$$= \sum\left[\hat{i}\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)\right]+\sum\left[\hat{i}\times\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)\right]$$

$$= \sum\left[\hat{i}\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)\right]-\sum\left[\hat{i}\times\left(\bar{B}\times\frac{\partial\bar{A}}{\partial x}\right)\right] \qquad (1)$$

Now $\hat{i}\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)=\left(\hat{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\bar{A}-(\hat{i}\cdot\bar{A})\frac{\partial\bar{B}}{\partial x}$

and so $\sum\left[\hat{i}\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)\right]=\left[\sum\left(\hat{i}\cdot\frac{\partial\bar{B}}{\partial x}\right)\right]\bar{A}-\bar{A}\cdot\left(\sum\hat{i}\frac{\partial}{\partial x}\right)\bar{B}$

$$=(\nabla\cdot\bar{B})\bar{A}-(\bar{A}\cdot\nabla)\bar{B} \qquad (2)$$

Similarly, on interchanging $\bar{A}$ and $\bar{B}$ in equation (2), we get

$$\sum\left[\hat{i}\times\left(\bar{B}\times\frac{\partial\bar{A}}{\partial x}\right)\right]=(\nabla\cdot\bar{A})\bar{B}-(\bar{B}\cdot\nabla)\bar{A} \qquad (3)$$

Substitution of equations (2) and (3) in equation (1) gives

$$\nabla\times(\bar{A}\times\bar{B})=(\bar{B}\cdot\nabla)\bar{A}-(\bar{A}\cdot\nabla)\bar{B}+\bar{A}(\nabla\cdot\bar{B})-\bar{B}(\nabla\cdot\bar{A})$$

(iii) $\nabla(\bar{A}\cdot\bar{B})=\sum\left[\hat{i}\frac{\partial}{\partial x}(\bar{A}\cdot\bar{B})\right]$

$$=\sum\left[\hat{i}\left(\bar{A}\cdot\frac{\partial\bar{B}}{\partial x}+\frac{\partial\bar{A}}{\partial x}\cdot\bar{B}\right)\right]$$

$$=\sum\left[\left(\bar{A}\cdot\frac{\partial\bar{B}}{\partial x}\right)\hat{i}\right]+\sum\left[\left(\bar{B}\cdot\frac{\partial\bar{A}}{\partial x}\right)\hat{i}\right]$$

We know that

$$\bar{A}\times(\bar{B}\times\bar{C})=(\bar{A}\cdot\bar{C})\bar{B}-(\bar{A}\cdot\bar{B})\bar{C}$$

or $(\bar{A}\cdot\bar{B})\bar{C}=(\bar{A}\cdot\bar{C})\bar{B}-\bar{A}\times(\bar{B}\times\bar{C})$

Thus $\left(\bar{A}\cdot\frac{\partial\bar{B}}{\partial x}\right)\hat{i}=(\bar{A}\cdot\hat{i})\frac{\partial\bar{B}}{\partial x}-\bar{A}\times\left(\frac{\partial\bar{B}}{\partial x}\times\hat{i}\right)$

$$=(\bar{A}\cdot\hat{i})\frac{\partial\bar{B}}{\partial x}+\bar{A}\times\left(\hat{i}\times\frac{\partial\bar{B}}{\partial x}\right)$$

and so $\sum\left(\bar{A}\cdot\frac{\partial\bar{B}}{\partial x}\right)\hat{i}=\bar{A}\cdot\left(\sum\hat{i}\frac{\partial}{\partial x}\right)\bar{B}+\bar{A}\times\sum\left(\hat{i}\times\frac{\partial\bar{B}}{\partial x}\right)$

$$=(\bar{A}\cdot\nabla)\bar{B}+\bar{A}\times(\nabla\times\bar{B}) \qquad (5)$$

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Similarly , interchanging $\vec{A}$ and $\vec{B}$ in equation (5) , we get

$$\sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i} = (\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A}) \qquad (6)$$

Substitution of equations (5) and (6) in equation (4) gives

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$$

(iv) We know that $\vec{B}\times(\vec{C}\times\vec{D}) = (\vec{B}\cdot\vec{D})\vec{C} - (\vec{B}\cdot\vec{C})\vec{D}$

Setting $\vec{B} = \vec{C} = \nabla$ and $\vec{D} = \vec{A}$, we get

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - (\nabla\cdot\nabla)\vec{A}$$

$$= \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

Note that rather than writing $(\nabla\cdot\vec{A})\nabla$, we must write $\nabla(\nabla\cdot\vec{A})$ to make sure that , ∇ operates on $\nabla\cdot\vec{A}$.

