he line segment C_2 from $(1,0)$ to $(1,1)$, $x = 1$, $dx = 0$, while y varies from 0 to

egral (1) over this part of the path is $\int_{C_2} \bar{A} \cdot d\bar{r} = \int_{y=0}^{1} 2y\,dy = 1$

equation (2), we get $\int_C \bar{A} \cdot d\bar{r} = 1 + 1 = 2$

Along the line segment C_1 from $(0,0)$ to $(0,1)$, $x = 0$, $dx = 0$, while y varies from

ntegral (1) over this part of the path is $\int_{C_1} \bar{A} \cdot d\bar{r} = \int_{x=0}^{1} 0\,dy = 0$

ng the line segment C_2 from $(0,1)$ to $(1,1)$, $y = 1$, $dy = 0$, while x varies from 0 to

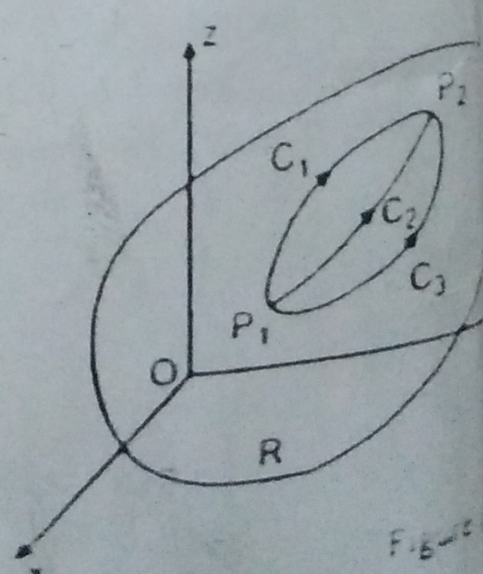
integral (1) over this part of the path is $\int_{C_2} \bar{A} \cdot d\bar{r} = \int_{x=0}^{1} (1 + x^2)\,dx = \left| x + \frac{x^3}{3} \right|_0^1 =$

om equation (2), we get $\int_C \bar{A} \cdot d\bar{r} = 0 + \frac{4}{3} = \frac{4}{3}$.

OTE: In example (4), we have seen that the value of a line integral $\int_C \bar{A} \cdot d\bar{r}$ in general, depe n the path C joining the points P_1 and P_2. We now show that for certain types of vector funct alue of the line integral will depend only on P_1 and P_2 but will not depend on the path C from P_2. We first state the following definition.

5.4 LINE INTEGRAL INDEPENDENT OF PATH (OR CONSERVATIVE FIEL

The line integral $\int_C \bar{A} \cdot d\bar{r}$ is said to be independent of the path C (or the vector field $\vec{A}$ is conservative) in a given region R, if the value of the line integral $\int_{P_1}^{P_2} \bar{A} \cdot d\bar{r}$ is the same for all paths C joining any two given points P_1 and P_2 in R. Thus as shown in figure (5.8), the line integral is independent of the path C if the integrals along C_1, C_2, C



Fig

LE (5): If $\overline{A} = 2xy^2\,\hat{i} + 2(x^2y + y)\,\hat{j}$, evaluate $\int_C \overline{A}\cdot d\overline{r}$ from $(0,0)$ to $(2,4)$ along the following paths C:

(i) the straight line $y = 2x$

(ii) the parabola $y = x^2$

(iii) the line segment C_1 from $(0,0)$ to $(2,0)$ and then the line segment C_2 from $(2,0)$ to $(2,4)$.

TION: Since integration is performed in the xy–plane , therefore $d\overline{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\int_C \overline{A}\cdot d\overline{r} = \int_C 2xy^2\,dx + 2(x^2y + y \quad (1)$$

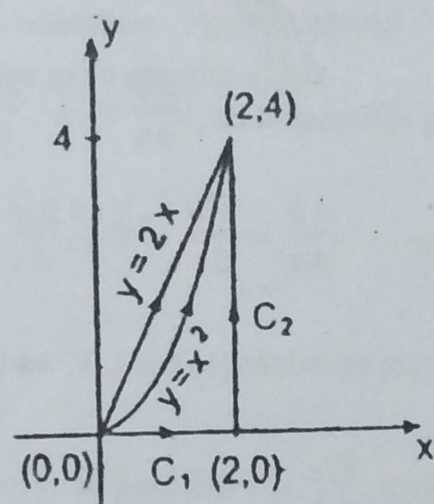


Figure (5.9)

the path C in each case is shown in figure (5.9).

