Equations (4) and (6) give alternative forms for $\nabla \phi$ and we see that equation (6) is usually more convenient when ϕ is specified explicitly as a function of x, y, and z.

4.11. DIVERGENCE OF A VECTOR POINT FUNCTION

Let $\overline{A}(x, y, z)$ be a differentiable vector point function in a certain region of space. Then the divergence of $\overline{A}$ written $\nabla . \overline{A}$ or div $\overline{A}$ is defined by

$$\nabla . \overline{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Note that $\nabla . \overline{A}$ is a scalar quantity. Also note that $\nabla . \overline{A} \neq \overline{A} . \nabla$.

If $\overline{A}$ is a constant vector, then $\nabla . \overline{A} = 0$. If $\nabla . \overline{A} = 0$ everywhere in some region R, then $\overline{A}$ is called a solenoidal vector point function in that region.

EXAMPLE (5): If $\overline{A} = x^2 z\hat{i} - 2y^3 z^2\hat{j} + xy^2 z\hat{k}$, find $\nabla . \overline{A}$ at the point $(1, -1, 1)$.

SOLUTION:

$$\nabla . \overline{A} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z),$$

$$= 2xz - 6y^2 z^2 + xy^2$$

Therefore, $(\nabla . \overline{A})_{(1,-1,1)} = -3$

4.12 PROPERTIES OF THE DIVERGENCE

THEOREM (4.5): If $\overline{A}$ and $\overline{B}$ are differentiable vector point functions, and ϕ is a differentiable scalar point function, then prove that

(i) $\nabla . (\overline{A} + \overline{B}) = \nabla . \overline{A} + \nabla . \overline{B}$

(ii) $\nabla . (\phi\overline{A}) = \phi(\nabla . \overline{A}) + \overline{A} . (\nabla\phi)$

PROOF: Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\overline{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\overline{A} + \overline{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$

Hence

$$\nabla . (\overline{A} + \overline{B}) = \frac{\partial}{\partial x}(A_1 + B_1) + \frac{\partial}{\partial y}(A_2 + B_2) + \frac{\partial}{\partial z}(A_3 + B_3)$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$$= \nabla . \overline{A} + \nabla . \overline{B}$$

(ii) We have $\phi\overline{A} = \phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k}$

Hence $\nabla . (\phi \bar{A}) = \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$

$$= \phi \frac{\partial A_1}{\partial x} + \frac{\partial \phi}{\partial x} A_1 + \phi \frac{\partial A_2}{\partial y} + \frac{\partial \phi}{\partial y} A_2 + \phi \frac{\partial A_3}{\partial z} + \frac{\partial \phi}{\partial z} A_3$$

$$= \phi \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + \left(\frac{\partial \phi}{\partial x} A_1 + \frac{\partial \phi}{\partial y} A_2 + \frac{\partial \phi}{\partial z} A_3 \right)$$

$$= \phi (\nabla . \bar{A}) + \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) . (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})$$

$$= \phi (\nabla . \bar{A}) + (\nabla \phi) . \bar{A}$$

$$= \phi (\nabla . \bar{A}) + \bar{A} . \nabla \phi$$

Note that if ϕ is constant, then $\nabla . (\phi \bar{A}) = \phi \, \nabla . \bar{A}$.

THEOREM (4.6): Prove that

(i) $\nabla . \bar{r} = 3$ (ii) $\nabla . [f(r) \bar{r}] = 3 f(r) + r f'(r)$

(iii) $\nabla . (r^n \bar{r}) = (n+3) r^n$ (iv) $\nabla . \left(\frac{\bar{r}}{r^3} \right) = 0$

