A normal to the surface $\phi_2 = 4$ is

$$\nabla \phi_2 = 8xy\,\hat{i} + 4x^2\,\hat{j} + 3z^2\,\hat{k}$$

$$= -8\,\hat{i} + 4\,\hat{j} + 12\,\hat{k} \quad \text{at } (1, -1, 2)$$

The two normals will be orthogonal to each other at the given point if $\nabla \phi_1 \cdot \nabla \phi_2 = 0$

i.e. $(a-2)(-8) + (-2b)(4) + b(12) = 0$

or $-2a + b = -4$ (1)

Another equation in a and b follows from the surface $\phi_1 = 0$ at the given point,

i.e. $b = 1$ (2)

Solving equations (1) and (2), we find that $a = \frac{5}{2}$, $b = 1$

PROBLEMS ON THE DIVERGENCE

PROBLEM (17): Determine the constant a so that $\overline{V} = (x + 3y)\,\hat{i} + (y - 2x)\,\hat{j} + (x + az)\,\hat{k}$ is solenoidal.

SOLUTION: A vector $\overline{V}$ is solenoidal if its divergence is zero. Now

$$\nabla \cdot \overline{V} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2x) + \frac{\partial}{\partial z}(x + az) = 1 + 1 + a$$

Then $\nabla \cdot \overline{V} = a + 2 = 0$ implies $a = -2$

PROBLEM (18): If ϕ and ψ are scalar point functions, show that

$$\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$$

SOLUTION: We know that

$$\nabla \cdot (\phi \overline{A}) = \phi(\nabla \cdot \overline{A}) + \overline{A} \cdot \nabla \phi \qquad (1)$$

Let $\overline{A} = \nabla \psi$ in equation (1), we get

$$\nabla \cdot (\phi \nabla \psi) = \phi(\nabla \cdot \nabla \psi) + \nabla \psi \cdot \nabla \phi = \phi \nabla^2 \psi + \nabla \psi \cdot \nabla \phi \qquad (2)$$

Interchanging ϕ and ψ yields,

$$\nabla \cdot (\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi \qquad (3)$$

Subtracting equation (3) from equation (2), we get

$$\nabla \cdot (\phi \nabla \psi) - \nabla \cdot (\psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$$

or $$\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$$

PROBLEM (19): For a constant vector $\overline{A}$, show that $\nabla \cdot [(\overline{A} \cdot \overline{r})\,\overline{r}] = 4(\overline{A} \cdot \overline{r})$

SOLUTION:

$$\nabla \cdot [(\overline{A} \cdot \overline{r})\,\overline{r}] = (\overline{A} \cdot \overline{r})(\nabla \cdot \overline{r}) + \overline{r} \cdot \nabla(\overline{A} \cdot \overline{r})$$

$$= 3(\overline{A} \cdot \overline{r}) + \overline{r} \cdot \overline{A} \quad [\text{since } \nabla(\overline{A} \cdot \overline{r}) = \overline{A}]$$

$$= 4(\overline{A} \cdot \overline{r})$$

PROBLEM (20): If $\overline{A} = \overline{A}(x, y, z)$, show that $(d\,\overline{r} . \nabla)\overline{A} = d\overline{A}$.

SOLUTION: Since $\overline{A} = \overline{A}(x, y, z)$, therefore

$$d\overline{A} = \frac{\partial \overline{A}}{\partial x}dx + \frac{\partial \overline{A}}{\partial y}dy + \frac{\partial \overline{A}}{\partial z}dz$$

$$= \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)\overline{A}$$

$$= \left[(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}).\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\right]\overline{A}$$

$$= (d\,\overline{r} . \nabla)\overline{A}$$

PROBLEM (21): If $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, show that $\nabla . \overline{A} = \nabla A_1 . \hat{i} + \nabla A_2 . \hat{j} + \nabla A_3 . \hat{k}$

SOLUTION: Since $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, therefore

$$\nabla . \overline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \qquad (1)$$

Now $\nabla A_1 . \hat{i} = \left(\frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_1}{\partial y}\hat{j} + \frac{\partial A_1}{\partial z}\hat{k}\right).\hat{i} = \frac{\partial A_1}{\partial x}$

Similarly $\nabla A_2 . \hat{j} = \frac{\partial A_2}{\partial y}$ and $\nabla A_3 . \hat{k} = \frac{\partial A_3}{\partial z}$

Thus equation (1) becomes

$$\nabla . \overline{A} = \nabla A_1 . \hat{i} + \nabla A_2 . \hat{j} + \nabla A_3 . \hat{k}$$

PROBLEM (22): Prove that $\nabla . \left[\frac{f(r)\,\overline{r}}{r}\right] = \frac{2}{r} f(r) + f'(r)$.

