apter 5

INE , SURFACE , AND VOLUME INTEGRALS AND RELATED INTEGRAL THEOREMS

INTRODUCTION

So far , we have dealt with derivative operations on vector fields . In this chapter , we shall defin tegrals , surface integrals , and volume integrals and consider some important applications of thes ls . We shall see that a line integral is a natural generalization of the definite integral , the surfac l is a generalization of a double integral , and volume integral is a generalization of a triple integra ulus .

Line integrals can be transformed into double integrals with the help of Green's theorem in the With the help of Stokes' theorem , line integrals can be transformed into surface integrals , and sely . Surface integrals can be transformed into triple integrals and conversely with the help o divergence theorem . These transformations are of great practical importance . The correspondin ns of Green's , Stokes' , and Gauss serve as powerful tools in many practical as well as theoretica ns .

TANGENTIAL LINE INTEGRAL

Let $\bar{A}(x, y, z) = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$ be a vector unction which is defined and continuous along the arc f the space curve C . Subdivide the arc AB into n nts by means of the points $P_1, P_2, \ldots, P_{n-1}$ chosen ily and write $A = P_0$ and $B = P_n$ as shown in (5.1) . Consider one such segment $P_{k-1} P_k$ and let length of this segment be Δs_k , $k = 1, 2, \ldots, n$. $_k(x_k, y_k, z_k)$ be any point on the segment $P_{k-1} P_k$ efine $\bar{A}(x_k, y_k, z_k) = \bar{A}_k$. Let $\hat{T}_k$ be unit vector to C at Q_k .

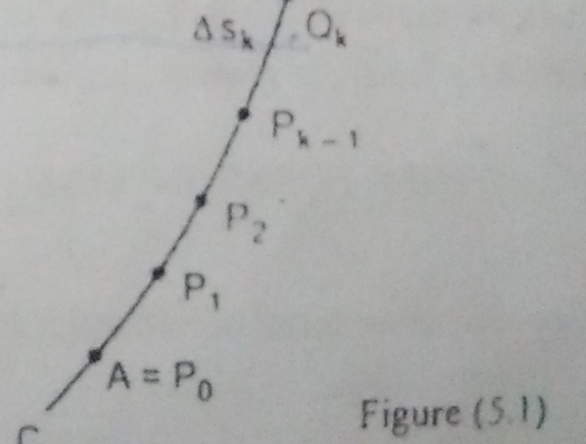


Figure (5.1)

ltiply the tangential component of $\bar{A}$ at Q_k with the arc length Δs_k of the correspondin t $P_{k-1} P_k$ and form the sum $\sum_{k=1}^{n} \bar{A}_k \cdot \hat{T}_k \, \Delta s_k$.

Now take the limit of this sum as $n \to \infty$ in such a way that the arc length of each segment Δs_k -

This limit, if it exists, is called the tangential line integral of $\overline{A}$ along C from A to B and is denot

by $\int_A^B \overline{A}.\hat{T}\,ds$ or $\int_C \overline{A}.\hat{T}\,ds$.

i.e. $$\underset{n \to \infty}{Lt} \sum_{k=1}^{n} \overline{A}_k.\hat{T}_k\,\Delta s_k = \int_A^B \overline{A}.\hat{T}\,ds$$

Since $\hat{T} = \frac{d\overline{r}}{ds}$ where $\overline{r}$ is the position vector of any point on C, it is usual to put $\hat{T}\,ds = d\overline{r}$

thus the line integral $$\int_A^B \overline{A}.\hat{T}\,ds = \int_A^B \overline{A}.d\overline{r} = \int_C \overline{A}.d\overline{r} = \int_C A_1 dx + A_2 dy + A_3 dz$$

where $d\overline{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ is called the differential displacement vector.

The line integral $\int_C \overline{A}.d\overline{r}$ is sometimes called a scalar line integral of a vector field A.

If C is a closed curve which we shall suppose a simple closed curve (i.e. a curve which does not inter itself anywhere), the line integral around C is often denoted by

$$\oint_C \overline{A}.d\overline{r} = \oint_C A_1 dx + A_2 dy + A_3 dz$$

If $\overline{A}$ is the force $\overline{F}$ on a particle moving along C, this line integral represents the work done by a forc

In fluid mechanics, this integral is called the circulation of $\overline{A}$ around C, where $\overline{A}$ represents velocity of a fluid. In general, any integral which is to be evaluated along a curve is called a line integra

OTHER FORMS OF LINE INTEGRALS

The other forms of line integrals are $\int_C \phi\,d\overline{r} = \hat{i}\int_C \phi\,dx + \hat{j}\int_C \phi\,dy + \hat{k}\int_C \phi\,dz$

and $$\int_C \overline{A} \times d\overline{r} = \int_C (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \times (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

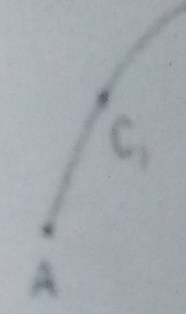
$$= \hat{i}\int_C (A_2 dz - A_3 dy) + \hat{j}\int_C (A_3 dx - A_1 dz) + \hat{k}\int_C (A_1 dy - A_2 dx)$$

GENERAL PROPERTIES OF LINE INTEGRALS

The follow... the properties of line integrals that are useful in computation...