PROOF: Let $\bar{r} = x \hat{i} + y \hat{j} + z \hat{k}$, then

(i) $\nabla . \bar{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$

(ii) Using theorem (4.5) part (ii), we have

$$\nabla . [f(r) \bar{r}] = f(r) (\nabla . \bar{r}) + \bar{r} . [\nabla f(r)]$$

$$= 3 f(r) + \bar{r} . \frac{f'(r) \bar{r}}{r} \quad \text{[using theorem (4.2)]}$$

$$= 3 f(r) + \frac{f'(r)}{r} \bar{r} . \bar{r}$$

$$= 3 f(r) + r f'(r) \quad (\text{since } \bar{r} . \bar{r} = r^2)$$

(iii) Setting $f(r) = r^n$ in part (ii), we have

$$\nabla . (r^n \bar{r}) = 3 r^n + r n r^{n-1} = 3 r^n + n r^n = (n+3) r^n$$

(iv) Let $n = -3$ in part (iii), then $\nabla . \left(\frac{\bar{r}}{r^3} \right) = (-3+3) r^{-3} = 0$

EXAMPLE (5): Find the most general differentiable function $f(r)$ so that $f(r) \bar{r}$ is solenoidal.

SOLUTION: If $f(r) \bar{r}$ is solenoidal, then $\nabla . [f(r) \bar{r}] = 0$

or $\quad [\nabla f(r)] \cdot \bar{r} + f(r)\, \nabla \cdot \bar{r} = 0 \qquad (1)$

But $\quad \nabla f(r) = \dfrac{f'(r)\,\bar{r}}{r}$, therefore equation (1) becomes

$$\frac{f'(r)\,\bar{r}\cdot\bar{r}}{r} + 3\,f(r) = 0 \quad \text{or} \quad r f'(r) + 3 f(r) = 0$$

or $\quad f'(r) = -\dfrac{3 f(r)}{r}$

or $\quad \dfrac{f'(r)}{f(r)} = -\dfrac{3}{r}$ (by seperating the variables)

Integrating we get $\ln f(r) = -3 \ln r + \ln C = \ln \dfrac{C}{r^3}$

or $\quad f(r) = \dfrac{C}{r^3}$ where C is an arbitrary constant .

4.13 PHYSICAL INTERPRETATION OF THE DIVERGENCE

Consider the motion of an incompressible fluid (i.e. fluid with constant density say oil or water). Figure (4.8) shows an imaginary small rectangular parallelopiped of dimensions Δx , Δy , Δz with edges parallel to the coordinate axes and having centre at the point $P(x, y, z)$. Let $\bar{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be the velocity of the fluid at P .

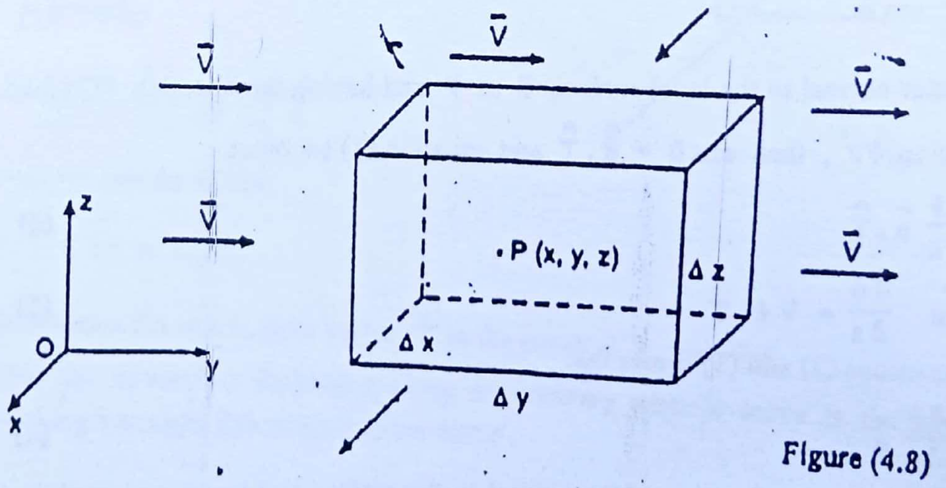

Figure (4.8)

We start by considering the flux (i.e. the amount of fluid crossing in unit time) through opposite faces of the parallelopiped . As the density is constant , either volume or mass can be used as a measure of the amount . For simplicity , volume will be considered here .

