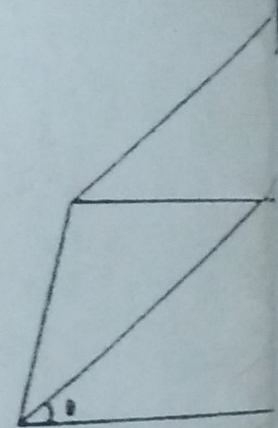
Choose a point (x_k, y_k, z_k) on each surface element ΔS_k and draw the unit normal $\hat{n}_k$ to the S at this point. Let γ_k be the acute angle between this unit normal and the positive z-axis. If this surface element is sufficiently small, it can be regarded as a plane as shown in figure (5.15)(b). We know from geometry, that if two planes intersect at an acute angle, an area in one plane are equa projected into the other by multiplying the cosine of the included angle as shown in figure (5.16). Since the angle between two planes is the angle between their normals,



Figure

therefore $\Delta S_k \cos\gamma_k \approx \Delta A_k$

or $$\Delta S_k \approx \sec\gamma_k \, \Delta A_k = \frac{\Delta x_k \Delta y_k}{|\hat{n}_k \cdot \hat{k}|}$$

Thus the sum (1) in the definition of the normal surface integral

$$\sum_{k=1}^{n} \bar{A}_k \cdot \hat{n}_k \, \Delta S_k \approx \sum_{k=1}^{n} \bar{A}_k \cdot \hat{n}_k \frac{\Delta x_k \Delta y_k}{|\hat{n}_k \cdot \hat{k}|}$$

and the limit of this sum can be written as

$$\iint_S \bar{A} \cdot \hat{n} \, dS = \iint_R \bar{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

NOTE: Similarly, we can prove that if R is the projection of the surface S on the yz-plane, then

$$\iint_S \bar{A} \cdot \hat{n} \, dS = \iint_R \bar{A} \cdot \hat{n} \frac{dy\,dz}{|\hat{n} \cdot \hat{i}|}$$

and if R is the projection of S on the zx-plane, then $\displaystyle\iint_S \bar{A} \cdot \hat{n} \, dS = \iint_R \bar{A} \cdot \hat{n} \frac{dz\,dx}{|\hat{n} \cdot \hat{j}|}$

EXAMPLE (11): Evaluate $\displaystyle\iint_S \bar{A} \cdot \hat{n} \, dS$ where $\bar{A} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ and S is the plane $2x + 3y + 6z = 12$ which is located in the first octant.

SOLUTION: The surface S and its projection R on the xy-plane are shown in figure ( )

We know that $$\iint_S \bar{A} \cdot \hat{n} \, dS = \iint_R \bar{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

Also we know that a normal vector to the surface $2x + 3y + 6z = 12$ is given by

$$\nabla(2x + 3y + 6z) = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

and $\iint_S \overline{A}\cdot\hat{n}\,dS = \iint_R \overline{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \int_{y=0}^{6}\int_{z=0}^{4}\frac{2}{\sqrt{5}}(y+x)\cdot\frac{\sqrt{5}}{2}\,dz\,dy$

$$= \int_0^6\int_0^4\left[y+\left(\frac{6-y}{2}\right)\right]dz\,dy = \int_0^6\int_0^4\left(3+\frac{y}{2}\right)dz\,dy$$

$$= \int_0^6\left(3+\frac{y}{2}\right)|z|_0^4\,dy = 4\int_0^6\left(3+\frac{y}{2}\right)dy = 4\left|3y+\frac{y^2}{4}\right|_0^6$$

$$= 4(18+9) = 108$$

5.8 VOLUME INTEGRAL

Let $\overline{A}(x,y,z)$ be a vector point function which is defined and continuous in a closed region R. Subdivide the region R into n subregions ΔR_k of volume ΔV_k, $k = 1, 2, \ldots, n$. Let $P_k(x_k, y_k, z_k)$ be any point on each subregion ΔR_k as shown in figure (5.19).

