrary, we have $B'_q = l_{qj} B_j$

$$B_j = l_{qj} B'_q \qquad (2)$$

[illegible], we get

[illegible] $l_{qj} A_{ij} B'_q$

[illegible] $A_{ij}) B'_q = 0$

Since the vector B_j is arbitrary, the vector B'_q is also arbitrary so that $B'_q \neq 0$ and the above relation is true only when

$$A'_{pq} - \ell_{pi}\ell_{qj} A_{ij} = 0$$

or

$$A'_{pq} = \ell_{pi}\ell_{qj} A_{ij}$$

showing that the 2 – suffix set A_{ij} is a tensor of rank 2.

GENERALIZATION

If $A_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_n} B_{j_1 j_2 \cdots j_n}$ is a tensor of order m, where $B_{j_1 j_2 \cdots j_n}$ is an arbitrary tensor of order n, then prove that $A_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_n}$ is a tensor of order $m + n$.

7.22 SYMMETRIC AND ANTI – SYMMETRIC TENSORS

A tensor $A_{i_1 i_2 \cdots i_n}$ is said to be symmetric in a pair of indices i_1 and i_2 (say) if

$$A_{i_1 i_2 \cdots i_n} = A_{i_2 i_1 \cdots i_n} \tag{1}$$

while it is said to be anti – symmetric in the indices i_1 and j_2 if

$$A_{i_1 i_2 \cdots i_n} = -A_{i_2 i_1 \cdots i_n} \tag{2}$$

A tensor is said to be symmetric (anti – symmetric) if it is symmetric (anti – symmetric) in all possible pairs of indices. Symmetric and anti – symmetric tensors occur frequently in mathematics and physics. For example, the interia tensor, the stress tensor, the strain tensor and the rate of strain tensor are all symmetric, while the spin tensor is an example of an anti – symmetric tensor.

THEOREM (7.13): **Prove that the Kronecker tensor δ_{ij} is a second order symmetric tensor and the alternating tensor ϵ_{ijk} is a third order anti – symmetric teosor.**

PROOF: We have $\delta_{ij} = \hat{e}_i \cdot \hat{e}_j = \hat{e}_j \cdot \hat{e}_i = \delta_{ji}$

which shows that δ_{ij} is a symmetric tensor.

Also, $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$

which shows that ϵ_{ijk} is an anti – symmetric tensor.

NOTE: A symmetric second order tensor A_{ij} can be written as a matrix in the form

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

while an anti – symmetric second order tensor has a matrix of the form

$$[A_{ij}] = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}$$

Thus a symmetric second order tensor has only 6 independent components, while an anti – symmetric second order tensor has only 3 independent components. Also in an anti – symmetric tensor the components on the leading diagonal are all zero. [$A_{ii} = -A_{ii}$ or $2A_{ii} = 0$ or $A_{ii} = 0$]

THEOREM (7.14): Prove that every second order tensor can be represented uniquely as the sum a symmetric and an anti – symmetric tensor.

PROOF: Let A_{ij} be a second order tensor, then we can write

$$A_{ij} = \frac{1}{2}(A_{ij}+A_{ji})+\frac{1}{2}(A_{ij}-A_{ji})$$
$$= B_{ij}+C_{ij}$$

where $B_{ij} = \frac{1}{2}(A_{ij}+A_{ji}) = B_{ji}$ is symmetric

and $C_{ij} = \frac{1}{2}(A_{ij}-A_{ji}) = -\frac{1}{2}(A_{ji}-A_{ij}) = -C_{ji}$, is anti – symmetric.

7.23 INVARIANCE OF SYMMETRIC AND ANTI-SYMMETRIC PROPERTY OF A TENSOR

THEOREM (7.15): If a tensor is symmetric (anti – symmetric) w.r.t. a pair of indices in one coordinate system, then it has the same property in any other coordinate system.

PROOF: We prove this theorem for the tensor A_{ijk}

If A_{ijk} is symmetric in i and j, then

$$A_{ijk} = A_{jik} \qquad (1)$$

Also $A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}A_{ijk}$

$= \ell_{mi}\ell_{nj}\ell_{pk}A_{jik}$ [using equation (1)]

$= \ell_{nj}\ell_{mi}\ell_{pk}A_{jik} = A'_{nmp}$

Thus $A'_{mnp} = A'_{nmp}$ showing that the tensor is symmetric w.r.t. the same pair of indices in the new coordinate system as well.

