Squaring equations (1) and (2) and adding , we get

$$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2 \qquad (4)$$

or $$r = \sqrt{x^2+y^2} \quad (\text{since } r \text{ is positive})$$

Dividing equation (2) by equation (1) , we get

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \qquad (5)$$

or $$\theta = \tan^{-1}\frac{y}{x}$$

Hence , the equations expressing the cylinderical polar coordinates in terms of Cartesian coordinates are:

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x}, \quad z = z \qquad (6)$$

NOTE: For points on the z–axis ($x = 0$, $y = 0$), θ is indeterminate . Such points are called singular points of the transformation .

EXAMPLE (2): If r, θ, z are cylinderical polar coordinates , describe each of the following loci and write the equation of each locus in rectangular coordinates :

(i) $r = 4$ (ii) $\theta = \frac{\pi}{2}$ (iii) $z = 3$

(iv) $\theta = \frac{\pi}{3}, z = 1$ (v) $r = 4, z = 0$ (vi) $r = 2, \theta = \frac{\pi}{6}$

SOLUTION: In cylinderical coordinates , $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x}, \quad z = z$$

(i) $r = 4$

Here r is fixed while θ and z vary . We can write the given equation as

$$\sqrt{x^2+y^2} = 4$$

or $$x^2+y^2 = 16$$

i.e. the given equation represents a cylinder with axis as the z–axis and radius 4 .

(ii) $\theta = \frac{\pi}{2}$

Here θ is fixed while r and z vary . We can write the given equation as

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{2}$$

or $$\frac{y}{x} = \tan\frac{\pi}{2} = \infty$$

This implies $x = 0$ i.e. the given equation represents the yz–plane where $y \geq 0$.

(iii) $z = 3$

Here z is fixed while r and θ vary . The given equation represents a plane parallel to the xy – plane at a distance 3 units from the origin

(iv) $\theta = \frac{\pi}{3}, \quad z = 1$

Here θ and z are fixed while only r varies. We can write $\theta = \frac{\pi}{3}$ as

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{3} \quad \text{or} \quad \frac{y}{x} = \tan\frac{\pi}{3} = \sqrt{3}$$

or $\quad y = \sqrt{3}\,x, \quad z = 1$

i.e. the given equations represent a straight line $y = \sqrt{3}\,x$, in the plane $z = 1$ where $x \geq 0$

(v) $r = 4, \quad z = 0$

Here r and z are fixed while only θ varies. We can write $r = 4$ as

$$\sqrt{x^2 + y^2} = 4 \quad \text{or} \quad x^2 + y^2 = 16, \quad z = 0$$

i.e. the given equations represent a circle with centre at the origin and radius 4 in the xy-plane

(vi) $r = 2, \quad \theta = \frac{\pi}{6}$

Here r and θ are fixed while only z varies. We can write $r = 2$ as

$$\sqrt{x^2 + y^2} = 2 \quad \text{or} \quad x^2 + y^2 = 4$$

and $\theta = \frac{\pi}{6}$ as $\tan^{-1}\frac{y}{x} = \frac{\pi}{6}$ or $\frac{y}{x} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, or $y = \frac{1}{\sqrt{3}}x$

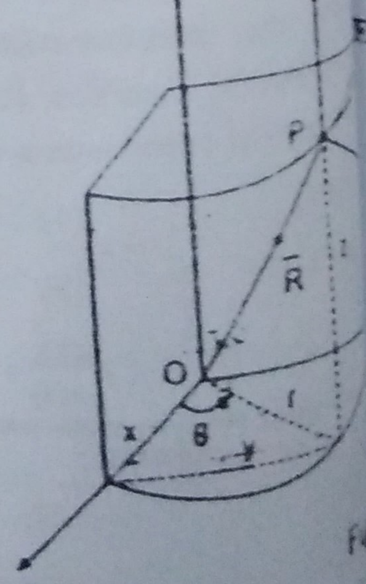
i.e. the given equations represent a straight line parallel to the z-axis and passing through intersection of the circle $x^2 + y^2 = 4$ and the straight line $y = \frac{1}{\sqrt{3}}x$.