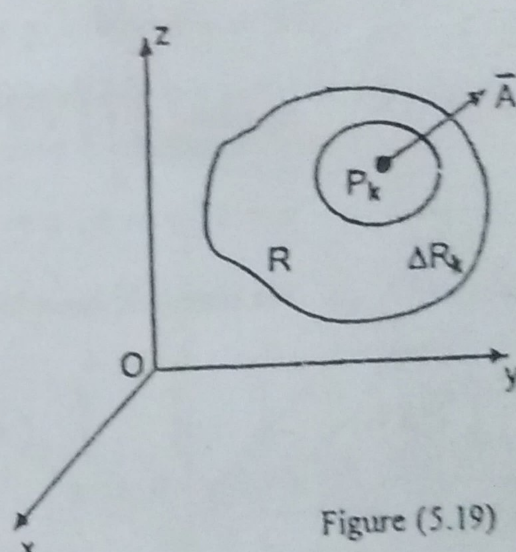


Figure (5.19)

Define $\overline{A}(x_k, y_k, z_k) = \overline{A}_k$. We multiply the value of $\overline{A}$ at the selected point (i.e. $\overline{A}_k$) with the volume of the corresponding subregion and form the sum

$$\sum_{k=1}^{n}\overline{A}_k\,\Delta V_k \qquad (1)$$

Now take the limit of sum (1) as $n\to\infty$ in such a way that each $\Delta V_k \to 0$. This limit, if it exists, is called the volume integral of $\overline{A}(x,y,z)$ over R and is denoted by $\iiint_R \overline{A}\,dV$

i.e. $$\iiint_R \overline{A}(x,y,z)\,dV = \underset{n\to\infty}{Lt}\sum_{k=1}^{n}A_k\,\Delta V_k \qquad (2)$$

If $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then the volume integral (2) can be written as

$$\iiint_R \overline{A}\,dV = \hat{i}\iiint_R A_1\,dV + \hat{j}\iiint_R A_2\,dV + \hat{k}\iiint_R A_3\,dV \qquad (3)$$

If we have a scalar point function $\phi(x, y, z)$ defined and continuous over the region R, then the integral becomes

$$\iiint_R \phi \, dV \tag{3}$$

In rectangular coordinate system, $dV = dx\,dy\,dz$ so the volume integral (3) can be written as

$$\iiint_R \phi(x, y, z)\, dx\,dy\,dz$$

which is the ordinary triple integral of $\phi(x, y, z)$ over the region R. If $\phi(x, y, z) = 1$, the V of the region R is given by $V = \iiint_R dV$.

EXAMPLE (13): **Evaluate $\iiint_R \bar{r}\, dV$ where R is the region bounded by the surfaces x $y = 0,\ y = 6,\ z = x^2,\ z = 4$.**

SOLUTION: The region R bounded by the given surfaces is shown in figure (5.20). Then

$$\iiint_R \bar{r}\, dV = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (x\hat{i} + y\hat{j} + z\hat{k})\, dz\,dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx$$

$$+ \hat{j}\int_0^2\int_0^6\int_{x^2}^4 y\,dz\,dy\,dx$$

$$+ \hat{k}\int_0^2\int_0^6\int_{x^2}^4 z\,dz\,dy\,dx$$

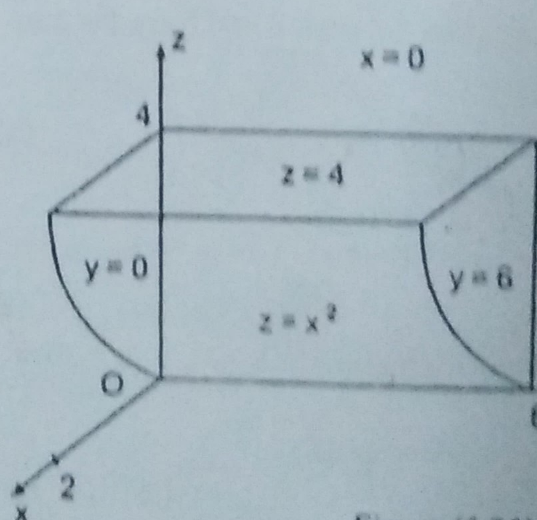


Figure (5.20)

$$= \hat{i}\int_0^2\int_0^6 x\,|z|_{x^2}^4\,dy\,dx + \hat{j}\int_0^2\int_0^6 y\,|z|_{x^2}^4\,dy\,dx + \hat{k}\int_0^2\int_0^6 \left|\frac{1}{2}z^2\right|_{x^2}^4 dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6 (4x - x^3)\,dy\,dx + \hat{j}\int_0^2\int_0^6 (4 - x^2)\,y\,dy\,dx + \hat{k}\int_0^2\int_0^6 \left(8 - \frac{1}{2}x^4\right) \ldots$$

