arly $\nabla \times (A_2 \hat{e}_2) = \frac{1}{h_1 h_2} \frac{\partial}{\partial u_1} (A_2 h_2) \hat{e}_3 - \frac{1}{h_2 h_3} \frac{\partial}{\partial u_3} (A_2 h_2) \hat{e}_1$

$$\nabla \times (A_3 \hat{e}_3) = \frac{1}{h_2 h_3} \frac{\partial}{\partial u_2} (A_3 h_3) \hat{e}_1 - \frac{1}{h_3 h_1} \frac{\partial}{\partial u_1} (A_3 h_3) \hat{e}_2$$

equation (10) becomes

$$\nabla \times \bar{A} = \frac{\hat{e}_1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right] + \frac{\hat{e}_2}{h_3 h_1} \left[\frac{\partial}{\partial u_3} (A_1 h_1) - \frac{\partial}{\partial u_1} (A_3 h_3) \right]$$
$$+ \frac{\hat{e}_3}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (A_2 h_2) - \frac{\partial}{\partial u_2} (A_1 h_1) \right]$$

can be written as

$$\nabla \times \bar{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \qquad (11)$$

EXPRESSION FOR LAPLACIAN

We know that

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \hat{e}_3$$

$$\nabla . \bar{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right]$$

$\bar{A} = \nabla \psi$, then $A_1 = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1}$, $A_2 = \frac{1}{h_2} \frac{\partial \psi}{\partial u_2}$, $A_3 = \frac{1}{h_3} \frac{\partial \psi}{\partial u_3}$ and thus

$$\nabla . \bar{A} = \nabla . \nabla \psi = \nabla^2 \psi$$
$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right] \qquad (12)$$

AMPLE (1): Consider the curvilinear coordinate system defined for $z \geq 0$ by

$x = u_1 - u_2$, $y = u_1 + u_2$, $z = u_3^2$

(i) Find the unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ and show that the system is orthogonal and right–handed. Also find the scale factors h_1, h_2, h_3.

(ii) Find the expressions for $(ds)^2$ and dV.

(iii) Find $\nabla \psi$ in this system for $\psi(u_1, u_2, u_3) = u_1 u_2 + u_3^2$.

Given that $\psi(u_1, u_2, u_3) = u_1 u_2 + u_3^2$, therefore $\frac{\partial \psi}{\partial u_1} = u_2$, $\frac{\partial \psi}{\partial u_2} = u_1$, $\frac{\partial \psi}{\partial u_3} = 2u_3$

that in an orthogonal curvilinear coordinate system

$$\nabla \psi = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}\hat{e}_3$$

$$= \frac{1}{\sqrt{2}}u_2\hat{e}_1 + \frac{1}{\sqrt{2}}u_1\hat{e}_2 + \frac{1}{2u_3}2u_3\hat{e}_3 = \frac{1}{\sqrt{2}}(u_2\hat{e}_1 + u_1\hat{e}_2) + \hat{e}_3$$

Given that $\bar{A} = u_3 u_1 \hat{e}_1 + u_3 u_2 \hat{e}_2 + u_1 u_2 \hat{e}_3$

that in orthogonal curvilinear coordinate system,

$$\nabla . \bar{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]$$

$$= \frac{1}{(\sqrt{2})(\sqrt{2})(2u_3)}\left[\frac{\partial}{\partial u_1}((u_3 u_1)(2\sqrt{2}u_3)) + \frac{\partial}{\partial u_2}(u_3 u_2 (2u_3)\sqrt{2}) + \frac{\partial}{\partial u_3}(u_1 u_2 (2))\right]$$

$$= \frac{1}{4u_3}(2\sqrt{2}u_3^2 + 2\sqrt{2}u_3^2 + 0) = \frac{1}{4u_3}(4\sqrt{2}u_3^2) = \sqrt{2}u_3$$

$$\nabla \times \bar{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}\sqrt{2}(2u_3)}\begin{vmatrix} \sqrt{2}\hat{e}_1 & \sqrt{2}\hat{e}_2 & 2u_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ \sqrt{2}u_3 u_1 & \sqrt{2}u_3 u_2 & 2u_1 u_2 u_3 \end{vmatrix}$$

$$= \frac{1}{4u_3}[\sqrt{2}\hat{e}_1(2u_1 u_3 - \sqrt{2}u_2) + \sqrt{2}\hat{e}_2(\sqrt{2}u_1 - 2u_2 u_3) + 0]$$

$$= \frac{1}{2u_3}[(\sqrt{2}u_1 u_3 - u_2)\hat{e}_1 + (u_1 - \sqrt{2}u_2 u_3)\hat{e}_2]$$