SOLUTION: $\nabla . \left[\frac{f(r)\,\overline{r}}{r}\right] = \nabla . \left[\frac{f(r)}{r}\overline{r}\right]$

$$= \frac{f(r)}{r}(\nabla . \overline{r}) + \overline{r} . \nabla\left[\frac{f(r)}{r}\right] \qquad [\text{Theorem (4.5) (ii)}]$$

$$= \frac{f(r)}{r}(3) + \overline{r} . \frac{1}{r}\frac{d}{dr}\left[\frac{f(r)}{r}\right]\overline{r} \qquad [\text{Theorem (4.2)}]$$

$$= \frac{3}{r}f(r) + \overline{r} . \frac{1}{r}\left[\frac{r f'(r) - f(r)}{r^2}\right]\overline{r}$$

$$= \frac{3}{r}f(r) + \frac{1}{r}\left[\frac{r f'(r) - f(r)}{r^2}\right](\overline{r} . \overline{r})$$

$$= \frac{3}{r}f(r) + \frac{1}{r}\left[\frac{r f'(r) - f(r)}{r^2}\right]r^2$$

$$= \frac{3}{r} f(r) + f'(r) - \frac{1}{r} f(r)$$

$$= \frac{2}{r} f(r) + f'(r)$$

PROBLEM (23): Show that

(I) $\nabla \cdot \left[r \nabla \left(\frac{1}{r^3} \right) \right] = \frac{3}{r^4}$ (II) $\nabla \cdot \left[\frac{1}{r} \nabla \left(\frac{1}{r} \right) \right] = \frac{1}{r^4}$

(III) $\nabla \left[\nabla \cdot \left(\frac{\vec{r}}{r} \right) \right] = -\frac{2 \vec{r}}{r^3}$

SOLUTION: We have

(I) $\nabla \cdot \left[r \nabla \left(\frac{1}{r^3} \right) \right] = \nabla \cdot \left[r (-3 r^{-5} \vec{r}) \right]$ [using theorem (4.2)]

$$= -3 \nabla \cdot (r^{-4} \vec{r})$$

$= -3(-4+3) r^{-4} = \frac{3}{r^4}$ [using theorem (4.6)]

(II) $\nabla \cdot \left[\frac{1}{r} \nabla \left(\frac{1}{r} \right) \right] = \nabla \cdot \left[\frac{1}{r} \left(-\frac{\vec{r}}{r^3} \right) \right]$ [using theorem (4.2)]

$$= -\nabla \cdot (r^{-4} \vec{r})$$

$= -(-4+3) r^{-4} = \frac{1}{r^4}$ [using theorem (4.6)]

(III) Using theorem (4.6), we have $\nabla \cdot \left(\frac{\vec{r}}{r} \right) = \frac{2}{r}$, therefore

$$\nabla \left[\nabla \cdot \left(\frac{\vec{r}}{r} \right) \right] = \nabla \left(\frac{2}{r} \right)$$

$= -2 r^{-3} \vec{r} = -\frac{2 \vec{r}}{r^3}$ [using theorem (4.2)]

PROBLEMS ON LAPLACIAN

PROBLEM (24): Prove that $\nabla^2 (\phi \psi) = \phi \nabla^2 \psi + 2 \nabla \phi \cdot \nabla \psi + \psi \nabla^2 \phi$

SOLUTION: $\nabla^2 (\phi \psi) = \nabla \cdot \nabla (\phi \psi)$

$$= \nabla \cdot (\phi \nabla \psi + \psi \nabla \phi)$$

$$= \nabla \cdot (\phi \nabla \psi) + \nabla \cdot (\psi \nabla \phi)$$

$$= \phi (\nabla \cdot \nabla \psi) + (\nabla \phi) \cdot (\nabla \psi) + \psi (\nabla \cdot \nabla \phi) + (\nabla \psi) \cdot (\nabla \phi)$$

$$= \phi \nabla^2 \psi + 2 \nabla \phi \cdot \nabla \psi + \psi \nabla^2 \phi$$

PROBLEM (25): Show that $\nabla^2\left[\nabla\cdot\left(\frac{\overline{r}}{r^3}\right)\right] = \frac{2}{r^4}$