EXAMPLE (12): If $\vec{A}$ is a constant vector , prove that

(i) $\nabla(\vec{A}\cdot\vec{r}) = \vec{A}$ (ii) $\nabla\cdot(\vec{A}\times\vec{r}) = 0$

(iii) $\nabla\times(\vec{A}\times\vec{r}) = 2\vec{A}$

SOLUTION: By theorem [4.13 (iii)] , we have

(i) $\nabla(\vec{A}\cdot\vec{r}) = (\vec{r}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{r} + \vec{r}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{r})$

Since $\vec{A}$ is constant vector , we have

$(\vec{r}\cdot\nabla)\vec{A} = \vec{0}$, $\nabla\times\vec{A} = \vec{0}$. Furthermore , $\nabla\times\vec{r} = \vec{0}$.

Thus $\nabla(\vec{A}\cdot\vec{r}) = (\vec{A}\cdot\nabla)\vec{r} = \vec{A}$ [using theorem (4.11) (iv)]

ALTERNATIVE METHOD

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$\vec{A}\cdot\vec{r} = A_1x + A_2y + A_3z$

and $\nabla(\vec{A}\cdot\vec{r}) = \hat{i}\frac{\partial}{\partial x}(A_1x + A_2y + A_3z) + \hat{j}\frac{\partial}{\partial y}(A_1x + A_2y + A_3\hat{z})$

$$+ \hat{k}\frac{\partial}{\partial z}(A_1x + A_2y + A_3\hat{z})$$

$$= A_1\hat{i} + A_2\hat{j} + A_3\hat{k} = \vec{A}$$

(ii) By theorem [4.13 (i)]

$$\nabla.(\overline{A}\times\overline{r}) = \overline{r}.(\nabla\times\overline{A}) - \overline{A}.(\nabla\times\overline{r}) = \overline{r}\,.\,\overline{0} - \overline{A}\,.\,\overline{0} = 0$$

(iii) By theorem [4.13 (ii)]

$$\nabla\times(\overline{A}\times\overline{r}) = (\overline{r}.\nabla)\overline{A} - \overline{r}(\nabla.\overline{A}) - (\overline{A}.\nabla)\overline{r} + \overline{A}(\nabla.\overline{r})$$

$$= -(\overline{A}.\nabla)\overline{r} + \overline{A}(\nabla.\overline{r})$$

$$= -\overline{A} + 3\overline{A} = 2\overline{A} \quad [\text{since } (\overline{A}.\nabla)\overline{r} = \overline{A}]$$

4.20 SOLVED PROBLEMS

PROBLEMS ON THE GRADIENT

PROBLEM (1): Let R be the distance from a fixed point $A(a, b, c)$ to any point $P(x, y, z)$. Show that $\nabla R = \hat{R}$ is a unit vector in the direction $\overline{AP} = \overline{R}$.

SOLUTION: If $\bar{r}_A$ and $\bar{r}_P$ are the position vectors of the points A and P respectively, then

$$\bar{r}_A = a\hat{i} + b\hat{j} + c\hat{k} \text{ and } \bar{r}_P = x\hat{i} + y\hat{j} + z\hat{k}$$

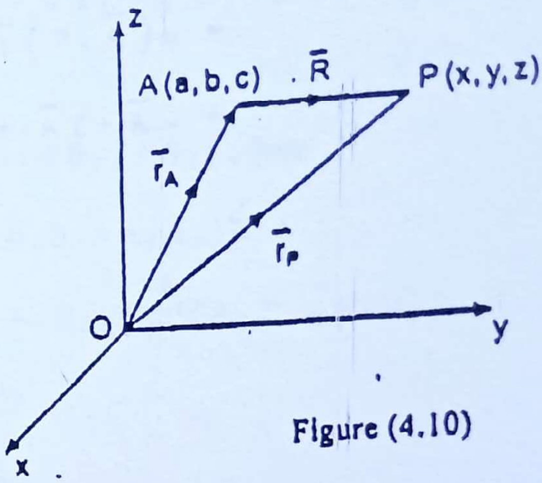

Figure (4.10)

Thus

$$\overline{R} = \bar{r}_P - \bar{r}_A$$

$$= (x-a)\hat{i} + (y-b)\hat{j} + (z-c)\hat{k}$$

so that

$$R = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

Then

$$\nabla R = \nabla\left[\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}\right]$$

$$= \frac{(x-a)\hat{i} + (y-b)\hat{j} + (z-c)\hat{k}}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

$$= \frac{\overline{R}}{R} = \hat{R},$$ a unit vector in the direction of $\overline{R}$

PROBLEM (2): If $\phi = x^n + y^n + z^n$, show that $\bar{r}\cdot\nabla\phi = n\phi$.