We know that in orthogonal curvilinear coordinate system,

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial u_3}\right)\right]$$

$$= \frac{1}{(\sqrt{2})(\sqrt{2})(2u_3)}\left[\frac{\partial}{\partial u_1}\left(\frac{2\sqrt{2}u_3}{\sqrt{2}}(3u_1^2)\right) + \frac{\partial}{\partial u_2}\left(\frac{2\sqrt{2}u_3}{\sqrt{2}}(3u_2^2)\right)\right.$$

$$\left. + \frac{\partial}{\partial u_3}\left(\frac{(\sqrt{2})(\sqrt{2})}{2u_3}(3u_3^2)\right)\right]$$

RECTANGULAR CARTESIAN COORDINATES

Let $P(x, y, z)$ be any point whose projection on the lane is $Q(x, y)$. Then the rectangular Cartesian dinates (x, y, z) of P are defined as $x = OR$, $y = RQ$, $z = QP$ as shown in figure (6.5). In rectangular Cartesian dinate system, the unit vectors are denoted by $\hat{i}, \hat{j}$, and $\hat{k}$ any vector $\overline{A}$ can be represented in terms of these unit ors as:

$$\overline{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

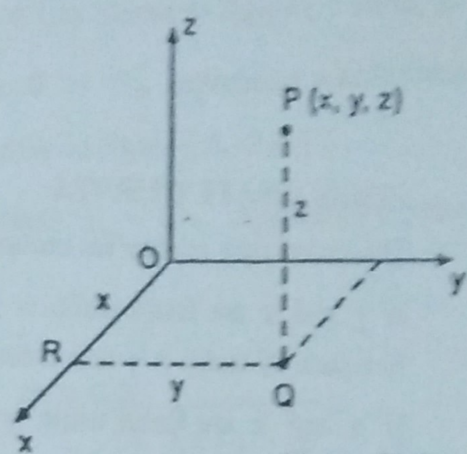


Figure (6.5)

position vector $\overline{r}$ in this system is given by $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$
he scale factors are given by

$$h_1 = \left|\frac{\partial \overline{r}}{\partial x}\right| = |\hat{i}| = 1, \quad h_2 = \left|\frac{\partial \overline{r}}{\partial y}\right| = |\hat{j}| = 1, \quad h_3 = \left|\frac{\partial \overline{r}}{\partial z}\right| = |\hat{k}| = 1$$ *

the rectangular Cartesian coordinate system is a particular case of an orthogonal curvilinear dinate system where $u_1 = x$, $u_2 = y$, $u_3 = z$ and $h_1 = 1$, $h_2 = 1$, $h_3 = 1$.

COORDINATE SURFACES

In rectangular Cartesian coordinate system, the coordinate surfaces are:

If x is held constant while y and z vary, then the equation $x = C_1$ represents a plane parallel to the yz–plane as shown in figure [6.6 (a)].

If y is held constant while x and z vary, then the equation $y = C_2$ represents a plane parallel to the zx–plane as shown in figure [6.6 (b)].

If z is held constant while x and y vary, then the equation $z = C_3$ represents a plane parallel to the xy–plane as shown in figure [6.6 (c)].

the coordinate surfaces are mutually orthogonal in the sense that any two of them intersect at right es. Furthermore, each point in this system is the intersection of the three coordinate surfaces C_1, $y = C_2$, and $z = C_3$.