Similarly, in case the tensor A_{ijk} is anti – symmetric in i and j we can show $A'_{mnp} = -A'_{nmp}$.

7.24 FUNDAMENTAL PROPERTY OF TENSOR EQUATIONS

THEOREM (7.16): A tensor equation which holds in one coordinate system holds in every coordinate system i.e. the form of a tensor equation remains the same in every rectangular coordinate system.

PROOF: We prove this property for the simple tensor equation.

Let $A_iB_{ijk} = C_{jk}$ (1)

be a tensor equation where A_i, B_{ijk}, C_{jk} represent the components of the three tensors w.r.t. the system K. We will prove that equation (1) has the same form in another coordinate system K'.

Multiplying both sides of equation (1) with $\ell_{mj}\ell_{nk}$, we get

$$\ell_{mj}\ell_{nk}A_iB_{ijk} = \ell_{mj}\ell_{nk}C_{jk}$$

If A'_p, B'_{pmn}, and C'_{mn} be the components of the same tensors in a [illegible]

$$C'_{mn} = \ell_{mj}\ell_{nk}C_{jk} \qquad (3)$$

Now since $A_i B_{ijk} = \delta_{iq} A_q B_{ijk}$

$$= \ell_{pi}\ell_{pq}A_q B_{ijk} \qquad (4)$$

Using equations (3) and (4), equation (2) becomes

$$\ell_{mj}\ell_{nk}\ell_{pi}\ell_{pq}A_q B_{ijk} = C'_{mn}$$

But $\quad A'_p = \ell_{pq}A_q$ and $B'_{pmn} = \ell_{pi}\ell_{mj}\ell_{nk}B_{ijk}$

Therefore $A'_p B'_{pmn} = C'_{mn} \qquad (5)$

which is of the same form as equation (1).

COROLLARY: From equations (1) and (5) writing $D_{jk} = A_i B_{ijk} - C_{jk}$, $D'_{mn} = A'_p B'_{pmn} - C'_{mn}$ we have $D_{jk} = 0$ and $D'_{mn} = 0$ for $m, n = 1, 2, 3$ which shows that if the components [illegible] in one coordinate system are all zero, then the components in every coordinate system are [illegible]

ZERO TENSOR

A tensor whose components relatively to one coordinate system and, therefore, [illegible] every coordinate system are all zero is known as a zero tensor.

7.25 ISOTROPIC TENSORS

A tensor is said to be isotropic if its components remain the same in all rectangular [illegible] coordinate systems under orthogonal rotation of axes.

Note that tensors of order zero (i.e. scalars) are all isotropic. Since there are no isotropic [illegible] order 1, therefore, we will discuss the isotropic tensors of second and third orders, which are of [illegible] importance in tensor analysis.

THEOREM (7.17): *Prove that the Kronecker tensor δ_{ij} is an isotropic tensor of order 2.*

PROOF: We know that the equation of transformation for the second order tensor [illegible]

$$A'_{mn} = \ell_{mi}\ell_{nj}A_{ij} \qquad (1)$$

Let $\quad A_{ij} = \delta_{ij}$ in equation (1), then

$$A'_{mn} = \ell_{mi}\ell_{nj}\delta_{ij} = \ell_{mi}\ell_{ni} = \delta_{mn}$$

which shows that the components δ_{ij} transform into themselves under the tensor [illegible] is an isotropic tensor of order 2. This tensor is the most important of all the isotropic tensors [illegible] the only isotropic tensor of order 2 is a scalar multiple of δ_{ij}.