SOLUTION: Using theorem (4.7), we have

$$\nabla\cdot\left(\frac{\overline{r}}{r^3}\right) = \nabla\cdot\left(r^{-2}\,\overline{r}\right) = (-2+3)\,r^{-2} = \frac{1}{r^2}\text{, therefore}$$

$$\nabla^2\left[\nabla\cdot\left(\frac{\overline{r}}{r^3}\right)\right] = \nabla^2\left(\frac{1}{r^2}\right) = \nabla^2(r^{-2})$$

$$= -2(-2+1)\,r^{-4} \qquad [\text{using theorem (4.8)}]$$

$$= \frac{2}{r^4}$$

PROBLEMS ON THE CURL

PROBLEM (26): If $\overline{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$ find $\nabla\times\nabla\times\overline{A}$.

SOLUTION: We have

$$\nabla\times\overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = (2x+2z)\,\hat{i} - (x^2+2z)\,\hat{k}$$

$$\nabla\times\nabla\times\overline{A} = \nabla\times\left[(2x+2z)\,\hat{i} - (x^2+2z)\,\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+2z & 0 & -(x^2+2z) \end{vmatrix} = (2x+2)\,\hat{j}$$

PROBLEM (27): If $\overline{A} = A_1\,\hat{i} + A_2\,\hat{j} + A_3\,\hat{k}$, show that

$$\nabla\times\overline{A} = \nabla A_1\times\hat{i} + \nabla A_2\times\hat{j} + \nabla A_3\times\hat{k}$$

SOLUTION: Since $\overline{A} = A_1\,\hat{i} + A_2\,\hat{j} + A_3\,\hat{k}$, then

$$\nabla\times\overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k} \qquad (1)$$

Now $\nabla A_1 \times \hat{i} = \left(\frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_1}{\partial y}\hat{j} + \frac{\partial A_1}{\partial z}\hat{k}\right) \times \hat{i}$

$$= \frac{\partial A_1}{\partial y}(\hat{j} \times \hat{i}) + \frac{\partial A_1}{\partial z}(\hat{k} \times \hat{i})$$

$$= -\frac{\partial A_1}{\partial y}\hat{k} + \frac{\partial A_1}{\partial z}\hat{j} \qquad (2)$$

$$\nabla A_2 \times \hat{j} = \left(\frac{\partial A_2}{\partial x}\hat{i} + \frac{\partial A_2}{\partial y}\hat{j} + \frac{\partial A_2}{\partial z}\hat{k}\right) \times \hat{j}$$

$$= \frac{\partial A_2}{\partial x}(\hat{i} \times \hat{j}) + \frac{\partial A_2}{\partial z}(\hat{k} \times \hat{j})$$

$$= \frac{\partial A_2}{\partial x}\hat{k} - \frac{\partial A_2}{\partial z}\hat{i} \qquad (3)$$

$$\nabla A_3 \times \hat{k} = \left(\frac{\partial A_3}{\partial x}\hat{i} + \frac{\partial A_3}{\partial y}\hat{j} + \frac{\partial A_3}{\partial z}\hat{k}\right) \times \hat{k}$$

$$= \frac{\partial A_3}{\partial x}(\hat{i} \times \hat{k}) + \frac{\partial A_3}{\partial y}(\hat{j} \times \hat{k})$$

$$= -\frac{\partial A_3}{\partial x}\hat{j} + \frac{\partial A_3}{\partial y}\hat{i} \qquad (4)$$

Adding equations (2), (3), and (4), we get

$$\nabla A_1 \times \hat{i} + \nabla A_2 \times \hat{j} + \nabla A_3 \times \hat{k} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k} \qquad (5)$$

From equations (1) and (5), we have

$$\nabla \times \overline{A} = \nabla A_1 \times \hat{i} + \nabla A_2 \times \hat{j} + \nabla A_3 \times \hat{k}$$

PROBLEM (28): Prove that

(i) $\nabla \times (\phi \nabla \phi) = \overline{0}$ (ii) $\nabla \times (\phi \nabla \psi) + \nabla \times (\psi \nabla \phi) = \overline{0}$

SOLUTION: We have

(i) $\nabla \times (\phi \nabla \phi) = \phi(\nabla \times \nabla \phi) + \nabla \phi \times \nabla \phi$

$$= \phi(\overline{0}) + \overline{0} = \overline{0} \qquad [\text{using theorem } (4.9)]$$

(ii) $\nabla \times (\phi \nabla \psi) + \nabla \times (\psi \nabla \phi) = \phi(\nabla \times \nabla \psi) + \nabla \phi \times \nabla \psi + \psi(\nabla \times \nabla \phi) + \nabla \psi \times \nabla \phi$

$$= \phi(\overline{0}) + \nabla \phi \times \nabla \psi + \psi(\overline{0}) - \nabla \phi \times \nabla \psi = \overline{0}$$

PROBLEM (29): If $\overline{A}$ is a constant vector, show that $\nabla \times [(\overline{A} \cdot \overline{r})\overline{A}] = \overline{0}$.