Along the straight line $y = 2x$, we have

$2\,dx$ while x varies from 0 to 2.

e integral (1) becomes

$$\int_C \overline{A}\cdot d\overline{r} = \int_{x=0}^{2} 2x(2x)^2\,dx + 2[x^2(2x) + 2x]\,2\,dx$$

$$= \int_0^2 (16x^3 + 8x)\,dx = \left|4x^4 + 4x^2\right|_0^2 = 64 + 16 = 80.$$

Along the parabola $y = x^2$, we have $dy = 2x\,dx$ while x varies from 0 to 2.

e integral (1) becomes

$$\int_C \overline{A}\cdot d\overline{r} = \int_{x=0}^{2} 2x(x^2)^2\,dx + 2[x^2(x^2) + x^2]\,2x\,dx$$

$$= \int_0^2 (6x^5 + 4x^3)\,dx = \left|x^6 + x^4\right|_0^2 = 64 + 16 = 80.$$

In this case we have
$$\int_C \overline{A}\cdot d\overline{r} = \int_{C_1} \overline{A}\cdot d\overline{r} + \int_{C_2} \overline{A}\cdot d\overline{r} \quad (2)$$

e C is the curve consisting of the line segments C_1 and C_2.

long the line segment C_1, $y = 0$, therefore $dy = 0$, while x varies from 0 to 2. Thus

$$\int_{C_1} \bar{A}\cdot d\bar{r} = \int_{x=0}^{2} 0\,dx = 0$$

Along the line segment C_2, $x = 2$, therefore $dx = 0$ while y varies from 0 to 4.

$$\int_{C_2} \bar{A}\cdot d\bar{r} = \int_{y=0}^{4} 2(4y+y)\,dy = \int_{0}^{4} 10y\,dy = 5\left|y^2\right|_0^4 = 5(16) = 80$$

From equation (2), we get $\int_{C} \bar{A}\cdot d\bar{r} = 0 + 80 = 80$

5.5 THEOREMS ON LINE INTEGRALS INDEPENDENT OF PATH

THEOREM (5.1): Prove that a necessary and sufficient condition for $\int_{P_1}^{P_2} \bar{A}\cdot d\bar{r}$ to be independent of the path joining any two points P_1 and P_2 (i.e. $\bar{A}$ to be conservative) in a given region is that $\oint_C \bar{A}\cdot d\bar{r} = 0$ for all closed paths C in the region.

PROOF: Let C be any simple closed curve, and let P_1 and P_2 be any two points on C as shown in figure (5.10). Then since by hypothesis, the integral is independent of the path

(i.e. $\bar{A}$ is conservative), we have $\int_{P_1AP_2} \bar{A}\cdot d\bar{r} = \int_{P_1BP_2} \bar{A}\cdot d\bar{r}$

Reversing the direction of integration in the integral on the right, we have

$$\int_{P_1AP_2} \bar{A}\cdot d\bar{r} = -\int_{P_2BP_1} \bar{A}\cdot d\bar{r}$$

or $$\int_{P_1AP_2} \bar{A}\cdot d\bar{r} + \int_{P_2BP_1} \bar{A}\cdot d\bar{r} = 0$$

or $$\oint_C \bar{A}\cdot d\bar{r} = 0$$

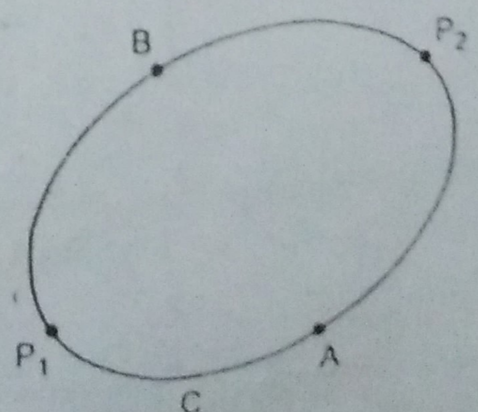


Figure (5.10)

Conversely, if $\oint_C \bar{A}\cdot d\bar{r} = 0$, then $\int_{P_1AP_2} \bar{A}\cdot d\bar{r} + \int_{P_2BP_1} \bar{A}\cdot d\bar{r} = 0$

or $\int_{P_1AP_2} \bar{A}\cdot d\bar{r} - \int_{P_1BP_2} \bar{A}\cdot d\bar{r} = 0$

or $\int_{P_1AP_2} \bar{A}\cdot d\bar{r} = \int_{P_1BP_2} \bar{A}\cdot d\bar{r}$

which shows that the line integral is independent of the path joining P_1 and P_2 as required.