(i) $\int_C K\bar{A}\cdot d\bar{r} = K\int_C \bar{A}\cdot d\bar{r}$ (K any real constant)

(ii) $\int_C (\bar{A}+\bar{B})\cdot d\bar{r} = \int_C \bar{A}\cdot d\bar{r} + \int_C \bar{B}\cdot d\bar{r}$



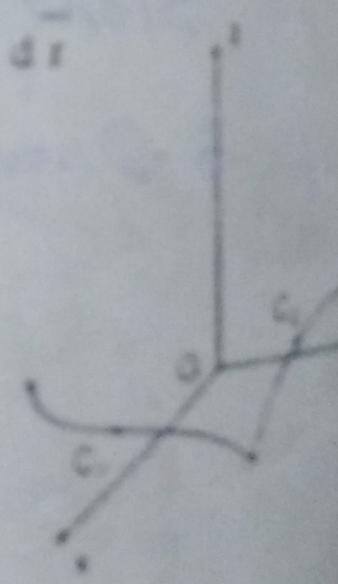
(iii) $\int_C \bar{A}\cdot d\bar{r} = \int_{C_1} \bar{A}\cdot d\bar{r} + \int_{C_2} \bar{A}\cdot d\bar{r}$

where the path C is subdivided into two arcs C_1 and C_2 that have the same orienta... in figure (5.2). If the sense of orientation along C is reversed, the value of the integral a...

(iv) If C is piecewise smooth, consisting of smooth curves $C_1, C_2, \ldots$

figure (5.3), the line integral of $\bar{A}$ over C is defined as the sum of the line integrals o... the smooth curves making up C :

$$\int_C \bar{A}\cdot d\bar{r} = \int_{C_1} \bar{A}\cdot d\bar{r} + \int_{C_2} \bar{A}\cdot d\bar{r} + \ldots + \int_{C_n} \bar{A}\cdot d\bar{r}$$



In this sum, the orientation along C must be maintained over the curves $C_1, C_2, \ldots, C_n$. That is, the initial point of C_j is the terminal point of C_{j-1}. This requirement is indicated by the arrows as shown in figure (5.3).

EXAMPLE (1): If $\bar{A} = 3xy\,\hat{i} - y^2\,\hat{j}$, evaluate $\int_C \bar{A}\cdot d\bar{r}$ where C i... x y - plane, $y = 2x^2$, from (0, 0) to (1, 2)

SOLUTION: The curve C defined by $y = 2x^2$ in the x y - plane is shown in figure (5.4). Since the integration is performed in the x y - plane (z = 0), we can take

$$\bar{r} = x\,\hat{i} + y\,\hat{j}, \text{ therefore } d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$$

Thus $\int \bar{A}\cdot d\bar{r} = \int (3 \ldots$

^{2}, then $dy = 4x\,dx$. Also x varies from 0 to 1.

$$\overline{A}\,.\,d\overline{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - 4x^4(4x)\,dx$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx = \left| \frac{3}{2}x^4 - \frac{8}{3}x^6 \right|_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

(2): **If $\overline{A} = (2x+y)\,\hat{i} + (3y-x)\,\hat{j}$, evaluate $\int_C \overline{A}\,.\,d\overline{r}$ where C is the curve in the xy–plane consisting of the line segment C_1 from $(0,0)$ to $(2,0)$ and then the line segment C_2 from $(2,0)$ to $(3,2)$.**

N: The path C consisting of line segments C_1 and C_2 is shown in figure (5.5).

ation is performed in the xy–plane, therefore $\overline{r} = x\,\hat{i} + y\,\hat{j}$ and so $d\overline{r} = dx\,\hat{i} + dy\,\hat{j}$.