$$= 6\hat{i}\int_0^2 (4x - x^3)\,dx + \hat{j}\int_0^2 (4 - x^2)\left|\frac{1}{2}y^2\right|_0^6 dx + 6\hat{k}\int_0^2 \left(8 - \frac{1}{2}x^4\right.$$

$$= 6\hat{i}\left|2x^2 - \frac{1}{4}x^4\right|_0^2 + 18\hat{j}\left|4x - \frac{1}{3}x^3\right|_0^2 + 6\hat{k}\left|8x - \frac{1}{10}x^5\right|_0^2$$

$$= 6\hat{i}(4) + 18\hat{j}\left(\frac{16}{3}\right) + 6\hat{k}\left(\frac{64}{5}\right)$$

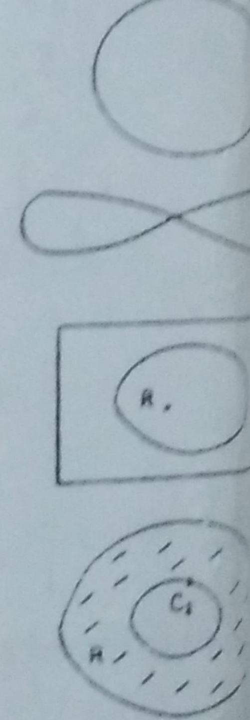
$$= 24\hat{i} + 96\hat{j} + \frac{384}{5}\hat{k}$$

5.9 SIMPLY AND MULTIPLY CONNECTED REGIONS

A simple closed curve is a closed curve which does not intersect itself anywhere. For example, the curve in figure [5.21 (a)] is a simple closed curve while the curve in figure [5.21 (b)] is not.

A region R is said to be simply connected if any simple closed curve lying in R can be continuously shrunk to a point. For example, the interior of a rectangle as shown in figure [5.21 (c)] is an example of a simply connected region.

A region R which is not simply connected is called multiply connected. For example, the region R exterior to C_2 and interior to C_1 is not simply connected because a circle drawn within R and enclosing C_2 cannot be shrunk to a point without crossing C_2 as shown in figure [5.21 (d)]. In other words, the regions which have holes are called multiply connected.



Figure

5.10 GREEN'S THEOREM IN THE PLANE

We will consider vector functions of just x and y and derive a relationship be... integral around a closed curve and a double integral over the part of the plane enclosed by the c...

THOREM (5.6): If R is a simply-connected region of the xy-plane bounded by a clos... and if M and N are continuous functions of x and y havin... derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is described in the positive (counterclockwise) direction.

PROOF: We prove the theorem for a closed curve C which has the property th... line parallel to the coordinate axes cuts C in at most two points as shown in figure (5.22). Let the equations of the curves AEB and AFB be $y = f_1(x)$ and $y = f_2(x)$ respectively

$$\iint_R \frac{\partial M}{\partial y}\, dx\, dy = \int_{x=a}^{b} \left[\int_{y=f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy \right] dx = \int_a^b \left| M(x,y) \right|_{y=f_1(x)}^{f_2(x)} dx$$

$$= \int_a^b \left[M(x, f_2) - M(x, f_1) \right] dx$$

$$= - \int_a^b M(x, f_1)\, dx - \int_b^a M(x, f_2)\, dx = - \oint_C M\, dx$$

en $$\oint_C M\, dx = - \iint_R \frac{\partial M}{\partial y}\, dx\, dy \qquad (1)$$

milarly let the equations of the curves EAF and EBF be $x = g_1(y)$ and $x = g_2(y)$ respectively.