6.15 UNIT VECTORS IN CYLINDERICAL COORDINATE SYSTEM

The position vector of any point P in cylinderical polar coordinates is

$$\bar{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k}$$



The tangent vectors in the directions of r, θ, and z respectively, are given by

$$\frac{\partial \bar{R}}{\partial r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\frac{\partial \bar{R}}{\partial \theta} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$$

$$\frac{\partial \bar{R}}{\partial z} = \hat{k}$$

The unit vectors in these directions of r, θ, and z are given by

$$\hat{e}_r = \frac{\frac{\partial \bar{R}}{\partial r}}{\left|\frac{\partial \bar{R}}{\partial r}\right|} = \frac{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}}{\sqrt{\cos^2\theta + \sin^2\theta}} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad (1)$$

$$\hat{e}_\theta = \frac{\frac{\partial \bar{R}}{\partial \theta}}{\left|\frac{\partial \bar{R}}{\partial \theta}\right|} = \frac{-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \quad (2)$$

$$\hat{e}_z = \frac{\frac{\partial \bar{R}}{\partial z}}{\left|\frac{\partial \bar{R}}{\partial z}\right|} = \hat{k} \quad (3)$$

In matrix notation, equations (1), (2), and (3) can be written as

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \quad (4)$$

SCALE FACTORS

The scale factors for the cylinderical coordinate system are given by

$$h_1 = h_r = \left|\frac{\partial \bar{R}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \quad (5)$$

$$h_2 = h_\theta = \left|\frac{\partial \bar{R}}{\partial \theta}\right| = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}$$

$$= r\sqrt{\sin^2\theta + \cos^2\theta} = r \quad (6)$$

$$h_3 = h_z = \left|\frac{\partial \bar{R}}{\partial z}\right| = |\hat{k}| = 1 \quad (7)$$

Hence the scale factors are : $h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$

THEOREM (6.2): In cylinderical polar coordinates, show that

$$\frac{\partial \hat{e}_r}{\partial r} = 0, \quad \frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_r}{\partial z} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r, \quad \frac{\partial \hat{e}_\theta}{\partial z} = 0$$

$$\frac{\partial \hat{e}_z}{\partial \theta} = 0, \quad \frac{\partial \hat{e}_z}{\partial z} = 0$$

PROOF:

$\hat{e}_r$ ... $\hat{e}_z = \hat{k}$

Then $\frac{\partial \hat{e}_r}{\partial r} = 0, \quad \frac{\partial \hat{e}_r}{\partial \theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_r}{\partial z} = 0$

$\frac{\partial \hat{e}_\theta}{\partial r} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{e}_r, \quad \frac{\partial \hat{e}_\theta}{\partial z} = 0$

$\frac{\partial \hat{e}_z}{\partial r} = 0, \quad \frac{\partial \hat{e}_z}{\partial \theta} = 0, \quad \frac{\partial \hat{e}_z}{\partial z} = 0$

THEOREM (6.3): Prove that in cylinderical polar coordinates

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\,\hat{e}_\theta, \quad \frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\,\hat{e}_r, \quad \frac{d}{dz}\hat{e}_z = \bar{0}$$

where dots denote differentiation w.r.t. time t.

PROOF: We know that $\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}, \quad \hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}, \quad \hat{e}_z = \hat{k}$

Then $\frac{d}{dt}\hat{e}_r = -\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{d\theta}{dt} = \dot{\theta}\,\hat{e}_\theta$

$\frac{d}{dt}\hat{e}_\theta = -\cos\theta\frac{d\theta}{dt}\hat{i} - \sin\theta\frac{d\theta}{dt}\hat{j} = -(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\frac{d\theta}{dt} = -\dot{\theta}\,\hat{e}_r$

and $\frac{d}{dz}\hat{e}_z = \frac{d}{dz}\hat{k} = \bar{0}$

6.16 ORTHOGONALITY OF CYLINDERICAL COORDINATE SYSTEM

We know that the unit vectors in cylinderical polar coordinates are

$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}, \quad \hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}, \quad$ and $\hat{e}_z = \hat{k}$

Then $\hat{e}_r \cdot \hat{e}_\theta = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\cdot(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$

$\hat{e}_\theta \cdot \hat{e}_z = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\cdot(\hat{k}) = 0$

$\hat{e}_r \cdot \hat{e}_z = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\cdot(\hat{k}) = 0$

and so $\hat{e}_r, \hat{e}_\theta$, and $\hat{e}_z$ are mutually perpendicular and the coordinate system is orthogonal

6.17 RELATIONSHIPS AMONG UNIT VECTORS IN CYLINDERICAL SYSTEM

THEOREM (6.4): Prove that for cylinderical coordinate system.