SCALAR POTENTIAL FUNCTION

A scalar potential function ϕ is a single–valued function for which there exists a co vector field $\bar{A}$ in a simply connected region R that satisfies the relation $\bar{A} = \nabla\phi$.

THEOREM (5.2): **Prove that a necessary and sufficient condition for $\int_C \bar{A}\cdot d\bar{r}$ to be indep of the path C joining any two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2,$ (i.e. $\bar{A}$ to be conservative) is that there exists a scalar function ϕ su $\bar{A} = \nabla\phi$, where ϕ is single valued and has continuous partial derivativ**

PROOF: Let $\bar{A} = \nabla\phi$, then

$$\int_C \bar{A}\cdot d\bar{r} = \int_{P_1}^{P_2} \bar{A}\cdot d\bar{r} = \int_{P_1}^{P_2} \nabla\phi\cdot d\bar{r}$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Conversely, let $\int_C \overline{A} \cdot d\overline{r}$ be independent of the path C joining any two points. We choose these points as a fixed point $P_1 \equiv (x_1, y_1, z_1)$ and a variable point $P_2 \equiv (x, y, z)$, so that the integral is a function only of the coordinates (x, y, z) of the variable end point. Then

$$\phi(x, y, z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \overline{A} \cdot d\overline{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \overline{A} \cdot \frac{d\overline{r}}{ds}\, ds$$

By differentiation, $\dfrac{d\phi}{ds} = \overline{A} \cdot \dfrac{d\overline{r}}{ds}$ (1)

But $\dfrac{d\phi}{ds} = \dfrac{\partial \phi}{\partial s} = \nabla \phi \cdot \dfrac{d\overline{r}}{ds}$ (2)

From equations (1) and (2), we have $(\nabla \phi - \overline{A}) \cdot \dfrac{d\overline{r}}{ds} = 0$ (3)

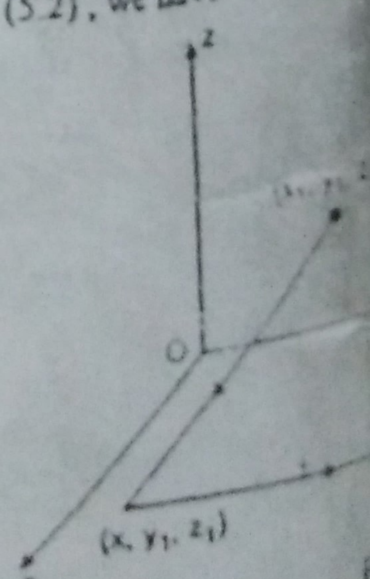
Since $\dfrac{d\overline{r}}{ds}$ is a unit tangent vector and $\neq \overline{0}$, therefore equation (3) implies that

$\nabla \phi - \overline{A} = 0$ or $\overline{A} = \nabla \phi$. Hence the theorem.

THEOREM (5.3): **Prove that a necessary and sufficient condition that a vector field be conservative is that $\nabla \times \overline{A} = \overline{0}$ (i.e. $\overline{A}$ is irrotational).**

PROOF: If $\overline{A}$ is a conservative field then by theorem (5.2), we have $\overline{A} = \nabla \phi$

Thus $\nabla \times \overline{A} = \nabla \times \nabla \phi = \overline{0}$



Conversely, if $\nabla \times \overline{A} = \overline{0}$, then

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \overline{0} \text{ and thus}$$

$$\frac{\partial A_3}{\partial y} = \frac{\partial A_2}{\partial z}, \quad \frac{\partial A_1}{\partial z} = \frac{\partial A_3}{\partial x}, \quad \frac{\partial A_2}{\partial x} = \frac{\partial A_1}{\partial y}$$

We must prove that $\overline{A} = \nabla \phi$ follows as a consequence of this

Now $\int_C \overline{A} \cdot d\overline{r} = \int_C A_1(x, y, z)\, dx + A_2(x, y, z)\, dy + A_3(x, y, z)\, dz$

$\int \bar{A} \cdot d\bar{r} = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \nabla \phi \cdot d\bar{r}$

$\Rightarrow \bar{A} = \nabla \phi$

EXAMPLE (6): Show that the vector field $\overline{A} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$ is conservative. Hence find the scalar potential function ϕ for which $\overline{A} = \nabla \phi$.

SOLUTION: We know that a necessary and sufficient condition for a vector field $\overline{A}$ to be conservative is $\nabla \times \overline{A} = \overline{0}$

Now
$$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= (-1+1)\,\hat{i} + (1-1)\,\hat{j} + (\cos y - \cos y)\,\hat{k} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \overline{0}$$

Thus the vector field $\overline{A}$ is a conservative.