THEOREM (7.18): *Prove that the alternating tensor ϵ_{ijk} is an isotropic tensor of order 3.*

PROOF: The equation of transformation for the third order tensor is

$$A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}A_{ijk} \qquad (1)$$

$A_{ijk} = \epsilon_{ijk}$ in equation (1), then

$A'_{mnp} = l_{mi} l_{nj} l_{pk} \epsilon_{ijk}$

$= l_{m1} l_{n2} l_{p3} + l_{m2} l_{n3} l_{p1} + l_{m3} l_{n1} l_{p2} - l_{m1} l_{n3} l_{p2} - l_{m2} l_{n1} l_{p3} - l_{n3} l_{n2} l_{p1}$

$$= \begin{vmatrix} l_{m1} & l_{n1} & l_{p1} \\ l_{m2} & l_{n2} & l_{p2} \\ l_{m3} & l_{n3} & l_{p3} \end{vmatrix} \qquad (2)$$

...y pair of suffixes m, n, p are equal, the determinant (2) has two equal columns and hence vanishes. ...m, n, and p can take only the values 1, 2 or 3. If the values of m, n, and p are in cyclic order ...m = 1, n = 2, and p = 3, the determinant reduces to the value 1. Since the interchange of any ...of columns in a determinant changes its sign, therefore if the values of m, n, and p are in an ...-cyclic order (say) m = 2, n = 1, and p = 3, then the determinant has the value − 1. ...the following are the combinations of values of m, n, and p for which the value of the determinant ...zero.

Cyclic Order	Anti – cyclic order
m = 1, n = 2, p = 3	m = 2, n = 1, p = 3
m = 2, n = 3, p = 1	m = 1, n = 3, p = 2
m = 3, n = 1, p = 2	m = 3, n = 2, p = 1

...rly, for the other combination of values of m, n, and p, the value of the determinant will be either ...−1. Thus it follows that, for all possible values of m, n, and p, the determinant has the same ...as ϵ_{mnp}.

...ce $A'_{mnp} = \epsilon_{mnp}$

...ich shows that under a rotation of axes, the tensor law is satisfied and each one of the set of the ...mbers ϵ_{ijk} transform into itself. Hence, ϵ_{ijk} is an isotropic tensor of order 3. Note that the only ...otropic tensors of order 3 are scalar multiples of ϵ_{ikm}.

ISOTROPIC TENSORS OF HIGHER ORDER

Similarly, it can be proved that any fourth order isotropic tensor with components A_{ijkl} can be expressed as a sum of the products of delta tensor in the form

$$A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \nu \delta_{il} \delta_{jk}$$

where λ, μ, and ν are arbitrary scalar invariants.

The most general isotropic tensor of order 5 is a linear combination of 10 terms of the form $\epsilon_{ijk} \delta_{lm}$.

The most general isotropic tensor of order 6 is a linear combination of 15 terms of the form $\delta_{ij} \delta_{kl} \delta_{mn}$.

The most general isotropic tensor of order 7 is a linear combination of 105 terms of the form

$\epsilon_{ijk} \delta_{lm} \delta_{np}$

7.26 TENSOR CALCULUS

DIFFERENTIATION OF TENSORS

THEOREM (7.19): If $A_{i_1 i_2 \cdots i_n}$ is a tensor of order n, then its partial derivative w.r.t. x_p, i.e. $\frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n}$ is also a tensor of order $n+1$.

PROOF: The law of transformation for the given tensor is

$$A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \ell_{j_2 i_2} \cdots \ell_{j_n i_n} A_{i_1 i_2 \cdots i_n} \qquad (1)$$

where all the symbols have the usual meanings. Differentiating both sides of equation (1) w.r.t. x'_k we get

$$\frac{\partial}{\partial x'_k} A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \ell_{j_2 i_2} \cdots \ell_{j_n i_n} \frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n} \frac{\partial x_p}{\partial x'_k}$$ where p is dummy.

Also we know that $x'_k = \ell_{kp} x_p$ or $x_p = \ell_{kp} x'_k$

so that $\frac{\partial x_p}{\partial x'_k} = \ell_{kp}$

Hence $$\frac{\partial}{\partial x'_k} A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} \frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n} \qquad (2)$$

which shows that $\frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n}$ is a tensor of order $n+1$.

NOTE: (i) If the partial derivative of $A_{i_1 i_2 \cdots i_n}$ w.r.t. x_p is denoted by $A_{i_1 i_2 \cdots i_n, p}$, then equation (2) can be written in the form

$$A'_{j_1 \cdots j_n, k} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} A_{i_1 \cdots i_n, p} \qquad (3)$$

Differentiating both sides of equation (3) w.r.t. x'_m we can show

$$A'_{j_1 j_2 \cdots j_n, km} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} \ell_{mq} A_{i_1 \cdots i_n, pq}$$

$\left(\text{where } A_{i_1 \cdots i_n, pq} = \frac{\partial^2}{\partial x_q \partial x_p} A_{i_1 \cdots i_n} \right)$ which shows that $A_{i_1 i_2 \cdots i_n, pq}$ is a tensor of order $n+2$.