SOLUTION: Since $\phi = x^n + y^n + z^n$, therefore

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^n + y^n + z^n) + \hat{j}\frac{\partial}{\partial y}(x^n + y^n + z^n) + \hat{k}\frac{\partial}{\partial z}(x^n + y^n + z^n)$$

$$= n x^{n-1}\hat{i} + n y^{n-1}\hat{j} + n z^{n-1}\hat{k}$$

and so

$$\bar{r}\cdot\nabla\phi = (x\hat{i} + y\hat{j} + z\hat{k})\cdot(n x^{n-1}\hat{i} + n y^{n-1}\hat{j} + n z^{n-1}\hat{k})$$

$$= n x^n + n y^n + n z^n$$

$$= n(x^n + y^n + z^n) = n\phi$$

PROBLEM (3): If $\phi = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2 + y^2 + z^2}}$, find $\nabla\phi$.

SOLUTION: Since $r = \sqrt{x^2 + y^2 + z^2}$, therefore, $\phi = r^2 e^{-r}$

Now

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r}$$

$$= (2re^{-r} - r^2 e^{-r})\hat{r}$$

$$= (2-r)re^{-r}\hat{r} = (2-r)e^{-r}\,\bar{r}$$

PROBLEM (4): If $\nabla\phi = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$, find $\phi(x,y,z)$ if $\phi(1,-2,2) = 4$.

SOLUTION: We know that

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \tag{1}$$

Since $\nabla\phi = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$ (2)

therefore comparing equations (1) and (2), we get

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \tag{3}$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \tag{4}$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2 \tag{5}$$

Integrating equation (3) partially w.r.t. x keeping y and z constants,

$$\phi = x^2yz^3 + f(y,z) \tag{6}$$

Similarly, from equations (4) and (5), we get

$$\phi = x^2yz^3 + g(x,z) \tag{7}$$

$$\phi = x^2yz^3 + h(x,y) \tag{8}$$

Comparison of equations (6), (7), and (8) shows that there will be a common value of ϕ if we choose

$f(y,z) = g(x,z) = h(x,y) = C$ where C is an arbitrary constant.

Thus $\phi = x^2yz^3 + C$ (9)

Using $\phi(1,-2,2) = 4$, we find from equation (9)

$$4 = (1)^2(-2)(2)^3 + C$$

or $4 = -16 + C$ or $C = 20$

Hence from equation (9), we get

$$\phi = x^2yz^3 + 20$$

PROBLEM (5): If $\nabla\phi = 2r^4\,\vec{r}$, find ϕ.

SOLUTION: We know that the directional derivative of ϕ w.r.t. the distance r in the direction the unit vector $\hat{r}$ is the component of $\nabla\phi$ in the direction of this unit vector,

i.e. $\frac{\partial\phi}{\partial r} = \nabla\phi\cdot\hat{r}$

$= 2r^4\,\vec{r}\cdot\hat{r} = 2r^5\,\hat{r}\cdot\hat{r} = 2r^5$ (since $\hat{r}\cdot\hat{r} = 1$)

Integrating w.r.t. r, we get

$$\phi = \frac{r^6}{3} + \text{constant}$$

PROBLEM (6): If $\phi = \phi(x,y,z)$ is a differentiable function, show that $d\phi = \nabla\phi \cdot d\overline{r}$.

SOLUTION: We know that if $\phi = \phi(x,y,z)$ is a differentiable function of x, y, z, then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\overline{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \nabla\phi \cdot d\overline{r}$$

PROBLEM (7): If $\overline{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, show that

$$d\overline{A} = (\nabla A_1 \cdot d\overline{r})\hat{i} + (\nabla A_2 \cdot d\overline{r})\hat{j} + (\nabla A_3 \cdot d\overline{r})\hat{k}$$

SOLUTION: Since $\overline{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, therefore

$$d\overline{A} = dA_1\hat{i} + dA_2\hat{j} + dA_3\hat{k} \qquad (1)$$

Now $$dA_1 = \frac{\partial A_1}{\partial x}dx + \frac{\partial A_1}{\partial y}dy + \frac{\partial A_1}{\partial z}dz$$

$$= \left(\frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_1}{\partial y}\hat{j} + \frac{\partial A_1}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \nabla A_1 \cdot d\overline{r}$$