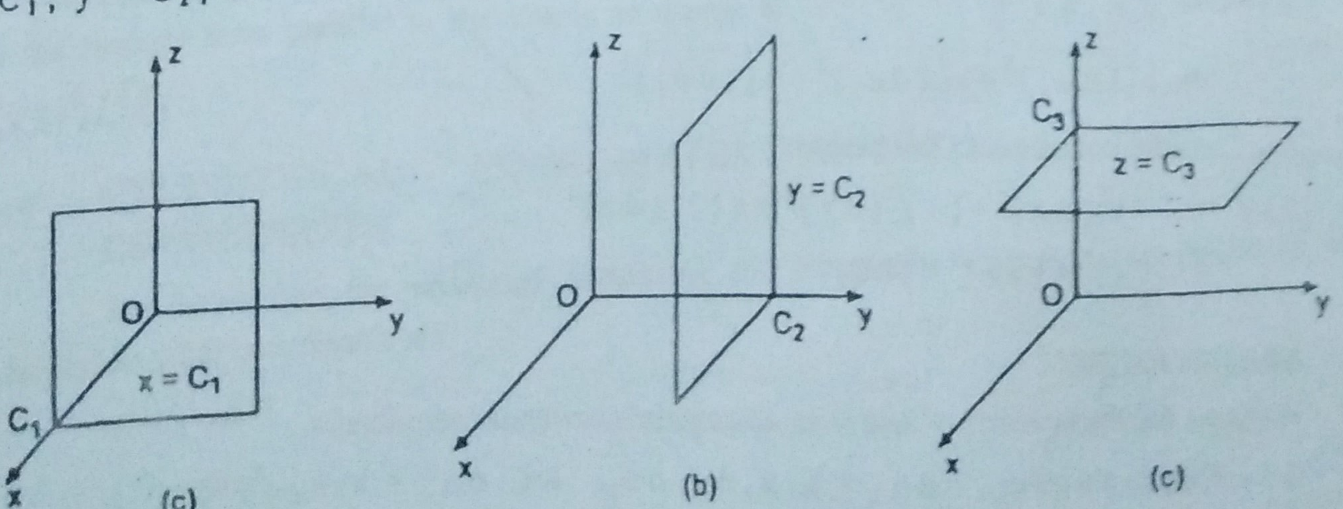


Figure (6.6)

COORDINATE CURVES

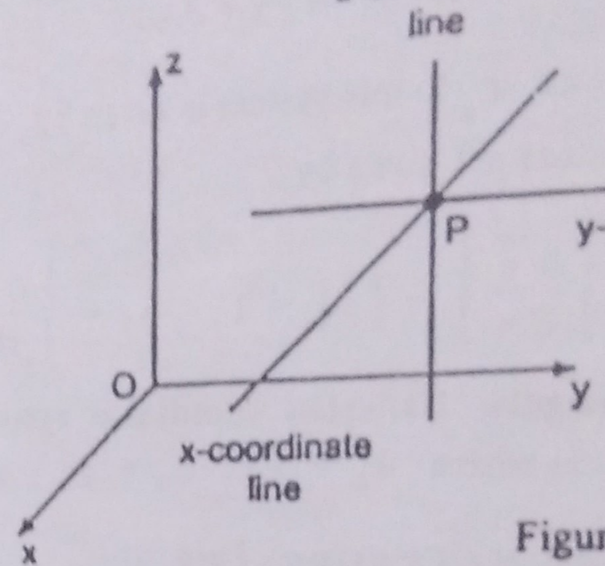
The coordinate curves for rectangular Cartesian coordinate system are :

(i) If y and z are fixed while x varies , then the intersection of $y = C_2$ and $z = C_3$ [...] line parallel to the x – axis called the x – coordinate curve (or line) .

(ii) If x and z are fixed while y varies , then the intersection of $x = C_1$ and $z = C_3$ is a straight line parallel to the y – axis called the y – coordinate curve (or line) .

(iii) If x and y are fixed while z varies , then the intersection of $x = C_1$ and $y = C_2$ is a straight line parallel to the z – axis called the z – coordinate curve (or line) .

Thus the coordinate curves of the rectangular Cartesian coordinate system are the straight lines passing through the point P as shown in figure (6.7).