Consider first the flux across the faces which are perpendicular to the x–axis . These faces are of area $\Delta y\, \Delta z$ and the flux will only depend on V_1 , this being the component of $\bar{V}$ in the x–direction . Thus flux into the parallelopiped through the back face $= V_1\left(x - \frac{1}{2}\Delta x, y, z\right)\Delta y\, \Delta z$

and the flux out of the parallelopiped through the front face $= V_1\left(x + \frac{1}{2}\Delta x, y, z\right)\Delta y\, \Delta z$.

Hence the net flux out of the parallelopiped through these two faces

$$= \left[V_1\left(x+\tfrac{1}{2}\Delta x, y, z\right) - V_1\left(x-\tfrac{1}{2}\Delta x, y, z\right) \right] \Delta y\, \Delta z$$

Using Taylor's series, we have

$$V_1\left(x+\tfrac{1}{2}\Delta x, y, z\right) = V_1(x, y, z) + \tfrac{1}{2}\Delta x \frac{\partial}{\partial x} V_1(x, y, z) + \ldots\ldots$$

and $$V_1\left(x-\tfrac{1}{2}\Delta x, y, z\right) = V_1(x, y, z) - \tfrac{1}{2}\Delta x \frac{\partial}{\partial x} V_1(x, y, z) + \ldots\ldots$$

Therefore, $$V_1\left(x+\tfrac{1}{2}\Delta x, y, z\right) - V_1\left(x-\tfrac{1}{2}\Delta x, y, z\right) = \Delta x \frac{\partial}{\partial x} V_1(x, y, z) + \ldots\ldots$$

Thus net flux in the x – direction $= \dfrac{\partial V_1}{\partial x}\Delta x\, \Delta y\, \Delta z + \ldots\ldots$ $\left[\text{by writing } \dfrac{\partial V_1}{\partial x} \text{ for } \dfrac{\partial}{\partial x} V_1(x, y, z)\right]$

Similarly, the net flux in the y – direction $= \dfrac{\partial V_2}{\partial y}\Delta x\, \Delta y\, \Delta z + \ldots\ldots$

and the net flux in the z – direction $= \dfrac{\partial V_3}{\partial z}\Delta x\, \Delta y\, \Delta z + \ldots\ldots$

Adding these three contributions, total flux out of the parallelopiped

$$= \left\{ \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right\} \Delta x\, \Delta y\, \Delta z \; + \text{terms involving higher powers of } \Delta x, \Delta y, \Delta z$$

such as $(\Delta x)^2 \Delta y\, \Delta z$, $\Delta x (\Delta y)^3 \Delta z$ etc.

Now as the volume of the parallelopiped is $\Delta x\, \Delta y\, \Delta z$, we have

$$\text{Flux per unit volume} = \left\{ \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial Z} \right\} + \text{terms in } \Delta x, \Delta y, \text{ or } \Delta z \text{ and their powers.} \qquad (1)$$

Finally, we let Δx, Δy, Δz all tend to zero so that the parallelopiped shrinks to the point P. The right hand side of equation (1) becomes $\dfrac{\partial V_1}{\partial x} + \dfrac{\partial V_2}{\partial y} + \dfrac{\partial V_3}{\partial z}$, which is precisely the divergence of the vector $\vec{V}$. Thus Fluid Mechanics affords one possible interpretation of the divergence as the amount of outward flux of the velocity field $\vec{V}$ per unit volume. Note that the divergence of $\vec{V}$ measures the expansion of the fluid at the point.

4.14 LAPLACIAN

If $\phi(x, y, z)$ is a scalar point function, then the divergence of the gradient of ϕ written as $\nabla . \nabla\phi = \nabla^2\phi$ is called the Laplacian of ϕ, and the equation $\nabla^2\phi = 0$ is called Laplace's equation. If a scalar function ϕ satisfies the Laplace's equation $\nabla^2\phi = 0$ in a certain region R, then ϕ is said to be a harmonic function in that region.