$$\overline{A}\,.\,d\overline{r} = \int_C [(2x+y)\,\hat{i} + (3y-x)\,\hat{j}]\,.\,[dx\,\hat{i} + dy\,\hat{j}]$$

$$= \int_C (2x+y)\,dx + (3y-x)\,dy \qquad (1)$$

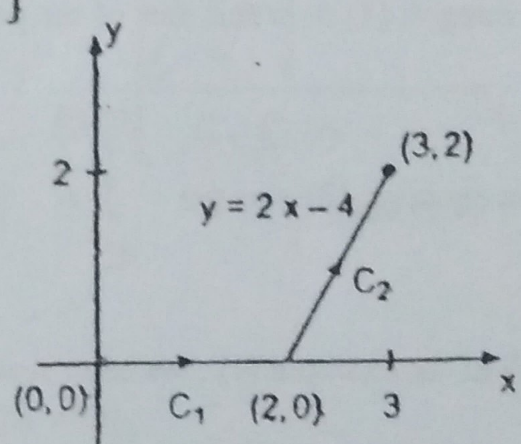


Figure (5.5)

consisting of the line segments C_1 and C_2, we have

$$\overline{A}\,.\,d\overline{r} = \int_{C_1} \overline{A}\,.\,d\overline{r} + \int_{C_2} \overline{A}\,.\,d\overline{r} \qquad (2)$$

segment C_1 from $(0,0)$ to $(2,0)$, $y = 0$ and so $dy = 0$, while x varies from 0 to 2.
al (1) over this part of the path is

$$\int_{C_1} \overline{A}\,.\,d\overline{r} = \int_{x=0}^{2} 2x\,dx = \left| x^2 \right|_0^2 = 4$$

e segment C_2 from $(2,0)$ to $(3,2)$, the equation is $y = 2x - 4$ and so $dy = 2\,dx$,
aries from 2 to 3. The integral (1) over this part of the path is

$$\int_{C_2} \overline{A}\,.\,d\overline{r} = \int_{x=2}^{3} [2x + (2x-4)]\,dx + [3(2x-4) - x]\,2\,dx$$

$$= \int_2^3 (14x - 28)\,dx = \left| 7x^2 - 28x \right|_2^3 = (63 - 84) - (28 - 56) = 7$$

From equation (2) we get

$$\int_C \overline{A}\cdot d\overline{r} = 4 + 7 = 11$$

EXAMPLE (3): If $\overline{A} = (x-3y)\,\hat{i} + (y-2x)\,\hat{j}$, evaluate $\oint_C \overline{A}\cdot d\overline{r}$ where C is an ellipse $\frac{x^2}{9}+\frac{y^2}{4} = 1$ in the xy-plane traversed in the positive (counterclockwise) direction.

SOLUTION: The curve C which is an ellipse with semi-major axis as 3 and semi-minor axis as 2 is shown in figure (5.6). Since the integration is performed in the xy-plane, we take $d\overline{r} = dx\,\hat{i} + dy\,\hat{j}$

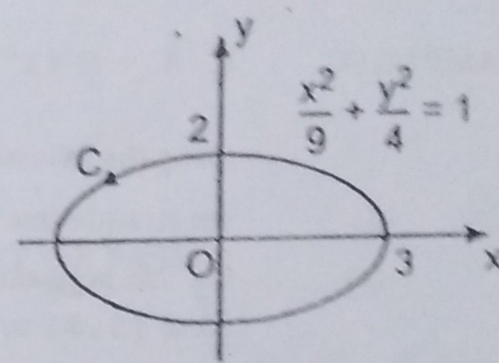


Figure (5.6)

Thus $$\oint_C \overline{A}\cdot d\overline{r} = \oint_C (x-3y)\,dx + (y-2x)\,dy \qquad (1)$$

The parametric equations of this ellipse are $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$ therefore, $dx = -3\sin t\,dt$, $dy = 2\cos t\,dt$. Hence from equation (1), we get

$$\oint_C \overline{A}\cdot d\overline{r} = \int_{t=0}^{2\pi} (3\cos t - 6\sin t)(-3\sin t\,dt) + (2\sin t - 6\cos t)(2\cos t\,dt)$$

$$= \int_0^{2\pi} (-5\sin t\cos t + 18\sin^2 t - 12\cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left[-\frac{5}{2}\sin 2t + 9(1-\cos 2t) - 6(1+\cos 2t)\right] dt$$

$$= \left|\frac{5}{4}\cos 2t + 9\left(t - \frac{\sin 2t}{2}\right) - 6\left(t + \frac{\sin 2t}{2}\right)\right|_0^{2\pi}$$

$$= \left|\frac{5}{4}\cos 2t + 3t - \frac{15}{2}\sin 2t\right|_0^{2\pi}$$

$$= \left(\frac{5}{4}\cos 4\pi + 6\pi\right) - \left(\frac{5}{4}\cos 0\right)$$

$$= \frac{5}{4} + 6\pi - \frac{5}{4} = 6\pi$$