en

$$\iint_R \frac{\partial N}{\partial x}\, dx\, dy = \int_{y=e}^{f} \left[\int_{x=g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\, dx \right] dy = \int_e^f \left| N(x,y) \right|_{g_1(y)}^{g_2(y)} dy$$

$$= \int_e^f \left[N(g_2, y) - N(g_1, y) \right] dy$$

$$= \int_f^e N(g_1, y)\, dy + \int_e^f N(g_2, y)\, dy$$

$$= \oint_C N\, dy$$

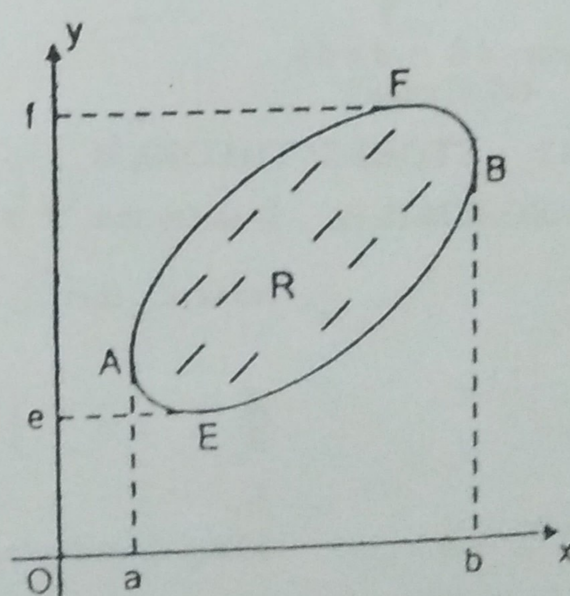


Figure (5.22)

en $$\oint_C N\, dy = \iint_R \frac{\partial N}{\partial x}\, dx\, dy \qquad (2)$$

dding equations (1) and (2), we get

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

OTE: (i) The proof can be extended to the curves C for which lines parallel to the coordinate axes may it C in more than two points as shown in figure (5.23) (a).

(ii) The theorem also holds for a multiply–connected region R such as shown in figure (5.23) (b).

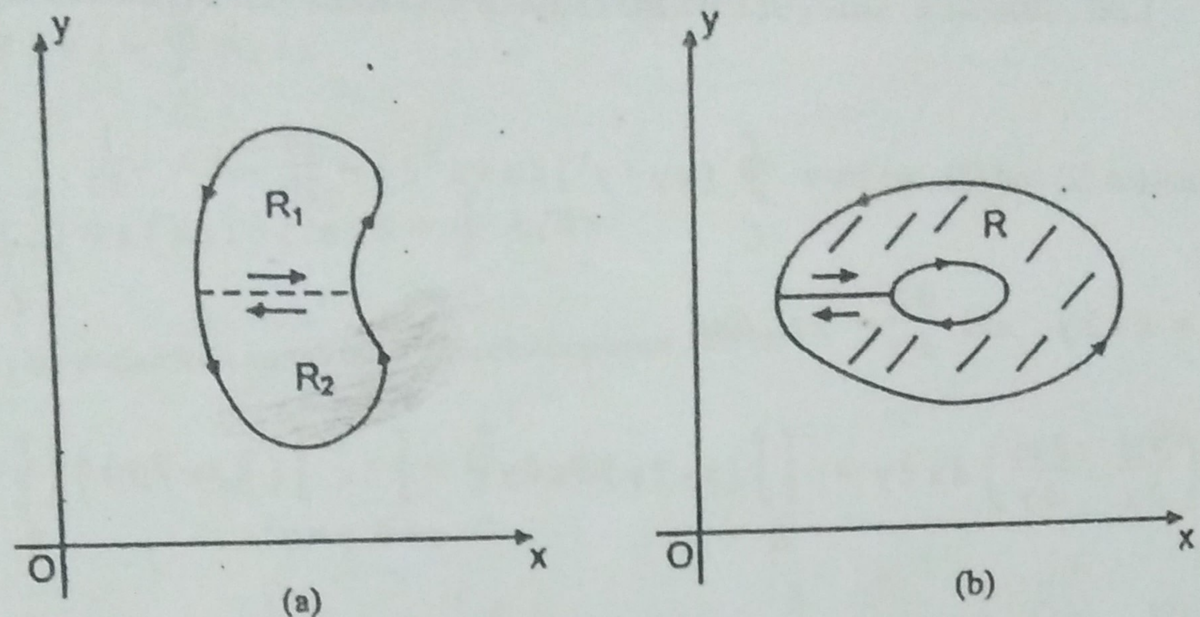


Figure (5.23)

EXAMPLE (14): **Verify Green's theorem in the plane for $M = xy + y^2$ and $N = x^2$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.**

SOLUTION: The plane curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$. Let C_1 be the curve $y = x^2$ and C_2 the curve $y = x$ and let the closed curve C be formed from C_1 and C_2. The positive direction in traversing C is shown in figure (5.24).

