Chapter 6

CURVILINEAR COORDINATES

6.1 INTRODUCTION

So far we have restricted ourselves completely to a rectangular Cartesian coordi... which has the advantage that all the three unit vectors $\hat{i}, \hat{j}, \hat{k}$ are constant unit vectors. In ... it is often useful to use other coordinate systems, for example, when a problem involves cy... spherical symmetry. In this chapter, we shall discuss the general orthogonal curvilinear coord... and show how the gradient, divergence, curl, and Laplacian can be transformed into th... particular, we shall discuss the two most important coordinate systems for space, i.e. the ... coordinate system and the spherical coordinate system. We shall see that the cylindrical ... simplify the equations of cylinders, while spherical coordinates simplify the equations of ... ...ques. We shall also derive the formulas for the gradient, divergence, curl, and Laplacian in c... and spherical coordinate systems.

6.2 TRANSFORMATION OF COORDINATES

Let the rectangular coordinates (x, y, z) of any point be expressed as functions of u, ...s that

$$\left.\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3) \end{aligned}\right] \qquad (1)$$

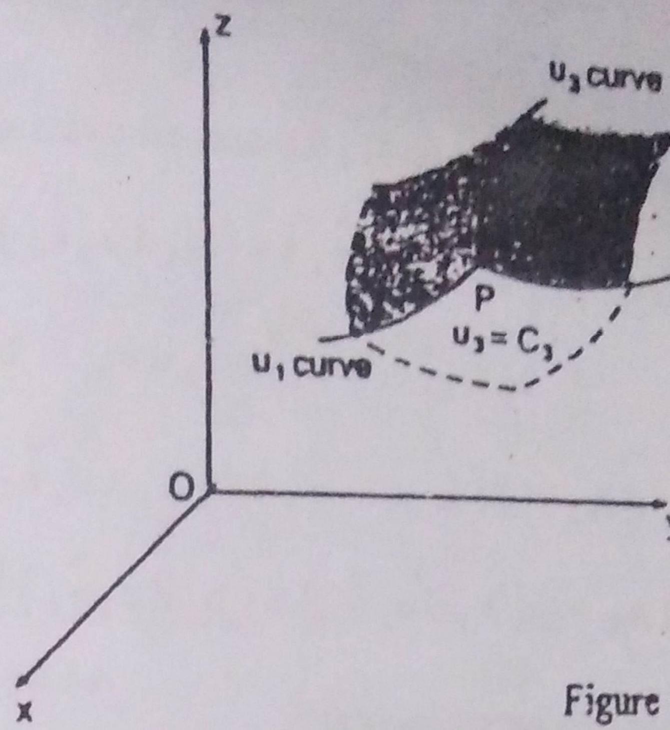
A theorem from elementary calculus shows that if the functions in equations (1) are single-valued ... continuous partial derivatives, then equations (1) can be solved uniquely for u_1, u_2, u_3 ... x, y, and z, i.e.

$$\left.\begin{aligned} u_1 &= u_1(x, y, z) \\ u_2 &= u_2(x, y, z) \\ u_3 &= u_3(x, y, z) \end{aligned}\right] \qquad (2)$$

...n a point P with rectangular coordin... ...ordinates (u ...

6.3 COORDINATE SURFACES AND COORDINATE CURVES

The coordinate surfaces (or level surfaces) are families of surfaces obtained by setting the coordinate equations equal to a constant. Thus if C_1, C_2, C_3 are constants, then the surfaces $u_1 = C_1$, $u_2 = C_2$, $u_3 = C_3$ are called coordinate surfaces. The coordinate surfaces are generally curved and each pair of these surfaces intersect in curves called coordinate curves in space. Thus u_1 - coordinate curve is that along which only u_1 varies while u_2 and u_3 are constants. Similarly, along u_2 - coordinate curve only u_2 varies while u_1 and u_3 are constants, and along u_3 - coordinate curve only u_3 varies while u_1 and u_2 are constants as shown in figure (6.1).



Figure

6.4 UNIT VECTORS IN CURVILINEAR COORDINATE SYSTEM

Since the three coordinate curves are generally not straight lines, as in the rectangular co[ordinate] system, such a coordinate system is called the curvilinear coordinate system.