Let
$$\overline{A} = \nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$$

Then
$$\frac{\partial \phi}{\partial x} = \sin y + z \qquad (1)$$
$$\frac{\partial \phi}{\partial y} = x\cos y - z \qquad (2)$$
$$\frac{\partial \phi}{\partial z} = x - y \qquad (3)$$

Integrating equations (1), (2), and (3), we get
$$\phi = x\sin y + xz + f(y, z)$$
$$\phi = x\sin y - yz + g(x, z)$$
$$\phi = xz - yz + h(x, y)$$

These agree if we choose $f(y, z) = -yz$, $g(x, z) = xz$, $h(x, y) = x\sin y$

so that $\phi = x\sin y + xz - yz + C$

where C is any constant.

THEOREM (5.4): Show that a necessary and sufficient condition that $A_1\,dx + A_2\,dy + A_3\,dz$ be an exact differential is that $\nabla \times \overline{A} = \overline{0}$ where $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$.

PROOF: Let $A_1\,dx + A_2\,dy + A_3\,dz = d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

be an exact differential of a scalar function $\phi(x, y, z)$. Then on comparing coefficients, we have
$$A_1 = \frac{\partial \phi}{\partial x}, \quad A_2 = \frac{\partial \phi}{\partial y}, \quad A_3 = \frac{\partial \phi}{\partial z}$$

and so
$$\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \nabla \phi$$

5.3 LINE INTEGRAL DEPENDENT ON PATH (SAME END POINTS)

We now show that the value of a line integral $\int_C \bar{A} \cdot d\bar{r}$ in general, depends not only on t points P_1 and P_2 of the path C but also on the geometric shape of the path C ; i.e. if we integrat P_1 to P_2 along different paths, we in general, obtain different values of the integral.

EXAMPLE (4): If $\bar{A} = (1 + x^2 y)\,\hat{i} + 2\,x\,y\,\hat{j}$, evaluate $\int_C \bar{A} \cdot d\bar{r}$ from (0 , 0) to (
along the following paths C :

(i) the straight line from (0 , 0) to (1 , 1).

(ii) the line segment C_1 from (0 , 0) to (1 , 0) and then the line segment from (1 , 0) to (1 , 1).

(iii) the line segment C_1 from (0 , 0) to (0 , 1) and then the line segment from (0 , 1) to (1 , 1).

SOLUTION: Since integration is performed in the xy–plane, therefore $d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$,

and so
$$\int_C \bar{A} \cdot d\bar{r} = \int_C (1 + x^2 y)\,dx + 2\,x\,y\,dy \qquad (1)$$

where the path C in each case is shown in figure (5.7).

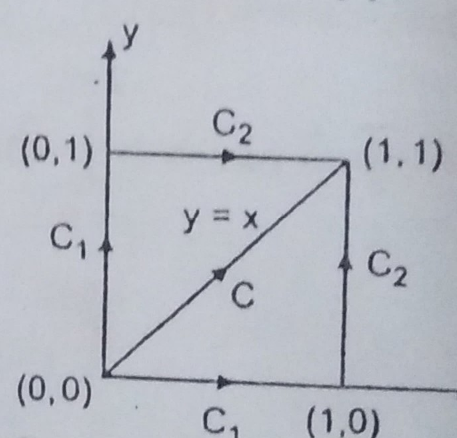


Figure (5.7)

(i) Along the straight line from (0 , 0) to (1 , 1), $y = x$, $dy = dx$ while x varies from 0 to 1. The line integral (1) becomes

$$\int_C \bar{A} \cdot d\bar{r} = \int_0^1 [1 + x^2 (x)]\,dx + 2\,x\,(x)\,dx = \int_0^1 (1 + 2x^2 + x^3)\,dx$$

$$= \left| x + \frac{2}{3}x^3 + \frac{1}{4}x^4 \right|_0^1 = 1 + \frac{2}{3} + \frac{1}{4} = \frac{23}{12}$$

(ii) In this case we have

$$\int_C \bar{A} \cdot d\bar{r} = \int_{C_1} \bar{A} \cdot d\bar{r} + \int_{C_2} \bar{A} \cdot d\bar{r} \qquad (2)$$

where C is the curve consisting of the line segments C_1 and C_2 as shown in figure (5.7).
Along the line segment C_1 from (0 , 0) to (1 , 0), $y = 0$, $dy = 0$, while x varies from 0 to 1

The integral (1) over this part of the path is $\int \bar{A} \cdot d\bar{r} = \int^1 dx = 1$