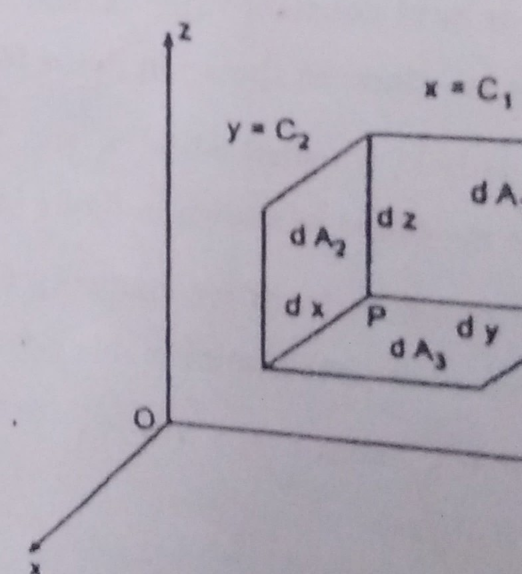
Figur

6.10 EXPRESSIONS FOR ARC LENGTH , AREA , AND VOLUME ELEM[...] RECTANGULAR CARTESIAN COORDINATES

We know that for rectangular Cartesian coordinates

$$u_1 = x, \quad u_2 = y, \quad u_3 = z$$

$$h_1 = 1, \quad h_2 = 1, \quad h_3 = 1$$



Figu

ARC LENGTH ELEMENT

In orthogonal curvilinear coordinate system , the element of arc length is determined from

$$(ds)^2 = d\vec{r} \cdot d\vec{r}$$

$$= h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2$$

In rectangular Cartesian coordinates , this becomes

$$(ds)^2 = (1)^2 (dx)^2 + (1)^2 (dy)^2 + (1)^2 (dz)^2$$

$$= (dx)^2 + (dy)^2 + (dz)^2 \qquad (1)$$

AREA ELEMENT

We know that the elements of area in an orthogonal curvilinear coordinates

$$dA_1 = h_2 h_3 du_2 du_3, \quad dA_2 = h_1 h_3 du_1 du_3, \quad \text{and} \quad dA_3 = h_1 h_2 du_1 du_2$$

In rectangular Cartesian coordinates , these become

$$dA_1 = (1)(1) dy dz = dy dz$$

$$dA_2 = (1)(1) dx dz = dx dz$$

VOLUME ELEMENT

We know that the volume element in orthogonal curvilinear coordinates is given by

$$dV = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3$$

In rectangular Cartesian coordinates , this becomes (3)

$$dV = (1)(1)(1)\, dx\, dy\, dz = dx\, dy\, dz$$

6.11 EXPRESSION FOR JACOBIAN IN RECTANGULAR CARTESIAN COORDINATES

We know that the Jacobian in orthogonal curvilinear coordinates is given by

$$J = h_1 h_2 h_3$$

In rectangular Cartesian coordinates , this becomes (4)

$$J = (1)(1)(1) = 1 .$$

6.12 EXPRESSIONS FOR GRADIENT , DIVERGENCE , CURL , AND LAPLACIAN IN RECTANGULAR CARTESIAN COORDINATES

We know that for rectangular Cartesian coordinates

$$u_1 = x, \quad u_2 = y, \quad u_3 = z; \quad \hat{e}_1 = \hat{i}, \quad \hat{e}_2 = \hat{j}, \quad \hat{e}_3 = \hat{k}$$

$$h_1 = 1, \quad h_2 = 1, \quad h_3 = 1; \quad \overline{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

EXPRESSION FOR GRADIENT

We know that in orthogonal curvilinear coordinates , we have

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \hat{e}_3$$

In rectangular Cartesian coordinates , this becomes

$$\nabla \psi = \frac{1}{1} \frac{\partial \psi}{\partial x} \hat{i} + \frac{1}{1} \frac{\partial \psi}{\partial y} \hat{j} + \frac{1}{1} \frac{\partial \psi}{\partial z} \hat{k} = \frac{\partial \psi}{\partial x} \hat{i} + \frac{\partial \psi}{\partial y} \hat{j} + \frac{\partial \psi}{\partial z} \hat{k} \qquad (5)$$

EXPRESSION FOR DIVERGENCE

We know that in orthogonal curvilinear coordinates , we have

$$\nabla \cdot \overline{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_3 h_1 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

In rectangular Cartesian coordinates , this becomes

$$\nabla \cdot \overline{A} = \frac{1}{(1)(1)(1)} \left[\frac{\partial}{\partial x} ((1)(1) A_1) + \frac{\partial}{\partial y} ((1)(1) A_2) + \frac{\partial}{\partial z} ((1)(1) A_3) \right]$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \qquad (6)$$

...ESSION FOR CURL

...ow that in orthogonal curvilinear coordinates, we have

$$... = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

...Cartesian coordinates, this becomes

$$... = \frac{1}{(1)(1)(1)}\begin{vmatrix} (1)\hat{i} & (1)\hat{j} & (1)\hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (1)A_1 & (1)A_2 & (1)A_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k} \qquad (7)$$