(ii) If ϕ is a scalar, then $\frac{\partial \phi}{\partial x_i}$ or $\phi_{,i}$ is a tensor of order 1 i.e. a vector.

INTEGRATION OF TENSORS

Integration of a tensor with respect to the coordinate direction yields a tensor of one order higher unless integration is combined with a contraction. For example,

$$\int A'_{mn} \, dx'_p = \int (\ell_{mi} \ell_{nj} A_{ij}) \ell_{pk} \, dx_k = \ell_{mi} \ell_{nj} \ell_{pk} \left(\int A_{ij} \, dx_k \right)$$ and thus $\left(\int A_{ij} \, dx_k \right)$ is a tensor of order 3. However $\left(\int A_{ij} \, dx_j \right)$ is a contraction of $\left(\int A_{ij} \, dx_k \right)$ and is thus of order 1 i.e. one less than A. Integration of a tensor w.r.t. a scalar, for example volume or surface, can be shown to yield a tensor of the same order.

7.27 APPLICATION TO VECTOR ANALYSIS

DOT PRODUCT

Let $\overline{A}$ and $\overline{B}$ be two vectors with components A_1, A_2, A_3 and B_1, B_2, B_3 respectively, then $\overline{A}.\overline{B} = A_1B_1 + A_2B_2 + A_3B_3 = A_iB_i$.

THEOREM (7.20): Prove that if A_i and B_j are the components of two first order tensors $\overline{A}$ and $\overline{B}$ respectively, then A_iB_i is a zeroth order tensor.

PROOF: Since A_i and B_j are the components of first order tensors, therefore under the transformation law from the system K to K', we have

$$A'_m = \ell_{mi}A_i \quad (1)$$

$$B'_n = \ell_{nj}B_j \quad (2)$$

Multiplying equations (1) and (2), we get

$$A'_m B'_n = \ell_{mi}\ell_{nj}A_iB_j \quad (3)$$

Setting $m = n$ in equation (3), we have

$$A'_m B'_m = \ell_{mi}\ell_{mj}A_iB_j = \delta_{ij}A_iB_j = A_iB_i \quad (4)$$

which shows that A_iB_i is a scalar or zeroth order tensor.

NOTE: (i) Equation (4) can be written as

$$A'_1B'_1 + A'_2B'_2 + A'_3B'_3 = A_1B_1 + A_2B_2 + A_3B_3$$

showing that the scalar product of two vectors is invariant under the orthogonal rotation of axes.

(ii) We have already proved that A_iB_j are the components of a second order tensor, whereas in the above theorem we have seen that A_iB_i is a zeroth order tensor. So the difference between A_iB_j and A_iB_i must be carefully observed.

CROSS PRODUCT

Let $\overline{A}$ and $\overline{B}$ be two vectors with components A_1, A_2, A_3 and B_1, B_2, B_3 respectively, then $\overline{C} = \overline{A}\times\overline{B} = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1)$.

We now show that the components of $\overline{C} = \overline{A}\times\overline{B}$ are given by

$$C_i = \epsilon_{ijk}A_jB_k \quad \text{for} \quad i = 1, 2, 3 \quad (1)$$

Now

$$C_1 = \epsilon_{1jk}A_jB_k = \epsilon_{123}A_2B_3 + \epsilon_{132}A_3B_2 = A_2B_3 - A_3B_2 = (\overline{A}\times\overline{B})_1$$

$$C_2 = \epsilon_{2jk}A_jB_k = \epsilon_{231}A_3B_1 + \epsilon_{213}A_1B_3 = A_3B_1 - A_1B_3 = (\overline{A}\times\overline{B})_2$$

$$C_3 = \epsilon_{3jk}A_jB_k = \epsilon_{312}A_1B_2 + \epsilon_{321}A_2B_1 = A_1B_2 - A_2B_1 = (\overline{A}\times\overline{B})_3$$

OTE: From equation (1), we have $C_i \hat{e}_i = \epsilon_{ijk} A_j B_k \hat{e}_i$

e. $\overline{C} = \epsilon_{ijk} A_j B_k \hat{e}_i$

r $\overline{A} \times \overline{B} = \epsilon_{ijk} A_j B_k \hat{e}_i$ where now the summation is over all the indices

HEOREM (7.21): Prove that

(i) the components of $\overline{A} \times \overline{B}$ i.e. $C_i = \epsilon_{ijk} A_j B_k$ transform as the components of a vector under a rotation of the coordinate axes.