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P. Then the set of equations

$$x = x(u_1, u_2, u_3)$$

$$y = y(u_1, u_2, u_3)$$

$$z = z(u_1, u_2, u_3)$$

can be written $\bar{r} = \bar{r}\,(u_1, u_2, u_3)$.

The vector $\dfrac{\partial \bar{r}}{\partial u_1}$ is tangent to the u_1 - coordinate curve at P. Then if $\hat{e}_1$ is the unit tangent vector at P in this direction we can write

$$\hat{e}_1 = \frac{\dfrac{\partial \bar{r}}{\partial u_1}}{\left|\dfrac{\partial \bar{r}}{\partial u_1}\right|}$$

so that $\dfrac{\partial \bar{r}}{\partial u_1} = h_1 \hat{e}_1$, where $h_1 = \left|\dfrac{\partial \bar{r}}{\partial u_1}\right|$

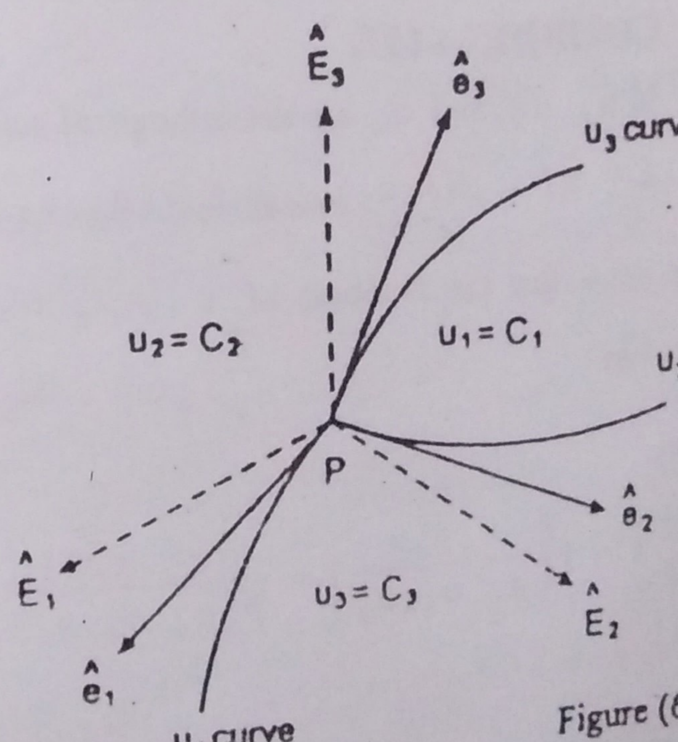


Figure (6

Similarly, if $\hat{e}_2$ and $\hat{e}_3$ are the unit tangent vectors to the u_2 and u_3 – curves at P respectively, then

$\dfrac{\partial \bar{r}}{\partial u_2} = h_2 \hat{e}_2$, $\dfrac{\partial \bar{r}}{\partial u_3} = h_3 \hat{e}_3$, where $h_2 = \left|\dfrac{\partial \bar{r}}{\partial u_2}\right|$, and $h_3 = \left|\dfrac{\partial \bar{r}}{\partial u_3}\right|$

The quantities h_1, h_2, and, h_3 are called, the scale factors. The unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are in the directions of increasing u_1, u_2, u_3 respectively. In general, h_1, h_2, h_3 are functions of u_1, u_2, u_3 and $h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$. Hence $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are also functions of u_1, u_2, u_3.
Since ∇u_1 is a vector at P normal to the surface $u_1 = C_1$, a unit vector in this direction is given by

$$\hat{E}_1 = \frac{\nabla u_1}{|\nabla u_1|}$$

Similarly, the unit vectors $\hat{E}_2 = \frac{\nabla u_2}{|\nabla u_2|}$ and $\hat{E}_3 = \frac{\nabla u_3}{|\nabla u_3|}$ at P are unit normal to the surfaces $u_2 = C_2$ and $u_3 = C_3$ respectively.

Thus at each point P of a curvilinear coordinate system there exist, in general, two sets of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ tangent to the coordinate curves and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ normal to the coordinate surfaces. These two sets of unit vectors generally vary in direction from point to point because the coordinate curves are curved. However, the two sets become identical if and only if the curvilinear coordinate system is orthogonal [see figure (6.3) below].