Similarly, $dA_2 = \nabla A_2 \cdot d\overline{r}$ and $dA_3 = \nabla A_3 \cdot d\overline{r}$

Thus from equation (1), we get

$$d\overline{A} = (\nabla A_1 \cdot d\overline{r})\hat{i} + (\nabla A_2 \cdot d\overline{r})\hat{j} + (\nabla A_3 \cdot d\overline{r})\hat{k}$$

PROBLEM (8): If $\phi = \phi(u)$, where $u = u(x,y,z)$, then prove that

(i) $\nabla\phi = \nabla\phi(u) = \phi'(u)\nabla u$ (ii) $\nabla\int f(u)\,du = f(u)\nabla u$

PROOF: (i) $$\nabla\phi = \nabla\phi(u) = \hat{i}\frac{\partial}{\partial x}\phi(u) + \hat{j}\frac{\partial}{\partial y}\phi(u) + \hat{k}\frac{\partial}{\partial z}\phi(u)$$

$$= \hat{i}\frac{d\phi}{du}\frac{\partial u}{\partial x} + \hat{j}\frac{d\phi}{du}\frac{\partial u}{\partial y} + \hat{k}\frac{d\phi}{du}\frac{\partial u}{\partial z}$$

$$= \frac{d\phi}{du}\left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right) = \phi'(u)\nabla u$$

(ii) Let $\phi(u) = \int f(u)\,du$, then $\phi'(u) = f(u)$. Hence

$$\nabla\int f(u)\,du = \nabla\phi(u) = \phi'(u)\nabla u \quad [\text{using part (i)}]$$

$$= f(u)\nabla u$$

PROBLEM (9): If $\phi = \phi(u, v, w)$, where $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$, show that

$$\nabla \phi = \frac{\partial \phi}{\partial u} \nabla u + \frac{\partial \phi}{\partial v} \nabla v + \frac{\partial \phi}{\partial w} \nabla w$$

SOLUTION: By definition $\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$ (1)

Since $\phi = \phi(u, v, w)$, therefore

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial x}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial y}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial z}$$

Substituting these in equation (1), we get

$$\nabla \phi = \left(\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial x} \right) \hat{i} + \left(\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial y} \right) \hat{j} + \left(\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial z} \right) \hat{k}$$

$$= \frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} \hat{i} + \frac{\partial u}{\partial y} \hat{j} + \frac{\partial u}{\partial z} \hat{k} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} \hat{i} + \frac{\partial v}{\partial y} \hat{j} + \frac{\partial v}{\partial z} \hat{k} \right) + \frac{\partial \phi}{\partial w} \left(\frac{\partial w}{\partial x} \hat{i} + \frac{\partial w}{\partial y} \hat{j} + \frac{\partial w}{\partial z} \hat{k} \right)$$

$$= \frac{\partial \phi}{\partial u} \nabla u + \frac{\partial \phi}{\partial v} \nabla v + \frac{\partial \phi}{\partial w} \nabla w$$

PROBLEM (10): Find the directional derivative of $\phi = x^2 y z + 4 x z^2$ at the point $(1, -2, -1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

Ex. P. 8

SOLUTION: We have

$$\nabla \phi = \hat{i} \frac{\partial}{\partial x} (x^2 y z + 4 x z^2) + \hat{j} \frac{\partial}{\partial y} (x^2 y z + 4 x z^2) + \hat{k} \frac{\partial}{\partial z} (x^2 y z + 4 x z^2)$$

$$= (2xyz + 4z^2) \hat{i} + x^2 z \hat{j} + (x^2 y + 8xz) \hat{k}$$

Thus at the point $(1, -2, -1)$, the value of the gradient is

$$(\nabla \phi)_{(1,-2,-1)} = 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is

$$\hat{T} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4 + 1 + 4}} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Then the required directional derivative of ϕ at $(1, -2, 1)$ is

$$\frac{\partial \phi}{\partial s} = \nabla \phi \cdot \hat{T} = (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k} \right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

PROBLEM (11): (i) In what direction from the point $(2, 1, -1)$ is the directional derivative of $\phi = x^2 y z^3$ a maximum ? (ii) What is the magnitude of this maximum ?