$\hat{e}_r \cdot \hat{e}_r = \hat{e}_\theta \cdot \hat{e}_\theta = \hat{e}_z \cdot \hat{e}_z = 1$

$\hat{e}_r \times \hat{e}_r = \hat{e}_\theta \times \hat{e}_\theta = \hat{e}_z \times \hat{e}_z = \bar{0}$

$\hat{e}_r \times \hat{e}_\theta = \hat{e}_z, \quad \hat{e}_\theta \times \hat{e}_z = \hat{e}_r, \quad \hat{e}_z \times \hat{e}_r = \hat{e}_\theta$

PROOF: We know that $\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}, \quad \hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}, \quad \hat{e}_z = \hat{k}$

$\hat{e}_r \cdot \hat{e}_r = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\cdot(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = \cos^2\theta + \sin^2\theta = 1$

$$\hat{e}_\theta \cdot \hat{e}_\theta = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \cdot (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) = \sin^2\theta + \cos^2\theta = 1$$

$$\hat{e}_z \cdot \hat{e}_z = \hat{k} \cdot \hat{k} = 1$$

$$\hat{e}_r \times \hat{e}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \bar{0}$$

$$\hat{e}_\theta \times \hat{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \bar{0}$$

$$\hat{e}_z \times \hat{e}_z = \hat{k} \times \hat{k} = \bar{0}$$

$$\hat{e}_r \times \hat{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + (\cos^2\theta + \sin^2\theta)\,\hat{k} = \hat{k} = \hat{e}_z$$

$$\hat{e}_\theta \times \hat{e}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 0\,\hat{k} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} = \hat{e}_r$$

$$\hat{e}_z \times \hat{e}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + 0\,\hat{k} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{e}_\theta$$

6.18 CARTESIAN UNIT VECTORS IN TERMS OF CYLINDERICAL UNIT VECTORS

We know that

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad (1)$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \qquad (2)$$

$$\hat{e}_z = \hat{k} \qquad (3)$$

Multiplying equation (1) by $\cos\theta$ and equation (2) by $\sin\theta$ and then subtracting, we get

$$\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta = (\cos^2\theta + \sin^2\theta)\,\hat{i}$$

or $$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \qquad (4)$$

Multiplying equation (1) by $\sin\theta$ and equation (2) by $\cos\theta$ and then adding, we get

$$\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = (\sin^2\theta + \cos^2\theta)\,\hat{j}$$

or $$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta \qquad (5)$$

Also $$\hat{k} = \hat{e}_z \qquad (6)$$

In matrix notation, equations (4), (5), and (6) can be written as

$$\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \end{bmatrix} \qquad (7)$$

6.19 POSITION VECTOR IN CYLINDERICAL COORDINATES

We know that in cylinderical polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, ...

and $\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$, $\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$, $\hat{k} = \hat{e}_z$

Then
$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= r\cos\theta\left(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta\right) + r\sin\theta\left(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta\right) + z\hat{e}_z$$
$$= r(\cos^2\theta + \sin^2\theta)\,\hat{e}_r + (-r\cos\theta\sin\theta + r\sin\theta\cos\theta)\,\hat{e}_\theta + z\hat{e}_z$$
$$= r\hat{e}_r + z\hat{e}_z$$

6.20 RELATIONSHIPS BETWEEN CARTESIAN AND CYLINDERICAL COMPONENTS OF A VECTOR

A vector $\bar{A}$ in rectangular Cartesian components is written as

$$\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k} \qquad (1)$$

Substituting the values of $\hat{i}, \hat{j}, \hat{k}$ in terms of $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$, we get

$$\bar{A} = A_1\left(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta\right) + A_2\left(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta\right) + A_3\hat{e}_z$$
$$= (A_1\cos\theta + A_2\sin\theta)\,\hat{e}_r + (-A_1\sin\theta + A_2\cos\theta)\,\hat{e}_\theta + A_3\hat{e}_z$$

But in cylinderical components, we have

$$\bar{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_z\hat{e}_z$$

Comparing coefficients, we get

$$\left.\begin{aligned} A_r &= A_1\cos\theta + A_2\sin\theta \\ A_\theta &= -A_1\sin\theta + A_2\cos\theta \\ A_z &= A_3 \end{aligned}\right] \qquad (2)$$

In matrix notation, equations (2) can be written as

$$\begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \qquad (3)$$

Similarly, the vector $\bar{A}$ in cylinderical components can be written as

$$\bar{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_z\hat{e}_z \qquad (4)$$

Substituting the values of $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ in terms of $\hat{i}, \hat{j}, \hat{k}$, we get

$$\bar{A} = A_r\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right) + A_\theta\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right) + A_z\hat{k}$$
$$= (A_r\cos\theta - A_\theta\sin\theta)\,\hat{i} + (A_r\sin\theta + A_\theta\cos\theta)\,\hat{j} + A_z\hat{k}$$

Again, because $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, comparing coefficients, we get

$$\left.\begin{aligned} A_1 &= A_r\cos\theta - A_\theta\sin\theta \\ A_2 &= A_r\sin\theta + A_\theta\cos\theta \end{aligned}\right] \qquad (5)$$