6.5 ORTHOGONAL CURVILINEAR COORDINATE SYSTEM

If the coordinate curves intersect at right angles, the curvilinear coordinate system is called orthogonal. The u_1, u_2, and u_3 coordinate curves of an orthogonal curvilinear system are similar to the x, y, and z coordinate axes of a rectangular Cartesian system. For this system, the two sets of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are the same. [see theorem (6.1) below]. In an orthogonal curvilinear coordinate system, the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are mutually orthogonal at every point,

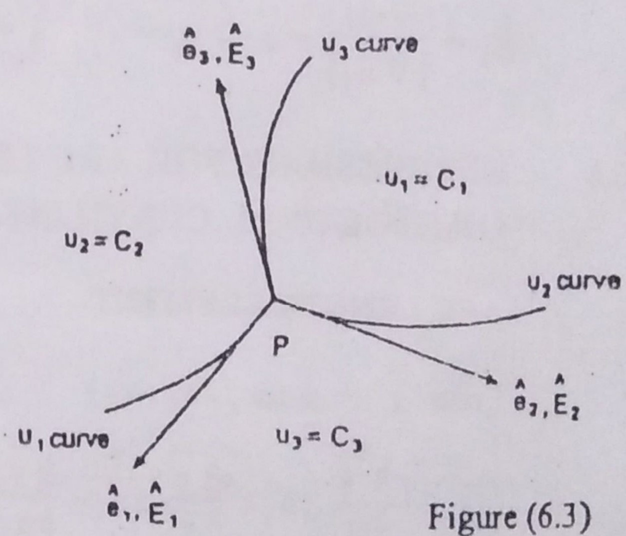


Figure (6.3)

i.e. $\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_3 \cdot \hat{e}_1 = 0$.

and $\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1$

Furthermore, if this system is right–handed, then

$$\hat{e}_1 \times \hat{e}_2 = -\hat{e}_2 \times \hat{e}_1 = \hat{e}_3$$

$$\hat{e}_2 \times \hat{e}_3 = -\hat{e}_3 \times \hat{e}_2 = \hat{e}_1$$

$$\hat{e}_3 \times \hat{e}_1 = -\hat{e}_1 \times \hat{e}_3 = \hat{e}_2$$

The vector $\overline{A}$ in an orthogonal curvilinear coordinate system can be expressed in terms of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ as $\overline{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$

where A_1, A_2, A_3 are the components of $\overline{A}$.

THEOREM (6.1): Prove that if u_1, u_2, u_3 are orthogonal curvilinear coordinates, then

(i) $|\nabla u_j| = h_j^{-1}$ (ii) $\hat{e}_j = \hat{E}_j,\ j = 1, 2, 3$

SOLUTION: (i) Since ∇u_1 is a vector normal to the surface $u_1 = C_1$, therefore it is parallel to $\hat{e}_1$.

Thus $\hat{e}_1 = h_1 \nabla u_1$ where h_1 is a scalar factor of proportionality between $\hat{e}_1$ and ∇u_1.

or $\nabla u_1 = \dfrac{\hat{e}_1}{h_1}$ and so $|\nabla u_1| = \dfrac{|\hat{e}_1|}{h_1} = \dfrac{1}{h_1}$ (since $|\hat{e}_1| = 1$)

or $|\nabla u_1| = h_1^{-1}$.

Similarly $|\nabla u_2| = h_2^{-1}$ and $|\nabla u_3| = h_3^{-1}$.

Combining the three equations, we can write $|\nabla u_j| = h_j^{-1},\ j = 1, 2, 3$.

(ii) By definition,

$$\hat{E}_j = \frac{\nabla u_j}{|\nabla u_j|} = h_j \nabla u_j = \hat{e}_j,\quad j = 1, 2, 3 \text{ and the result is proved.}$$