...RESSION FOR LAPLACIAN

...now that in orthogonal curvilinear coordinates, we have

$$... = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial u_3}\right)\right]$$

...Cartesian coordinates, this becomes

$$... = \frac{1}{(1)(1)(1)}\left[\frac{\partial}{\partial x}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial z}\right)\right]$$

$$= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \qquad (8)$$

...INDERICAL POLAR COORDINATES

... $P(x, y, z)$ be any point whose ... the xy-plane is $Q(x, y)$. Then the ...oordinates of P are (r, θ, z) in ... $Q, \theta = \angle XOQ$ and $z = QP$. From ... (6.9), the transformation equations ... rectangular Cartesian coordinates in ...derical polar coordinates are:

$... r\cos\theta$ (1)

$... r\sin\theta$ (2)

$... z$ (3)

~~$0 \le \theta \le 2\pi$ and $-\infty < z < \infty$~~

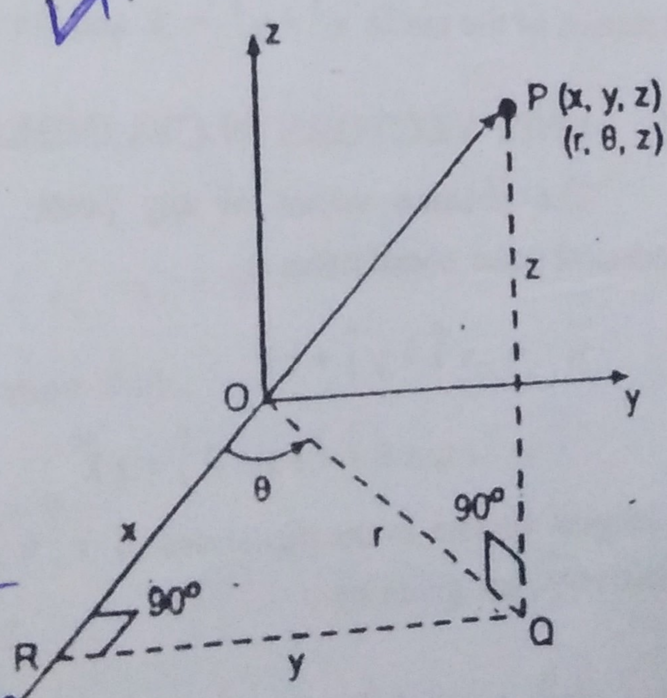


Figure (6.9)

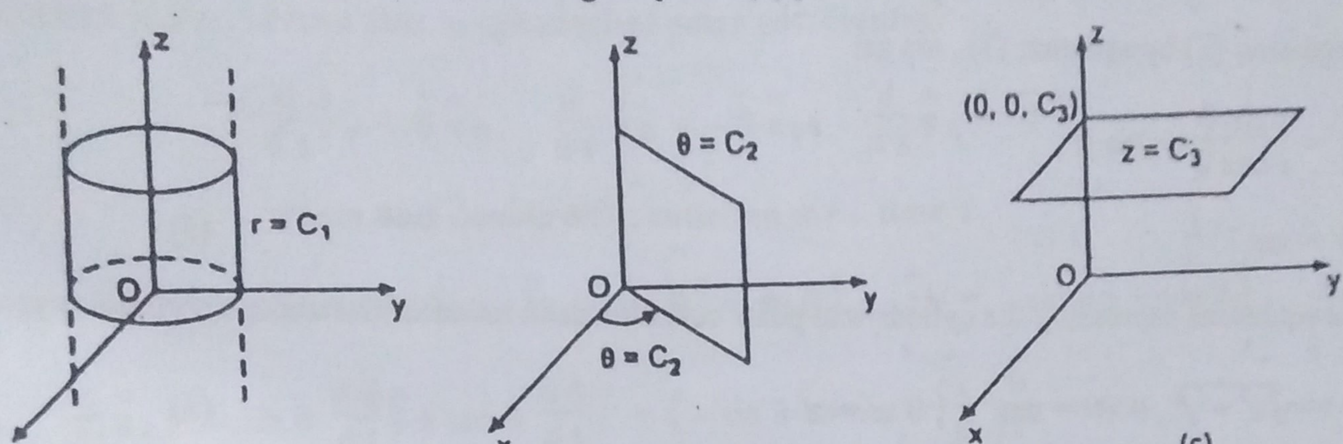
...ACES

...system, the coordinate surfaces are:

If r is held constant while θ and z vary, then the equation $r = C_1$ represents a right circular cylinder of radius C_1 and axis along z–axis (or z–axis if $C_1 = 0$) as shown in figure [6.10 (a)].