(ii) $\overline{A} \times \overline{B}$ is invariant under the rotation of coordinate axes.

ROOF: (i) Let ϵ_{ijk}, A_j, B_k be the components of a third order, and two first order nsors in the system $Ox_1x_2x_3$ and $\epsilon'_{mnp}, A'_n, B'_p$ be their corresponding components in the system $x'_1x'_2x'_3$. Then the laws of transformation are

$$\epsilon'_{mnp} = \ell_{mi} \ell_{nj} \ell_{pk} \epsilon_{ijk} \quad (1)$$

$$A'_n = \ell_{nj} A_j = \ell_{nr} A_r \quad (2)$$

$$B'_p = \ell_{pk} B_k = \ell_{ps} B_s \quad (3)$$

rom equations (1), (2), and (3), we get

$$\begin{aligned} \epsilon'_{mnp} A'_n B'_p &= \ell_{mi} \ell_{nj} \ell_{pk} \epsilon_{ijk} \ell_{nr} A_r \ell_{ps} B_s \\ &= \ell_{mi} (\ell_{nj} \ell_{nr})(\ell_{pk} \ell_{ps}) \epsilon_{ijk} A_r B_s \\ &= \ell_{mi} \delta_{jr} \delta_{ks} \epsilon_{ijk} A_r B_s \\ &= \ell_{mi} \epsilon_{ijk} (\delta_{jr} A_r)(\delta_{ks} B_s) \\ &= \ell_{mi} \epsilon_{ijk} A_j B_k \end{aligned}$$

$$C'_m = \ell_{mi} C_i \quad (4)$$

here $C'_m = \epsilon'_{mnp} A'_n B'_p$

uations (4) shows that the components of $\overline{A} \times \overline{B}$ transform as the components of a vector

) $\overline{A} \times \overline{B} = \epsilon_{ijk} A_j B_k \hat{e}_i$

w $\epsilon'_{mnp} A'_n B'_p = \ell_{mi} \epsilon_{ijk} A_j B_k \quad (5)$

so $\hat{e}'_m = \ell_{mi} \hat{e}_i = \ell_{mr} \hat{e}_r \quad (6)$

om equations (5) and (6), we get

$$\begin{aligned} \epsilon'_{mnp} A'_n B'_p \hat{e}'_m &= \ell_{mi} \epsilon_{ijk} A_j B_k \ell_{mr} \hat{e}_r \\ &= \ell_{mi} \ell_{mr} \epsilon_{ijk} A_j B_k \hat{e}_r \\ &= \delta_{ir} \epsilon_{ijk} A_j B_k \hat{e}_r \\ &= \epsilon_{ijk} A_j B_k (\delta_{ir} \hat{e}_r) = \epsilon_{ijk} A_j B_k \hat{e}_i \end{aligned}$$

... shows that $\overline{A} \times \overline{B}$ is invariant under the rotation of coordinate axes

SCALAR TRIPLE PRODUCT

Considering $\bar{A} \cdot \bar{B} \times \bar{C}$ as the scalar product of $\bar{A}$ and $\bar{B} \times \bar{C}$, we get

$$\bar{A} \cdot \bar{B} \times \bar{C} = A_i (B \times C)_i$$

$$= A_i \epsilon_{ijk} B_j C_k = \epsilon_{ijk} A_i B_j C_k \qquad (1)$$

PROBLEM (7.22): Prove that

(i) $\bar{A} \cdot \bar{B} \times \bar{C} = \bar{B} \cdot \bar{C} \times \bar{A} = \bar{C} \cdot \bar{A} \times \bar{B}$

