*Since*

$$\sum_{j=1}^{3}|a_{1j} = |1| + |2| + |-1| = 4$$

$$\sum_{j=1}^{3}|a_{2j} = |0| + |3| + |-1| = 4$$

$$\sum_{j=1}^{3}|a_{3j} = |5| + |-1| + |1| = 7$$

## 3.5 Unstable or ill-conditioned system

A system of equations $AX = B$ whose solution is extremely sensitive to small changes in the coefficients (matrix $A$ or $B$) is called ill-conditioned system. It is observed that even with the best available algorithm, the error due to round-off is some times large, because the problem itself may be very sensitive to the effects of small errors in the matrix $A$ or in matrix $B$. An ill-conditioned system cannot be tested for the accuracy of the computed solution by substituting values of solution into equations.

A system of equations is called stable if relatively small changes in the coefficients produce small changes in the solution vector.

**Example 3.5.1.** *Consider the system* $AX = B$ *as given below*

$$\begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.00 \\ 2.00 \end{bmatrix}$$

*Solution of the above system is* $x = 1$ *and* $y = 1$. *Now, modify matrix* $B$ *just slightly*

$$\begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.02 \\ 1.98 \end{bmatrix}$$

*Solution of the above system is* $x = 2$ *and* $y = 0$. *Now, modify matrix* $B$ *just slightly*

$$\begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.98 \\ 2.02 \end{bmatrix}$$

*Solution of the above system is $x = 0$ and $y = 2$.*

*Remark* 3.5.1. Ill-Conditioning of a matrix $A$ can usually be expected if $|A|$ is small.

### 3.5.1 Effect on Eigenvalues

Small changes in matrix $A$ do not necessarily lead to small changes in the eigenvalues of $A$. To illustrate, consider the following example

$$A = \begin{bmatrix} 1 & 1000 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1000 \\ 0.001 & 1 \end{bmatrix}$$

matrix $A$ has eigenvalues 1, 1 and matrix $B$ has eigenvalues 0 and 2. A diffrence of 0.001 in an entry (second row, first column) of the matrices led the eigenvalues to differ by 1. Thus a change of 0.001 in one entry of matrix $A$ led to 100% change in eigenvalues. If

$$A = \begin{bmatrix} 1 & 1000 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1000 \\ -0.001 & 1 \end{bmatrix}$$

then matrix $B$ have no real eigenvalue as its characteristic equation $\lambda^2 - 2\lambda + 2$ will give complex values.

*Remark* 3.5.2. For Symmetric matrices, small changes will generally not lead to large changes in eigenvalues.

### 3.5.2 Condition Number

Let A be a non singular matrix then condition number of matrix A relative to a norm is denoted by $k(A)$ or $cond(A)$ and is calculated as

$$cond(A) = \|A\| \, . \left\|A^{-1}\right\|$$

If $cond(A)$ is close to 1, then the system is well conditioned and stable. Otherwise system is ill conditional or unstable if $cond(A)$ is significantly larger than 1. For example, consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1.01 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 50.5 & -50 \\ -100 & 100 \end{bmatrix}$$

$$\|A\| = 4, \quad \text{and} \quad \left\|A^{-1}\right\| = 150.5$$

$$\because \qquad cond\ (A) = 602$$

Therefore, given matrix is ill-conditioned.

**Example 3.5.2.** *Determine the condition number of the matrix given below*

$$A = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}$$

**Solution.**

$$A^{-1} = \begin{bmatrix} -10000 & 10000 \\ 5000.5 & -5000 \end{bmatrix}$$

$$\|A\|_\infty = \max(|1| + |2|, |1.0001| + |2|) = \max(3, 3.0001) = 3.0001$$

$$\left\|A^{-1}\right\|_\infty = \max(|-10000|+|10000|, |5000.5|+|-5000|) = \max(20000, 10000.5) = 20000$$

$$k(A) = \|A\|_\infty . \left\|A^{-1}\right\|_\infty = (20000)(3.0001) = 60002.000 > 1$$

**Example 3.5.3.** *Determine the system is ill conditioned or not?*

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/2 & 1/4 & 1/5 \end{bmatrix}$$

**Solution.**

$$\|A\|_\infty = \max(|1| + |1/2| + |1/3|, |1/2| + |1/3| + |1/4|, |1/2| + |1/4| + |1/5|)$$

$$\|A\|_\infty = \max(1.8333, 1.08333, 0.950) = 1.8333$$

*By using Gaussian Elimination method*

$$A^{-1} = \begin{bmatrix} 3/2 & -6 & 5 \\ 9 & 12 & -30 \\ -15 & 0 & 30 \end{bmatrix}$$

$$\left\|A^{-1}\right\|_\infty = \max(|3/2| + |-6| + |5|, |9| + |12| + |-30|, |-15| + |0| + |30|)$$

$$= \max(12.5, 51, 45) = 51$$

*Now*

$$cond(A) = \|A\|_\infty . \left\|A^{-1}\right\|_\infty = (1.8333)(51)$$

$$cond(A) = 93.4983 > 1$$

*Hence matrix is ill conditioned.*

## 3.6 Exercise

Find the dominant and the least eigenvalues of the following matrices

1.
$$A = \begin{bmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix}$$

......Answer: Dominant eigenvalue of $A = 12.994$ and dominant eigenvalue of $B = -4.995$, Least eigenvalue of $A = 7.999$

2.
$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

....Answer: Performing 17 iterations, dominant eigenvalue of $A = 5.9979$ and dominant eigenvalue of $B = -1.9979$, Least eigenvalue of $A = 5.9979 - 1.9979$