SOLUTION: We know that

$$\nabla \phi = \hat{i}\,\frac{\partial}{\partial x}(x^2 y z^3) + \hat{j}\,\frac{\partial}{\partial y}(x^2 y z^3) + \hat{k}\,\frac{\partial}{\partial z}(x^2 y z^3)$$

$$= 2xyz^3\,\hat{i} + x^2 z^3\,\hat{j} + 3x^2 y z^2\,\hat{k}$$

Thus at the point $(2, 1, -1)$ the values of the gradient is

$$\nabla \phi = -4\,\hat{i} - 4\,\hat{j} + 12\,\hat{k}$$

(i) The directional derivative is maximum in the direction

$$\nabla \phi = -4\,\hat{i} - 4\,\hat{j} + 12\,\hat{k}$$

(ii) The magnitude of this maximum is

$$|\nabla \phi| = \sqrt{16 + 16 + 144} = \sqrt{176} = 4\sqrt{11}$$

PROBLEM (12): Find the values of the constants a , b , c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis .

SOLUTION: Since $\phi = axy^2 + byz + cz^2x^3$, therefore

$$\nabla \phi = (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$$

At the point $(1, 2, -1)$, the value of this gradient is

$$= (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k}$$

We know that the maximum directional derivative takes place in the direction of $\nabla \phi$ and has the magnitude of $\nabla \phi$. This maximum directional derivative will be parallel to the z-axis if

$$4a + 3c = 0 \qquad (1)$$

and

$$4a - b = 0 \qquad (2)$$

Therefore $\nabla \phi = (2b - 2c)\,\hat{k}$

Since the magnitude of this maximum directional derivative is 64 , therefore

$$2b - 2c = 64 \qquad (3)$$

Solving equations (1) , (2) , and (3) , we find that

$$a = 6 , \; b = 24 , \; c = -8 .$$

PROBLEM (13): Find the normal derivative $\left(\text{i.e. } \frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n}\right)$ of $\phi = xy^2 + yz^3$ at $(-1, 2, 1)$, where $\hat{n}$ is a unit normal vector to the surface $x \ln z - y^2 + 4 = 0$ at $(-1, 2, 1)$.

SOLUTION: Since $\phi = xy^2 + yz^3$, therefore

$$\nabla \phi = \hat{i}\frac{\partial}{\partial x}(xy^2+yz^3)+\hat{j}\frac{\partial}{\partial y}(xy^2+yz^3)+\hat{k}\frac{\partial}{\partial z}(xy^2+yz^3)$$

$$= y^2\hat{i}+(2xy+z^3)\hat{j}+3yz^2\hat{k}$$

Thus at the point $(-1,2,1)$ the value of the gradient is $\nabla\phi = 4\hat{i}-3\hat{j}+6\hat{k}$

Let the surface be represented by $\phi_1(x,y,z) = x\ln z - y^3 + 4 = 0$

Then $$\nabla\phi_1 = \hat{i}\frac{\partial}{\partial x}(x\ln z - y^3+4)+\hat{j}\frac{\partial}{\partial y}(x\ln z - y^3+4)+\hat{k}\frac{\partial}{\partial z}(x\ln z - y^3+4)$$

$$= \ln z\,\hat{i} - 3y^2\hat{j}+\frac{x}{z}\hat{k}$$

At the point $(-1,2,1)$ the value of this gradient is $\nabla\phi_1 = -12\hat{j}-\hat{k}$

and $|\nabla\phi_1| = \sqrt{(-12)^2+(1)^2} = \sqrt{145}$

The unit normal vector $\hat{n}$ to this surface is

$$\hat{n} = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{-12\hat{j}-\hat{k}}{\sqrt{145}}$$

Thus the required normal derivative of ϕ at $(-1,2,1)$ is given by

$$\frac{\partial\phi}{\partial n} = \nabla\phi\cdot\hat{n}$$

$$= (4\hat{i}-3\hat{j}+6\hat{k})\cdot\left(\frac{-12\hat{j}-\hat{k}}{\sqrt{145}}\right) = \frac{36-6}{\sqrt{145}} = \frac{30}{\sqrt{145}}$$

PROBLEM (14): Find the angle between the normals to the surface $xy = z^2$ at the points $(1,4,2)$ and $(-3,-3,3)$.