SOLUTION: We have

$$\nabla \times [(\overline{A} \cdot \overline{r})\overline{A}] = (\overline{A} \cdot \overline{r})(\nabla \times \overline{A}) + \nabla(\overline{A} \cdot \overline{r}) \times \overline{A} \qquad [\text{using theorem } (4.9)]$$

$$= (\overline{A} \cdot \overline{r})(\nabla \times \overline{A}) + \overline{A} \times \overline{A} \qquad [\text{since } \nabla(\overline{A} \cdot \overline{r}) = \overline{A}] \qquad (1)$$

Since $\overline{A}$ is a constant vector, therefore $\nabla \times \overline{A} = \overline{0}$. Also $\overline{A} \times \overline{A} = \overline{0}$.

Thus equation ( ) becomes

$$\nabla \times [(\overline{A} \cdot \overline{r})\overline{A}] = (\overline{A} \cdot \overline{r})(\overline{0}) + \overline{0} = \overline{0} + \overline{0} = \overline{0}$$

PROBLEM (30): (i) Find constants a, b, c so that

$$\overline{V} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k} \text{ is irrotational.}$$

(ii) Show that $\overline{V}$ can be expressed as the gradient of a scalar function.

SOLUTION: We have

(i)
$$\nabla \times \overline{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= (c + 1)\hat{i} + (a - 4)\hat{j} + (b - 2)\hat{k}$$

This equals zero when $a = 4$, $b = 2$, $c = -1$ and thus

$$\overline{V} = (x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + 2z)\hat{k} \tag{1}$$

(ii) Assume that $\overline{V} = \nabla \phi = \dfrac{\partial \phi}{\partial x}\hat{i} + \dfrac{\partial \phi}{\partial y}\hat{j} + \dfrac{\partial \phi}{\partial z}\hat{k}$ (2)

Comparing equations (1) and (2), we get

$$\frac{\partial \phi}{\partial x} = x + 2y + 4z \tag{3}$$

$$\frac{\partial \phi}{\partial y} = 2x - 3y - z \tag{4}$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z \tag{5}$$

Integrating equation (3) partially w.r.t. x keeping y and z constants,

$$\phi = \frac{x^2}{2} + 2xy + 4xz + f(y, z) \tag{6}$$

Similarly, from equations (4) and (5), we get

$$\phi = 2xy - \frac{3y^2}{2} - yz + g(x, z) \tag{7}$$

$$\phi = 4xz - yz + z^2 + h(x, y) \tag{8}$$

Comparison of equations (6), (7), and (8) shows that there will be a common value of ϕ if we choose

$$f(y, z) = -\frac{3y^2}{2} + z^2, \quad g(x, z) = \frac{x^2}{2} + z^2, \quad h(x, y) = \frac{x^2}{2} - \frac{3y^2}{2} \text{ so that}$$

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + \text{constant}$$

PROBLEM (31): Show that $\overline{V} = \frac{\overline{r}}{r^3}$ is irrotational. Find ϕ such that $\overline{V} = -\nabla\phi$ and such that $\phi(a) = 0$, where $a > 0$.

SOLUTION: A vector $\overline{V}$ is called irrotational if $\nabla \times \overline{V} = \overline{0}$, otherwise rotational.

Since $\nabla \times \overline{V} = \nabla \times \left(\frac{\overline{r}}{r^3}\right) = \nabla \times (r^{-3}\,\overline{r}) = \overline{0}$, therefore $\overline{V}$ is irrotational.

Now $\overline{V} = -\nabla\phi = \frac{\overline{r}}{r^3}$ or $\nabla\phi = -\frac{\overline{r}}{r^3}$

or $\frac{\partial\phi}{\partial r} = \nabla\phi \cdot \hat{r} = -\frac{\overline{r}}{r^3} \cdot \hat{r} = -\frac{1}{r}\hat{r} \cdot \hat{r} = -\frac{1}{r}$ (since $\hat{r} \cdot \hat{r} = 1$)

Integrating w.r.t. r, we get $\phi = -\ln r + C$

Using $\phi(a) = 0$, we have $C = \ln a$

therefore $\phi = -\ln r + \ln a = \ln\frac{a}{r}$

PROBLEM (32): Show that if $\phi(x, y, z)$ is any solution of Laplace's equation, then $\nabla\phi$ is a vector which is both solenoidal and irrotational.