...rix notation , equations (5) can be written as

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix} \qquad (6)$$

1 EXPRESSIONS FOR ARC LENGTH , AREA , AND VOLUME ELEMENTS IN CYLINDERICAL POLAR COORDINATES

In cylinderical polar coordinates , we have

$u_1 = r, \quad u_2 = \theta, \quad u_3 = z, \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1$

ARC LENGTH ELEMENT

In orthogonal curvilinear coordinates , the element of arc length is determined from

$$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2$$

...cylinderical polar coordinates , this becomes

$$(ds)^2 = (1)^2(dr)^2 + (r)^2(d\theta)^2 + (1)^2(dz)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$$

ALTERNATIVE METHOD

In cylinderical polar coordinates , $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dz = dz$$

Then
$$\begin{aligned} ds^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 + (dz)^2 \\ &= (\cos^2\theta + \sin^2\theta)(dr)^2 + (r^2\sin^2\theta + r^2\cos^2\theta)(d\theta)^2 \\ &\qquad - 2r\sin\theta\cos\theta\, dr\, d\theta + 2r\sin\theta\cos\theta\, dr\, d\theta + (dz')^2 \\ &= (dr)^2 + r^2(d\theta)^2 + (dz)^2 \end{aligned}$$

AREA ELEMENT

We know that the elements of area in orthogonal curvilinear coordinates are :

$dA_1 = h_2 h_3\, du_2\, du_3, \quad dA_2 = h_1 h_3\, du_1\, du_3$ and $dA_3 = h_1 h_2\, du_1\, du_2$

In cylinderical polar coordinates , these becomes

$$dA_1 = (r)(1)\,d\theta\, dz = r\, d\theta\, dz$$

$$dA_2 = (1)(1)\,dr\, dz = dr\, dz$$

and
$$dA_3 = (1)(r)\,dr\, d\theta = r\, dr\, d\theta .$$

VOLUME ELEMENT

The volume element in orthogonal curvilinear coordinates is :

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3$$

rical polar coordinates , this becomes

$$dV = (1)(r)(1)\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$$

EXPRESSION FOR JACOBIAN IN CYLINDERICAL POLAR COORDINATES

We know that the Jacobian in orthogonal curvilinear coordinates u_1, u_2, u_3 is given by

$$J = J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = h_1 h_2 h_3$$

derical coordinates $h_1 = 1,\ h_2 = r,\ h_3 = 1$ and $u_1 = r,\ u_2 = \theta,\ u_3 = z$

ore $\quad J\left(\frac{x, y, z}{r, \theta, z}\right) = (1)(r)(1) = r.$

EXPRESSIONS FOR $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ IN CYLINDERICAL COORDINATES

We know that $x = r\cos\theta,\ y = r\sin\theta,\ z = z$ and $r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right), z = z$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{r\sin\theta}{r} = \sin\theta\ ,$$

$$\frac{\partial r}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}\ ,$$

$$\frac{\partial \theta}{\partial z} = 0$$

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial z} = 1$$

w by the chain rule , we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial z}$$

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad (1)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial z}$$

$$= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \qquad (2)$$

nd

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z} = (0)\frac{\partial}{\partial r} + (0)\frac{\partial}{\partial \theta} + (1)\frac{\partial}{\partial z}$$

$$= \frac{\partial}{\partial z} \qquad (3)$$

Equations (1) , (2) , and (3) are the required expressions in terms of cylinderical polar coordinates cylinderical polar coordinates, this becomes

$$dV = (1)(r)(1)\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$$

22 EXPRESSION FOR JACOBIAN IN CYLINDERICAL POLAR COORDINATES

We know that the Jacobian in orthogonal curvilinear coordinates u_1, u_2, u_3 is given by

$$J = J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = h_1 h_2 h_3$$

cylinderical coordinates $h_1 = 1, \quad h_2 = r, \quad h_3 = 1$ and $u_1 = r, \quad u_2 = \theta, \quad u_3 = z$

herefore $J\left(\frac{x, y, z}{r, \theta, z}\right) = (1)(r)(1) = r.$

.23 EXPRESSIONS FOR $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ IN CYLINDERICAL COORDINATES

We know that $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $z = z$

Then
$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta,$$

$$\frac{\partial r}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r},$$

$$\frac{\partial \theta}{\partial z} = 0$$

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial z} = 1$$

Now by the chain rule, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial z}$$

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad (1)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial z}$$

$$= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \qquad (2)$$

and
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z} = (0)\frac{\partial}{\partial r} + (0)\frac{\partial}{\partial \theta} + (1)\frac{\partial}{\partial z}$$

$$= \frac{\partial}{\partial z} \qquad (3)$$

...d (3) are the required expressions in terms of cylinderical polar coordinates.