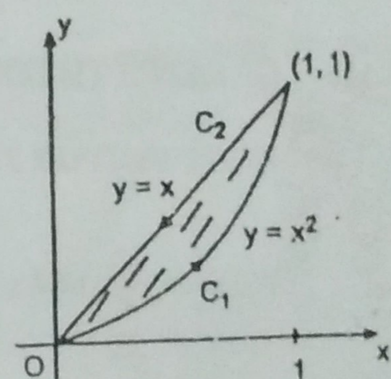


Figure (5.24)

Then we must show that

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Now
$$\oint_C M\,dx + N\,dy = \oint_C (xy + y^2)\,dx + x^2\,dy \qquad (1)$$

Along the curve C_1: $y = x^2$, $dy = 2x\,dx$, while x varies from 0 to 1. The line integral (1) equals

$$\int_{C_1} M\,dx + N\,dy = \int_0^1 (x^3 + x^4)\,dx + 2x^3\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx = \left| \frac{3}{4}x^4 + \frac{x^5}{5} \right|_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \qquad (2)$$

Along the curve C_2: $y = x$, $dy = dx$, while x varies from 1 to 0. The line integral (1) equals

$$\int_{C_2} M\,dx + N\,dy = \int_1^0 2x^2\,dx + x^2\,dx = \int_1^0 3x^2\,dx = \left|x^3\right|_1^0 = -1 \qquad (3)$$

Then from equations (2) and (3), we have $\oint_C (xy+y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = -\frac{1}{20}$

Since $\frac{\partial M}{\partial y} = x + 2y$, and $\frac{\partial N}{\partial x} = 2x$, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (x-2y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dy$$

$$= \int_0^1 \left| xy - y^2 \right|_{x^2}^{x} dx = \int_0^1 (x^4 - x^3)\,dx$$

$$= \left| \frac{x^5}{5} - \frac{x^4}{4} \right|_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

so that the theorem is verified.

5.11 GREEN'S THEOREM IN THE PLANE IN VECTOR NOTATION

FIRST VECTOR FORM (OR TANGENTIAL FORM) OF GREEN'S THEOREM

We have $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad (1)$$

Now $M\,dx + N\,dy = (M\hat{i} + N\hat{j}).(dx\,\hat{i} + dy\,\hat{j}) = \bar{A}.d\bar{r}$

where $\bar{A} = M\hat{i} + N\hat{j}$ and $d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$. Also, if $\bar{A} = M\hat{i} + N\hat{j}$ then

$$\nabla \times \bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\hat{i} + \frac{\partial M}{\partial z}\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

so that $(\nabla \times \bar{A}).\hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

Then from equation (1) Green's theorem in the plane can be written

$$\oint_C \bar{A}.d\bar{r} = \iint_R (\nabla \times \bar{A}).\hat{k}\,dR \quad \text{where } dR = dx\,dy$$

A generalization of this to surface S in space having C as boundary leads quite naturally t theorem. This form of Green's theorem is sometimes called Stokes' theorem in the plane Thu

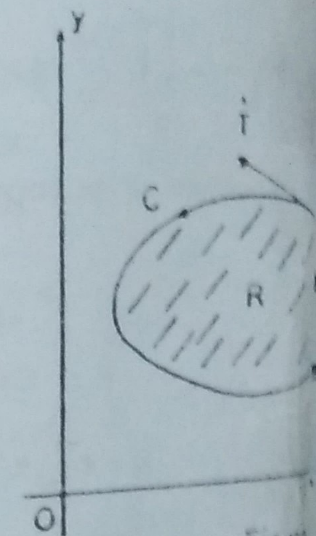
SECOND VECTOR FORM (OR NORMAL FORM) OF GREEN'S THEOREM

As above , $M\,dx + N\,dy = \bar{A}\,.\,d\bar{r} = \bar{A}\,.\,\hat{T}\,ds$

where $\dfrac{d\bar{r}}{ds} = \hat{T} =$ Unit tangent vector to C [see the figure (5.25)] .