SOLUTION: Since ϕ satisfies Laplace's equation therefore, $\nabla^2\phi = 0$ or $\nabla \cdot (\nabla\phi) = 0$.

Thus $\nabla\phi$ is solenoidal. Also by theorem [4.9 (iii)] $\nabla \times (\nabla\phi) = \overline{0}$ so that $\nabla\phi$ is irrotational.

PROBLEM (33): Show that $\nabla \times \left[\frac{1}{r}\nabla\left(\frac{1}{r}\right)\right] = \overline{0}$

SOLUTION: We have

$$\nabla \times \left[\frac{1}{r}\nabla\left(\frac{1}{r}\right)\right] = \nabla \times \left[\frac{1}{r}\left(-\frac{\overline{r}}{r^3}\right)\right] \quad \text{[using theorem (4.2)]}$$

$$= -\nabla \times (r^{-4}\,\overline{r}) = \overline{0} \quad \text{[using theorem (4.10)]}$$

PROBLEM (34): Prove that

(i) $(\overline{A} \times \nabla) \times \overline{r} = -2\overline{A}$ (ii) $(\overline{A} \times \nabla) \cdot \overline{r} = 0$

SOLUTION: (i) Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then

$$\overline{A} \times \nabla = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\hat{i} + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\hat{j} + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\hat{k}$$

$$\text{Thus} \quad (A \times \nabla) \times \overline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y} & A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z} & A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x} \\ x & y & z \end{vmatrix}$$

$$= (-A_1 - A_1)\hat{i} + (-A_2 - A_2)\hat{j} + (-A_3 - A_3)\hat{k}$$

$$= -2A_1\hat{i} - 2A_2\hat{j} - 2A_3\hat{k}$$

$$= -2(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) = -2\overline{A}$$

(ii). We know from theorem (4.12) that

$$(\overline{A} \times \nabla) \cdot \overline{B} = \overline{A} \cdot (\nabla \times \overline{B}) \qquad (1)$$

Let $\overline{B} = \overline{r}$ in equation (1), then

$$(\overline{A} \times \nabla) \cdot \overline{r} = \overline{A} \cdot (\nabla \times \overline{r})$$

$$= 0 \qquad [\text{using theorem (4.10)}]$$

PROBLEMS ON VECTOR IDENTITIES

PROBLEM (35): Prove that $(\overline{V} \cdot \nabla)\overline{V} = \frac{1}{2}\nabla V^2 - \overline{V} \times (\nabla \times \overline{V})$

SOLUTION: We have

$$\nabla V^2 = \nabla(\overline{V} \cdot \overline{V})$$

$$= (\overline{V} \cdot \nabla)\overline{V} + (\overline{V} \cdot \nabla)\overline{V} + \overline{V} \times (\nabla \times \overline{V}) + \overline{V} \times (\nabla \times \overline{V})$$

(Using theorem [4.13 (iii)])

$$= 2(\overline{V} \cdot \nabla)\overline{V} + 2\overline{V} \times (\nabla \times \overline{V})$$

or

$$(\overline{V} \cdot \nabla)\overline{V} = \frac{1}{2}\nabla V^2 - \overline{V} \times (\nabla \times \overline{V})$$

PROBLEM (36): If $\overline{A}$ is a constant vector, prove that

(i) $\nabla(\overline{A} \cdot \overline{u}) = (\overline{A} \cdot \nabla)\overline{u} + \overline{A} \times (\nabla \times \overline{u})$

(ii) $\nabla \cdot (\overline{A} \times \overline{u}) = -\overline{A} \cdot (\nabla \times \overline{u})$

(iii) $\nabla \times (\overline{A} \times \overline{u}) = \overline{A}(\nabla \cdot \overline{u}) - (\overline{A} \cdot \nabla)u$

SOLUTION: Since $\overline{A}$ is a constant vector, we have

$$\nabla \cdot \overline{A} = 0, \quad \nabla \times \overline{A} = \overline{0} \quad \text{and} \quad (\overline{u} \cdot \nabla)\overline{A} = \overline{0}$$