If θ is held constant while r and z vary, then the equation $\theta = C_2$ represents a half plane through the z–axis making an angle θ with the xz – plane as shown in figure [6.10 (b)].

If z is held constant, while r and θ vary, then the equation $z = C_3$ represents a plane perpendicular to z–axis as shown in figure [6.10 (c)].



(a) (b) (c)

Figure (6.10)

COORDINATE CURVES

The coordinate curves for cylinderical polar coordinate system are :

If θ and z are fixed while r varies, then the intersection of $\theta = C_2$ and $z = C_3$ is a straight line called the r–coordinate curve or simply the r–curve.

If r and z are fixed while θ varies, then the intersection of $r = C_1$ and $z = C_3$ is a circle (or point) called the θ–coordinate curve or simply the θ–curve.

i) If r and θ are fixed while z varies, then the intersection of $r = C_1$ and $\theta = C_2$ is a straight line called the z–coordinate curve or simply the z–curve.

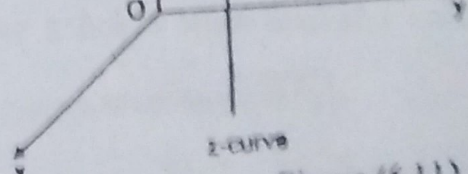


Figure (6.11)

bus the r–curves are straight lines radiating from and ormal to the z–axis, the θ–curves are circles centered on e z–axis and parallel to the xy–plane ; and the z–curves re the straight lines parallel to the z–axis as shown in igure (6.11).

5.14 CYLINDERICAL COORDINATES IN TERMS OF CARTESIAN COORDINATES

We know that the equations expressing the rectangular Cartesian coordinates in terms of cylinderical polar coordinates are:

$$x = r\cos\theta \quad (1)$$

$$y = r\sin\theta \quad (2)$$

$$z = z \quad (3)$$

Squaring equations (1) and (2) and adding, we get

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2 \quad (4)$$

or $r = \sqrt{x^2 + y^2}$ (since r is positive)

Dividing equation (2) by equation (1), we get

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \quad (5)$$

or $\theta = \tan^{-1}\frac{y}{x}$

Hence, the equations expressing the cylinderical polar coordinates in terms of Cartesian coordinates are:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x}, \quad z = z \quad (6)$$

NOTE: For points on the z–axis ($x = 0$, $y = 0$), θ is indeterminate. Such points are called singular points of the transformation.

EXAMPLE (2): If r, θ, z are cylinderical polar coordinates, describe each of the following loci and write the equation of each locus in rectangular coordinates :

(i) $r = 4$ (ii) $\theta = \frac{\pi}{2}$ (iii) $z = 3$

(iv) $\theta = \frac{\pi}{3}, z = 1$ (v) $r = 4, z = 0$ (vi) $r = 2, \theta = \frac{\pi}{6}$

SOLUTION: In cylinderical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x}, \quad z = z$$

(i) $r = 4$

Here r is fixed while θ and z vary. We can write the given equation as

$$\sqrt{x^2 + y^2} = 4$$

or $x^2 + y^2 = 16$

i.e. the given equation represents a cylinder with axis as the z–axis and radius 4.

(ii) $\theta = \frac{\pi}{2}$

Here θ is fixed while r and z vary. We can write the given equation as

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{2}$$

or $\frac{y}{x} = \tan\frac{\pi}{2} = \infty$

This implies $x = 0$ i.e. the given equation represents the yz–plane where $y \geq 0$.