If $\hat{n}$ is the outward drawn unit normal to C , then $\hat{T} = \hat{k} \times \hat{n}$ so that

$$M\,dx + N\,dy = \bar{A}\,.\,\hat{T}\,ds = \bar{A}\,.\,(\hat{k} \times \hat{n})\,ds = (\bar{A} \times \hat{k})\,.\,\hat{n}\,ds$$



Since $\bar{A} = M\hat{i} + N\hat{j}$, therefore

$$\bar{B} = \bar{A} \times \hat{k} = (M\hat{i} + N\hat{j}) \times \hat{k} = N\hat{i} - M\hat{j}$$

and $\nabla\,.\,\bar{B} = \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$

then equation (1) becomes $\oint_C \bar{B}\,.\,\hat{n}\,ds = \iint_R \nabla\,.\,\bar{B}\,dR$

where $dR = dx\,dy$

5.12 STOKES' THEOREM

THEOREM (5.7): It states that if S is an open , two–sided surface bounded by a simple closed curve C , then if $\bar{A}$ has continuous first partial derivatives

$$\oint_C \bar{A}\,.\,d\bar{r} = \iint_S (\nabla \times \bar{A})\,.\,\hat{n}\,dS$$

where C is traversed in the positive direction

In words the line integral of the tangential component of a vector function $\bar{A}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\bar{A}$ taken over any surface S having C as its boundary

PROOF: Let $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then Stokes' theorem can be written as

$$\iint_S [\nabla \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})]\,.\,\hat{n}\,dS = \oint_C A_1\,dx + A_2\,dy + A_3\,dz$$

We prove this theorem for a surface S which has the property that its projections on the xy, yz and xz planes are regions bounded by simple closed curves as shown in figure (5.26). Assume S to have representation $z = f(x, y)$ or $x = g(y, z)$ or $y = h(z, x)$, where f, g, h are for both curves , we must have

$$\oint_{\Gamma} F\,dx = \oint_{C} A_1\,dx$$

or
$$\iint_{S} [\nabla \times (A_1 \hat{i})] . \hat{n}\,dS = \oint_{C} A_1\,dx \qquad (4)$$

Similarly , by projections on the other coordinate planes , we have

$$\iint_{S} [\nabla \times (A_2 \hat{j})] . \hat{n}\,dS = \oint_{C} A_2\,dy \qquad (5)$$

$$\iint_{S} [\nabla \times (A_3 \hat{k})] . \hat{n}\,dS = \oint_{C} A_3\,dz \qquad (6)$$

Addition of equations (4) , (5) , and (6) completes the proof of the theorem .

RECTANGULAR FORM OF STOKES' THEOREM

Let $\overline{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$ and $\hat{n} = n_1 \hat{i} + n_2 \hat{j} + n_3 \hat{k}$ be the outward drawn unit to the surface S . If α , β , and γ are the angles which the unit normal $\hat{n}$ makes with the directions of x , y , and z axes respectively , then

$$n_1 = \hat{n} . \hat{i} = \cos\alpha$$

$$n_2 = \hat{n} . \hat{j} = \cos\beta$$

and
$$n_3 = \hat{n} . \hat{k} = \cos\gamma$$

The quantities $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the directions cosines of $\hat{n}$. Then

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

Thus
$$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

and
$$(\nabla \times \overline{A}) . \hat{n} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \text{co}$$

Also
$$\overline{A} . d\overline{r} = (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) . (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = A_1\,dx + A_2\,dy + A_3\,d$$

and Stokes' theorem becomes

$$\oint_C A_1\,dx + A_2\,dy + A_3\,dz$$

$$= \iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma\right] dS$$

EXAMPLE (15): Verify Stokes' theorem for $\overline{A} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}$, where S i upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

SOLUTION: The surface S and its projection R on the xy–plane is shown in figure (5.27). The boundary C of S is a circle in the xy–plane of radius 1 and centre at the origin.

Let $x = \cos\theta$, $y = \sin\theta$, $z = 0$, $0 \le \theta \le 2\pi$ be the parametric equations of C.