SOLUTION: Let the surface be represented by $\phi(x,y,z) = xy - z^2 = 0$

A normal to this surface is given by

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy-z^2)+\hat{j}\frac{\partial}{\partial y}(xy-z^2)+\hat{k}\frac{\partial}{\partial z}(xy-z^2) = y\hat{i}+x\hat{j}-2z\hat{k}$$

Let $\overline{N}_1$ be the normal to the surface at $(1,4,2)$, then

$$\overline{N}_1 = (\nabla\phi)_{(1,4,2)} = 4\hat{i}+\hat{j}-4\hat{k}$$

Let $\overline{N}_2$ be the normal to the surface at $(-3,-3,3)$, then

$$\overline{N}_2 = (\nabla\phi)_{(-3,-3,3)} = -3\hat{i}-3\hat{j}-6\hat{k}$$

Let θ be the angle between the normals to the surface at $(1,4,2)$ and $(-3,-3,3)$,

then $$\theta = \cos^{-1}\left(\frac{\overline{N}_1\cdot\overline{N}_2}{|\overline{N}_1|\,|\overline{N}_2|}\right) \quad (1)$$

Now $\bar{N}_1 . \bar{N}_2 = (4\hat{i}+\hat{j}-4\hat{k}).(-3\hat{i}-3\hat{j}-6\hat{k}) = -12-3+24 = 9$

$$|\bar{N}_1| = \sqrt{(4)^2+(1)^2+(-4)^2} = \sqrt{16+1+16} = \sqrt{33}$$

$$|\bar{N}_2| = \sqrt{(-3)^2+(-3)^2+(-6)^2} = \sqrt{9+9+36} = \sqrt{54}$$

then from equation (1) $\theta = \cos^{-1}\left(\frac{9}{\sqrt{33}\sqrt{54}}\right) = 77.69^\circ$ Approximately

PROBLEM (15): Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

SOLUTION: The angle between the surfaces at the point is the angle between the normals to the surfaces at that point. A normal to the surface $\phi_1(x,y,z) = x^2+y^2+z^2 = 9$ is

$$\nabla\phi_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$= 4\hat{i}-2\hat{j}+4\hat{k} \quad \text{at the point } (2,-1,2)$$

A normal to the surface $\phi_2(x,y,z) = x^2+y^2-z = 3$

is $\nabla\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$

$$= 4\hat{i}-2\hat{j}-\hat{k} \quad \text{at the point } (2,-1,2).$$

Now at the point $(2,-1,2)$, we have

$$(\nabla\phi_1).(\nabla\phi_2) = |\nabla\phi_1||\nabla\phi_2|\cos\theta$$

where θ is the required angle.

or $(4\hat{i}-2\hat{j}+4\hat{k}).(4\hat{i}-2\hat{j}-\hat{k})$

$$= |4\hat{i}-2\hat{j}+4\hat{k}|\,|4\hat{i}-2\hat{j}-\hat{k}|\cos\theta$$

or $16 = 6\sqrt{21}\cos\theta$

and $\cos\theta = \frac{8}{3\sqrt{21}} = 0.5819.$

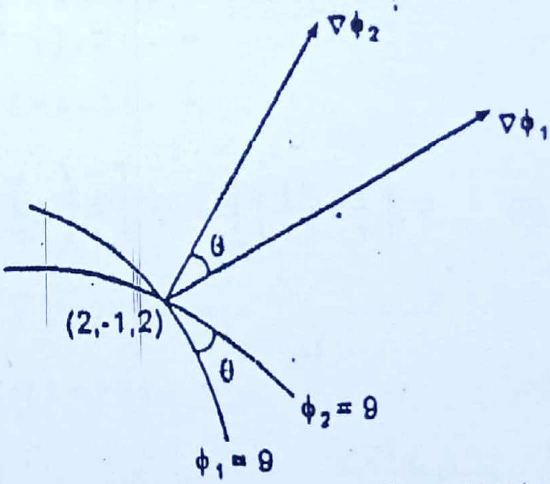

Figure (4.11)

Thus the acute angle is $\theta = \cos^{-1} 0.5819 = 54^\circ$ approximately.

PROBLEM (16): Find the constants a and b so that the surface $ax^2-byz-(a+2)x$ will be orthogonal to the surface $4x^2y+z^3 = 4$ at the point $(1,-1,2)$.

SOLUTION: Let $\phi_1(x,y,z) = ax^2-byz-(a+2)x = 0$ and $\phi_2(x,y,z) = 4x^2y+z^3 = 4$

The two surfaces will be orthogonal to each other at a given point if the normals to these surfaces are orthogonal to each other at that point.

A normal to the surface $\phi_1 = 0$ is

$$\nabla\phi_1 = [2ax-(a+2)]\hat{i}-bz\hat{j}-by\hat{k}$$

$$= (a-2)\hat{i}-2b\hat{j}+b\hat{k} \quad \text{at } (1,-1,2)$$