(ii) $\bar{A} \cdot \bar{B} \times \bar{C} = \bar{A} \times \bar{B} \cdot \bar{C}$

(i) Since $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, therefore,

$$\epsilon_{ijk} A_i B_j C_k = \epsilon_{jki} B_j C_k A_i = \epsilon_{kij} C_k A_i B_j$$

Using equation (1) above, we get

$$\bar{A} \cdot \bar{B} \times \bar{C} = \bar{B} \cdot \bar{C} \times \bar{A} = \bar{C} \cdot \bar{A} \times \bar{B}$$

(ii) $$\bar{A} \cdot \bar{B} \times \bar{C} = \epsilon_{ijk} A_i B_j C_k$$

$$= (\epsilon_{ijk} A_i B_j) C_k = (\bar{A} \times \bar{B})_k C_k = \bar{A} \times \bar{B} \cdot \bar{C}$$

VECTOR TRIPLE PRODUCT

PROBLEM (7.23): Prove that $\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C}) \bar{B} - (\bar{A} \cdot \bar{B}) \bar{C}$

We have

$$[\bar{A} \times (\bar{B} \times \bar{C})]_i = \epsilon_{ijk} A_j (\bar{B} \times \bar{C})_k$$

$$= \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m = \epsilon_{ijk} \epsilon_{lmk} A_j B_l C_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j B_l C_m$$

$$= \delta_{il} \delta_{jm} A_j B_l C_m - \delta_{im} \delta_{jl} A_j B_l C_m$$

$$= A_j B_i C_j - A_j B_j C_i = (A_j C_j) B_i - (A_j B_j) C_i$$

$$= (\bar{A} \cdot \bar{C}) B_i - (\bar{A} \cdot \bar{B}) C_i$$

This gives the three components of the required formula for $i = 1, 2, 3$.

$$\therefore \bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C}) \bar{B} - (\bar{A} \cdot \bar{B}) \bar{C}$$

THEOREM (7.24): Prove that

(i) the components of the del–operator $\nabla \left(\text{i.e. } \frac{\partial}{\partial x_i} \right)$ transform as the component of a vector under a rotation of the coordinate axes.

(ii) the vector del–operator ∇ is invariant under the rotation of the coordinate axes.

PROOF: (i) Let $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j'}$ be the components of the del – operator ∇ in the system $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$ respectively. Let x_i and x_j' be the coordinates of a point in these systems, then we know that

$$x_1 = \ell_{11} x_1' + \ell_{21} x_2' + \ell_{31} x_3'$$

$$x_2 = \ell_{12} x_1' + \ell_{22} x_2' + \ell_{32} x_3'$$

$$x_3 = \ell_{13} x_1' + \ell_{23} x_2' + \ell_{33} x_3'$$

Using the chain rule, we have

$$\frac{\partial}{\partial x_1'} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x_1'} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x_1'} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x_1'}$$

$$= \ell_{11}\frac{\partial}{\partial x_1} + \ell_{12}\frac{\partial}{\partial x_2} + \ell_{13}\frac{\partial}{\partial x_3}$$

$$= \ell_{1i}\frac{\partial}{\partial x_i} \qquad (1)$$

Similarly,

$$\frac{\partial}{\partial x_2'} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x_2'} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x_2'} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x_2'}$$

$$= \ell_{21}\frac{\partial}{\partial x_1} + \ell_{22}\frac{\partial}{\partial x_2} + \ell_{23}\frac{\partial}{\partial x_3}$$

$$= \ell_{2i}\frac{\partial}{\partial x_i} \qquad (2)$$

and

$$\frac{\partial}{\partial x_3'} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x_3'} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x_3'} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x_3'}$$

$$= \ell_{31}\frac{\partial}{\partial x_1} + \ell_{32}\frac{\partial}{\partial x_2} + \ell_{33}\frac{\partial}{\partial x_3}$$

$$= \ell_{3i}\frac{\partial}{\partial x_i} \qquad (3)$$

From equations (1), (2), and (3), we get

$$\frac{\partial}{\partial x_j'} = \ell_{ji}\frac{\partial}{\partial x_i} \qquad (4)$$

which shows that under a rotation of the coordinate axes, the components of the del operator ∇ transform as components of a vector. It is often called a vector operator.