Then $$\oint_C \overline{A}\cdot d\overline{r} = \oint_C (2x - y)\,dx - yz^2\,dy - y^2 z\,dz$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta)\,d\theta$$

$$= \int_0^{2\pi} \left[-\sin 2\theta + \left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$$

$$= \left|\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right|_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi$$

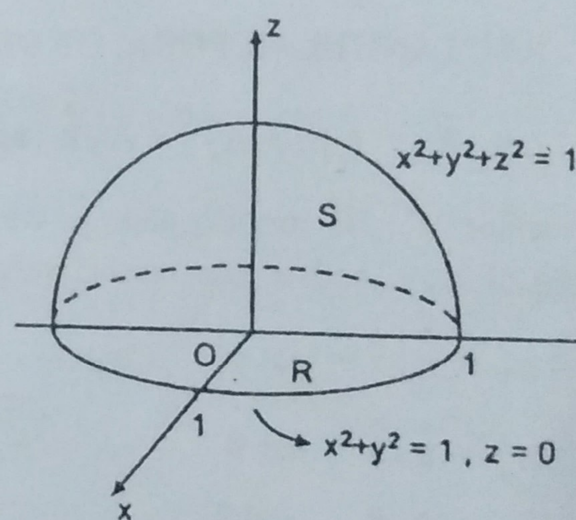


Figure (5.2

Also, $$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \hat{k}$$

Then $$\iint_S (\nabla \times \overline{A})\cdot\hat{n}\,dS = \iint_S \hat{k}\cdot\hat{n}\,dS = \iint_R dx\,dy \quad (\text{since } \hat{n}\cdot\hat{k}\,dS = dx\,dy)$$

$$= 4\int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \sqrt{1 - x^2}\,dx$$

Let $x = \sin\theta$, $dx = \cos\theta \, d\theta$, $0 \le \theta \le \pi/2$. Then

$$\iint_S (\nabla \times \bar{A}) \cdot \hat{n} \, dS = 4 \int_0^{\pi/2} \cos^2\theta \, d\theta = 4\left(\frac{1}{2}\right) \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta$$

$$= 2 \left| \theta + \frac{\sin 2\theta}{2} \right|_0^{\pi/2} = 2\left(\frac{\pi}{2}\right) = \pi$$

and Stokes' theorem is verified.

5.13 GAUSS' DIVERGENCE THEOREM

Gauss' divergence theorem has wide applications in mathematics, physics and en used to derive equations governing the flow of fluids, heat conduction, wave propagation fields.

THEOREM (5.8): It states that if R is the region bounded by a closed surface S vector point function with continuous first partial derivatives, the

$$\iint_S \bar{A} \cdot \hat{n} \, dS = \iiint_R \nabla \cdot \bar{A} \, dV$$

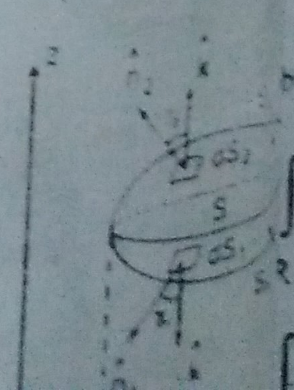
where $\hat{n}$ is the outward drawn unit normal to S.

In words the surface integral of the normal component of a vector function $\bar{A}$ taken surface S is equal to the integral of the divergence of $\bar{A}$ taken over the region R enclosed by

PROOF: If $\bar{A}$ is expressed in terms of components as $\bar{A} = A_1 \hat{i} + A_2 \hat{j} + A$ divergence theorem can be written as

$$\iint_S (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \cdot \hat{n} \, dS = \iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV$$

To establish this, we prove that the respective integrals on each side are equal



We prove this for a closed surface S, which has the property that any line parallel to the coordinate axes cuts S in atmost two points. Under this assumption, it follows that S is a double valued surface over its proj

$$\frac{\partial A_3}{\partial z} dV = \iiint_R \frac{\partial A_3}{\partial z} dz\,dy\,dx$$

$$= \iint_{R'} \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z} dz \right] dy\,dz$$

$$= \iint_{R'} \left| A_3(x,y,z) \right|_{f_1(x,y)}^{f_2(x,y)} dy\,dx$$

$$= \iint_{R'} \{ A_3[x,y,f_2(x,y)] - A_3[x,y,f_1(x,y)] \} dy\,dx \qquad (1)$$

part S_2, $dy\,dx = \cos\gamma_2\, dS_2 = \hat{k}.\hat{n}_2\, dS_2$, since the normal $\hat{n}_2$ to S_2 makes an

. For the lower part S_1, $dy\,dx = \cos\gamma_1\, dS_1 = -\hat{k}.\hat{n}_1\, dS_1$, since the normal

angle γ_1 with $-\hat{k}$.