6.6 EXPRESSIONS FOR ARC LENGTH, AREA, AND VOLUME ELEMENTS IN ORTHOGONAL CURVILINEAR COORDINATES

ARC LENGTH ELEMENT

From $\bar{r} = \bar{r}(u_1, u_2, u_3)$

$$d\bar{r} = \frac{\partial \bar{r}}{\partial u_1} du_1 + \frac{\partial \bar{r}}{\partial u_2} du_2 + \frac{\partial \bar{r}}{\partial u_3} du_3$$

$$= h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

Then the differential of arc length ds is determined from

$$(ds)^2 = d\bar{r} \cdot d\bar{r} \qquad (1)$$

For an orthogonal curvilinear coordinate system, we have

$$\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1$$

$$\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_3 \cdot \hat{e}_1 = 0$$

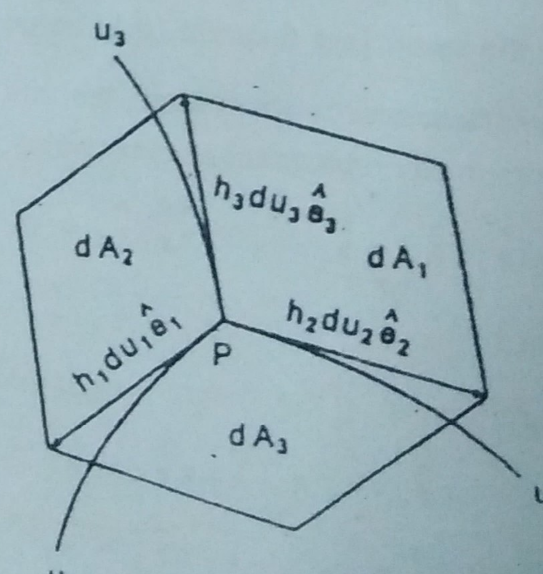


Figure (6.4)

Thus equation (1) gives

$$(ds)^2 = d\bar{r} \cdot d\bar{r} = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2 \qquad (2)$$

and the element of arc length ds is obtained by taking the square root of equation (2).

Along u_1– curve, u_2 and u_3 are constants so that $d\bar{r} = h_1 du_1 \hat{e}_1$. Then the differential of arc length ds_1 along u_1 at P is $h_1 du_1$. Similarly, the differential of arc lengths along u_2 and u_3– curves at P are $ds_2 = h_2 du_2$ and $ds_3 = h_3 du_3$, respectively.

AREA ELEMENT

From the figure (6.4), the area elements are given by

$$dA_1 = \left|(h_2 du_2 \hat{e}_2) \times (h_3 du_3 \hat{e}_3)\right|$$

$$= h_2 h_3 \left|\hat{e}_2 \times \hat{e}_3\right| du_2 du_3 = h_2 h_3 du_2 du_3 . \quad (\text{since} \left|\hat{e}_2 \times \hat{e}_3\right| =$$

Similarly, $dA_2 = \left|(h_1 du_1 \hat{e}_1) \times (h_3 du_3 \hat{e}_3)\right| = h_1 h_3 du_1 du_3$

and $dA_3 = \left|(h_1 du_1 \hat{e}_1) \times (h_2 du_2 \hat{e}_2)\right| = h_1 h_2 du_1 du_2$

VOLUME ELEMENT

We know that the absolute value of the scalar triple product gives th
parallelopiped. Thus

$$dV = \left|(h_1 du_1 \hat{e}_1) . (h_2 du_2 \hat{e}_2) \times (h_3 du_3 \hat{e}_3)\right|$$

$$= h_1 h_2 h_3 du_1 du_2 du_3 \left|\hat{e}_1 . \hat{e}_2 \times \hat{e}_3\right| = h_1 h_2 h_3 du_1 du_2 du_3$$

$$(\text{since} \left|\hat{e}_1 . \hat{e}_2 \times \hat{e}_3\right| =$$

6.7 EXPRESSION FOR JACOBIAN IN ORTHOGONAL COORDINATES

If u_1, u_2, and u_3 are the orthogonal curvilinear coordinates, and

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3)$$

then we know that the Jacobian of x, y, z w.r.t. u_1, u_2, u_3 denoted by J or
is defined as

$$J = J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix}$$

From the definition of a scalar triple product, the above determinant can be written as

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = \left(\frac{\partial x}{\partial u_1}\hat{i} + \frac{\partial y}{\partial u_1}\hat{j} + \frac{\partial z}{\partial u_1}\hat{k}\right) . \left(\frac{\partial x}{\partial u_2}\hat{i} + \frac{\partial y}{\partial u_2}\hat{j} + \frac{\partial z}{\partial u_2}\hat{k}\right) \times \left(\frac{\partial x}{\partial u_3}\hat{i} + \right.$$

$$= \frac{\partial \bar{r}}{\partial u_1} . \frac{\partial \bar{r}}{\partial u_2} \times \frac{\partial \bar{r}}{\partial u_3} = h_1 \hat{e}_1 . h_2 \hat{e}_2 \times h_3 \hat{e}_3$$

GRADIENT, DIVERGENCE, CURL, AND LAPLACIAN IN ORTHOGONAL CURVILINEAR COORDINATES

EXPRESSION FOR GRADIENT

Let $\nabla\psi = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$ (1)

where f_1, f_2, and f_3 are to be determined.