(iii) $z = 3$

Here z is fixed while r and θ vary. The given equation represents a plane parallel to the x y – plane at a distance 3 units from the origin

(iv) $\theta = \frac{\pi}{3}, \quad z = 1$

Here θ and z are fixed while only r varies. We can write $\theta = \frac{\pi}{3}$ as

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{3} \quad \text{or} \quad \frac{y}{x} = \tan\frac{\pi}{3} = \sqrt{3}$$

or $\quad y = \sqrt{3}\,x, \quad z = 1$

i.e. the given equations represent a straight line $y = \sqrt{3}\,x$, in the plane $z = 1$ where $x \geq 0$

(v) $r = 4, \quad z = 0$

Here r and z are fixed while only θ varies. We can write $r = 4$ as

$$\sqrt{x^2 + y^2} = 4 \quad \text{or} \quad x^2 + y^2 = 16, \quad z = 0$$

i.e. the given equations represent a circle with centre at the origin and radius 4 in the xy-plane

(vi) $r = 2, \quad \theta = \frac{\pi}{6}$

Here r and θ are fixed while only z varies. We can write $r = 2$ as

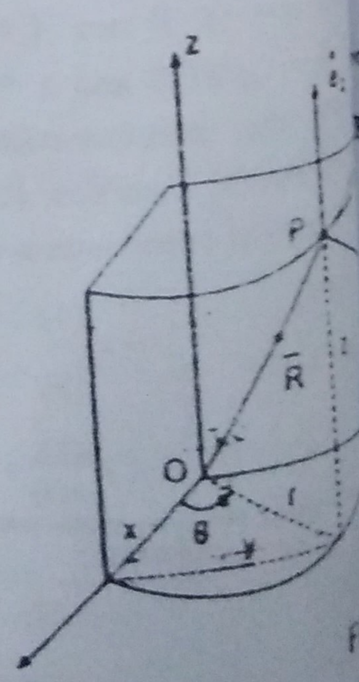
$$\sqrt{x^2 + y^2} = 2 \quad \text{or} \quad x^2 + y^2 = 4$$

and $\theta = \frac{\pi}{6}$ as $\tan^{-1}\frac{y}{x} = \frac{\pi}{6}$ or $\frac{y}{x} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, or $y = \frac{1}{\sqrt{3}}x$

i.e. the given equations represent a straight line parallel to the z-axis and passing through intersection of the circle $x^2 + y^2 = 4$ and the straight line $y = \frac{1}{\sqrt{3}}x$.

6.15 UNIT VECTORS IN CYLINDERICAL COORDINATE SYSTEM

The position vector of any point P in cylinderical polar coordinates is



$$\bar{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k}$$

The tangent vectors in the directions of r, θ, and z respectively, are given by

$$\frac{\partial \bar{R}}{\partial r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\frac{\partial \bar{R}}{\partial \theta} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$$

$$\frac{\partial \bar{R}}{\partial z} = \hat{k}$$

The unit vectors in these directions of r, θ, and z are given by

$$\hat{e}_r = \frac{\frac{\partial \bar{R}}{\partial r}}{\left|\frac{\partial \bar{R}}{\partial r}\right|} = \frac{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}}{\sqrt{\cos^2\theta + \sin^2\theta}} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad (1)$$

$$\hat{e}_\theta = \frac{\frac{\partial \bar{R}}{\partial \theta}}{\left|\frac{\partial \bar{R}}{\partial \theta}\right|} = \frac{-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \qquad (2)$$

$$\hat{e}_z = \frac{\frac{\partial \bar{R}}{\partial z}}{\left|\frac{\partial \bar{R}}{\partial z}\right|} = \hat{k} \qquad (3)$$

In matrix notation, equations (1), (2), and (3) can be written as

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \qquad (4)$$

SCALE FACTORS

The scale factors for the cylinderical coordinate system are given by

$$h_1 = h_r = \left|\frac{\partial \bar{R}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \qquad (5)$$

$$h_2 = h_\theta = \left|\frac{\partial \bar{R}}{\partial \theta}\right| = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}$$

$$= r\sqrt{\sin^2\theta + \cos^2\theta} = r \qquad (6)$$

$$h_3 = h_z = \left|\frac{\partial \bar{R}}{\partial z}\right| = |\hat{k}| = 1 \qquad (7)$$

Hence the scale factors are : $h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$

THEOREM (6.2): In cylinderical polar coordinates, show that

$$\frac{\partial \hat{e}_r}{\partial r} = 0, \quad \frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_r}{\partial z} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r, \quad \frac{\partial \hat{e}_\theta}{\partial z} = 0$$

$$\frac{\partial \hat{e}_z}{\partial \theta} = 0, \quad \frac{\partial \hat{e}_z}{\partial z} = 0$$

PROOF:

$\hat{e}_r$ ... $\hat{e}_z = \hat{k}$