$$A_3[x,y,f_2(x,y)]\,dy\,dx = \iint_{S_2} A_3 \hat{k}.\hat{n}_2\, dS_2$$

$$A_3[x,y,f_1(x,y)]\,dy\,dx = -\iint_{S_1} A_3 \hat{k}.\hat{n}_1\, dS_1$$

e equation (1) becomes

$$\iiint \frac{\partial A_3}{\partial z} dV = \iint_{S_2} A_3 \hat{k}.\hat{n}_2\, dS_2 + \iint_{S_1} A_3 \hat{k}.\hat{n}_1\, dS_1$$

$$= \iint_S A_3 \hat{k}.\hat{n}\, dS \qquad (2)$$

by projecting S on the yz and zx coordinate planes, we obtain respectively,

$$\int \frac{\partial A_3}{\partial z} dV = \iiint_R \frac{\partial A_3}{\partial z} dz\,dy\,dx$$

$$= \iint_{R'} \left[\int_{z = f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z} dz \right] dy\,dz$$

$$= \iint_{R'} A_3(x,y,z) \Big|_{f_1(x,y)}^{f_2(x,y)} dy\,dx$$

$$= \iint_{R'} \{ A_3[x,y,f_2(x,y)] - A_3[x,y,f_1(x,y)] \} dy\,dx \qquad (1)$$

part S_2, $dy\,dx = \cos\gamma_2\, dS_2 = \hat{k}\cdot\hat{n}_2\, dS_2$, since the normal $\hat{n}_2$ to S_2 makes an acute . For the lower part S_1, $dy\,dx = \cos\gamma_1\, dS_1 = -\hat{k}\cdot\hat{n}_1\, dS_1$, since the normal $\hat{n}_1$ to angle γ_1 with $-\hat{k}$.

$$A_3[x,y,f_2(x,y)]\,dy\,dx = \iint_{S_2} A_3 \hat{k}\cdot\hat{n}_2\, dS_2$$

$$A_3[x,y,f_1(x,y)]\,dy\,dx = -\iint_{S_1} A_3 \hat{k}\cdot\hat{n}_1\, dS_1$$

equation (1) becomes

$$\int \frac{\partial A_3}{\partial z} dV = \iint_{S_2} A_3 \hat{k}\cdot\hat{n}_2\, dS_2 + \iint_{S_1} A_3 \hat{k}\cdot\hat{n}_1\, dS_1$$

$$= \iint_S A_3 \hat{k}\cdot\hat{n}\, dS \qquad (2)$$

projecting S on the ... coordinate planes, we obtain respectively, tion of equations (2), (3), and (4) completes the proof of the theorem.

ce that Gauss' divergence theorem is a generalization of Green's theorem in the plane where the ne) region R and its boundary (curve) C are replaced by a (space) region R and its closed boundary face) S. For this reason the divergence theorem is often called Green's theorem in space.

RECTANGULAR FORM OF GAUSS'S DIVERGENCE THEOREM

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, and $\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

$$\nabla . \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\vec{A} . \hat{n} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})$$

$$= A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma$$

d the Gauss' divergence theorem can be written as

$$\iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dx\,dy\,dz = \iint_S (A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma)\, dS$$

XAMPLE (16) **Verify the divergence theorem for $\vec{A} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ where S is the surface of the cube bounded by**

$x = 0, \quad x = 1, \quad y = 0, \quad y = 1, \quad z = 0, \quad z = 1.$

OLUTION: The given cube is shown in figure (5.29). By the divergence theorem, we have

$$\iint_S \vec{A} . \hat{n}\, dS = \iiint_R \nabla . \vec{A}\, dV$$

Now

$$\iiint_R \nabla . \vec{A}\, dV = \iiint_R \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_R (4z - y)\, dV = \int_0^1\int_0^1\int_0^1 (4z - y)\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left|2z^2 - yz\right|_0^1 dy\, dx = \int_0^1\int_0^1 (2 - y)\, dy\, dx$$

$$= \int_0^1 \left|2y - \frac{y^2}{2}\right|_0^1 dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}$$

$$\iint \vec{A} . \hat{n}\, dS = \iint \vec{A} . \hat{n}\, dS_1 + \iint \vec{A} . \hat{n}\, dS_2$$