Since
$$d\bar{r} = \frac{\partial \bar{r}}{\partial u_1}du_1 + \frac{\partial \bar{r}}{\partial u_2}du_2 + \frac{\partial \bar{r}}{\partial u_3}du_3$$
$$= h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3$$

we have
$$d\psi = \nabla\psi . d\bar{r} = h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3 \quad (2)$$

But
$$d\psi = \frac{\partial\psi}{\partial u_1}du_1 + \frac{\partial\psi}{\partial u_2}du_2 + \frac{\partial\psi}{\partial u_3}du_3 \quad (3)$$

From equations (2) and (3) equating the coefficients of du_1, du_2, and du_3, we get
$$f_1 = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}, \quad f_2 = \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}, \quad f_3 = \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}$$

Then from equation (1), we have
$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{e}_3 \quad (4)$$

This indicates the operator equivalence
$$\nabla = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3} \quad (5)$$

which is the expression for the del operator in orthogonal curvilinear coordinates.

ALTERNATIVE FORM

Note that by taking $\psi = u_1$ in equation (4), we get
$$\nabla u_1 = \frac{1}{h_1}\hat{e}_1 \quad \text{i.e.} \quad \hat{e}_1 = h_1\nabla u_1$$

Similarly, by taking $\psi = u_2$ and $\psi = u_3$ we get
$$\hat{e}_2 = h_2\nabla u_2 \quad \text{and} \quad \hat{e}_3 = h_3\nabla u_3$$

Thus equation (4) takes an alternative form
$$\nabla\psi = \frac{\partial\psi}{\partial u_1}\nabla u_1 + \frac{\partial\psi}{\partial u_2}\nabla u_2 + \frac{\partial\psi}{\partial u_3}\nabla u_3 \quad (6)$$

EXPRESSION FOR DIVERGENCE

$$\nabla . \bar{A} = \nabla . (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3) \quad (7)$$

This is
$$= \nabla . (A_1\hat{e}_1) + \nabla . (A_2\hat{e}_2) + \nabla . (A_3\hat{e}_3)$$

Since $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ form a right–handed system, therefore

$$\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 = h_2 h_3 \nabla u_2 \times \nabla u_3$$
$$\hat{e}_2 = \hat{e}_3 \times \hat{e}_1 = h_3 h_1 \nabla u_3 \times \nabla u_1$$
$$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2 = h_1 h_2 \nabla u_1 \times \nabla u_2$$

Now
$$\nabla . (A_1 \hat{e}_1) = \nabla . (A_1 h_2 h_3 \nabla u_2 \times \nabla u_3)$$
$$= \nabla (A_1 h_2 h_3) . \nabla u_2 \times \nabla u_3 + A_1 h_2 h_3 \nabla . (\nabla u_2 \times \nabla u_3) \quad (8)$$

Using the formulas, $\nabla . (\overline{A} \times \overline{B}) = \overline{B} . (\nabla \times \overline{A}) - \overline{A} . (\nabla \times \overline{B})$ and $\nabla \times \nabla \phi = \overline{0}$, we get

$$\nabla . (\nabla u_2 \times \nabla u_3) = \nabla u_3 . (\nabla \times \nabla u_2) - \nabla u_2 . (\nabla \times \nabla u_3) = 0$$

Thus equation (8) becomes

$$\nabla . (A_1 \hat{e}_1) = \nabla (A_1 h_2 h_3) . \frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3} + 0$$
$$= \nabla (A_1 h_2 h_3) . \frac{\hat{e}_1}{h_2 h_3}$$
$$= \left[\frac{1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_2 h_3) \hat{e}_1 + \frac{1}{h_2} \frac{\partial}{\partial u_2} (A_1 h_2 h_3) \hat{e}_2 + \frac{1}{h_3} \frac{\partial}{\partial u_3} (A_1 h_2 h_3) \hat{e}_3 \right] . \frac{\hat{e}_1}{h_2 h_3}$$
$$= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (A_1 h_2 h_3)$$

Similarly $\nabla . (A_2 \hat{e}_2) = \dfrac{1}{h_1 h_2 h_3} \dfrac{\partial}{\partial u_2} (A_2 h_3 h_1)$

and $\nabla . (A_3 \hat{e}_3) = \dfrac{1}{h_1 h_2 h_3} \dfrac{\partial}{\partial u_3} (A_3 h_1 h_2)$

Then from equation (7), we get

$$\nabla . \overline{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right] \quad (9)$$

EXPRESSION FOR CURL

We have
$$\nabla \times \overline{A} = \nabla \times (A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3)$$
$$= \nabla \times (A_1 \hat{e}_1) + \nabla \times (A_2 \hat{e}_2) + \nabla \times (A_3 \hat{e}_3) \quad (10)$$

Now
$$\nabla \times (A_1 \hat{e}_1) = \nabla \times (A_1 h_1 \nabla u_1) \quad (\text{since } \hat{e}_1 = h_1 \nabla u_1)$$
$$= \nabla (A_1 h_1) \times \nabla u_1 + A_1 h_1 \nabla \times \nabla u_1$$
$$= \nabla (A_1 h_1) \times \frac{\hat{e}_1}{h_1} + \overline{0} \quad (\text{since } \nabla \times \nabla u_1 = \overline{0})$$
$$= \left[\frac{1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_1) \hat{e}_1 + \frac{1}{h_2} \frac{\partial}{\partial u_2} (A_1 h_1) \hat{e}_2 + \frac{1}{h_3} \frac{\partial}{\partial u_3} (A_1 h_1) \hat{e}_3 \right] \times \frac{\hat{e}_1}{h_1}$$
$$= \frac{1}{h_3 h_1} \frac{\partial}{\partial u_3} (A_1 h_1) \hat{e}_2 - \frac{1}{h_1 h_2} \frac{\partial}{\partial u_2} (A_1 h_1) \hat{e}_3$$

...arly $\nabla \times (A_2 \hat{e}_2) = \frac{1}{h_1 h_2} \frac{\partial}{\partial u_1}(A_2 h_2)\hat{e}_3 - \frac{1}{h_2 h_3} \frac{\partial}{\partial u_3}(A_2 h_2)\hat{e}_1$

$$\nabla \times (A_3 \hat{e}_3) = \frac{1}{h_2 h_3} \frac{\partial}{\partial u_2}(A_3 h_3)\hat{e}_1 - \frac{1}{h_3 h_1} \frac{\partial}{\partial u_1}(A_3 h_3)\hat{e}_2$$

...equation (10) becomes

$$\nabla \times \bar{A} = \frac{\hat{e}_1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{\partial}{\partial u_3}(A_2 h_2)\right] + \frac{\hat{e}_2}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(A_1 h_1) - \frac{\partial}{\partial u_1}(A_3 h_3)\right] + \frac{\hat{e}_3}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(A_2 h_2) - \frac{\partial}{\partial u_2}(A_1 h_1)\right]$$

...can be written as

$$\nabla \times \bar{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \qquad (11)$$

EXPRESSION FOR LAPLACIAN

We know that

$$\nabla \psi = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}\hat{e}_3$$

$$\nabla \cdot \bar{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]$$

$\bar{A} = \nabla\psi$, then $A_1 = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}$, $A_2 = \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}$, $A_3 = \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}$ and thus

$$\nabla \cdot \bar{A} = \nabla \cdot \nabla \psi = \nabla^2 \psi$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial u_3}\right)\right] \qquad (12)$$

...AMPLE (1): Consider the curvilinear coordinate system defined for $z \geq 0$ by $x = u_1 - u_2$, $y = u_1 + u_2$, $z = u_3^2$

(i) Find the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and show that the system is orthogonal and right–handed. Also find the scale factors h_1, h_2, h_3.

(ii) Find the expressions for $(ds)^2$ and dV.

(iii) Find $\nabla\psi$ in this system for $\psi(u_1, u_2, u_3) = u_1 u_2 + u_3^2$.