THEOREM (4.7): If ϕ is a differentiable scalar point function, then show that

$$\nabla \cdot \nabla \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a Laplacian operator.

PROOF: We have

$$\nabla \cdot \nabla \phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2 \phi$$

EXAMPLE (7): Find $\nabla^2 \phi$ if $\phi = 2x^3y^2z^4$.

SOLUTION: We know that

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad (1)$$

Now $\frac{\partial \phi}{\partial x} = 6x^2y^2z^4, \quad \frac{\partial \phi}{\partial y} = 4x^3yz^4 \quad \frac{\partial \phi}{\partial z} = 8x^3y^2z^3$

and $\frac{\partial^2 \phi}{\partial x^2} = 12xy^2z^4, \quad \frac{\partial^2 \phi}{\partial y^2} = 4x^3z^4, \quad \frac{\partial^2 \phi}{\partial z^2} = 24x^3y^2z^2$

Thus from equation (1) $\nabla^2 \phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

THEOREM (4.8): Prove that

(I) $\nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r)$

(II) $\nabla^2 r^n = n(n+1)r^{n-2}$, where n is a real constant.

(III) $\nabla^2\left(\frac{1}{r}\right) = 0$

PROOF: (I) $\nabla^2 f(r) = \nabla \cdot \nabla f(r) = \nabla \cdot \left[\frac{f'(r)\,\vec{r}}{r}\right]$

$$= \frac{f'(r)}{r} \nabla \cdot \vec{r} + \vec{r} \cdot \nabla\left[\frac{f'(r)}{r}\right] \qquad [\text{Theorem (4.5) (ii)}]$$

$$= \frac{f'(r)}{r}(3) + \vec{r} \cdot \frac{1}{r}\frac{d}{dr}\left[\frac{f'(r)}{r}\right]\vec{r} \qquad [\text{Theorem (4.2)}]$$

$$= \frac{3}{r} f'(r) + \frac{1}{r}\left[\frac{r f''(r) - f'(r)}{r^2}\right]\vec{r} \cdot \vec{r}$$

$$= \frac{3}{r} f'(r) + \frac{1}{r}\left[\frac{r f''(r) - f'(r)}{r^2}\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r)$$

$$= \frac{2}{r} f'(r) + f''(r)$$

(ii) Setting $f(r) = r^n$ in part (i), we get

$$\nabla^2 r^n = \frac{2}{r} n r^{n-1} + n(n-1) r^{n-2}$$

$$= 2n r^{n-2} - n r^{n-2} + n^2 r^{n-2}$$

$$= n(n+1) r^{n-2}$$

(iii) Let $n = -1$ in part (ii), then

$$\nabla^2\left(\frac{1}{r}\right) = -1(-1+1) r^{-3} = 0$$

Any function ϕ which satisfies the Laplace's equation $\nabla^2 \phi = 0$ is called a solution of this equation. It follows that $\phi = \frac{1}{r}$ is a solution of Laplace's equation and is therefore harmonic.

EXAMPLE (8): Show that

(i) $\nabla^2(\ln r) = \frac{1}{r^2}$ (ii) $\nabla^2(e^r) = e^r\left(1 + \frac{2}{r}\right)$

(iii) $\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = e^r\left(1 + \frac{4}{r}\right)$

SOLUTION: We know that $\nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r)$ (1)

(i) Let $f(r) = \ln r$, then $f'(r) = \frac{1}{r}$, $f''(r) = -\frac{1}{r^2}$ and from equation (1), we get

$$\nabla^2(\ln r) = \frac{2}{r}\left(\frac{1}{r}\right) - \frac{1}{r^2} = \frac{2}{r^2} - \frac{1}{r^2} = \frac{1}{r^2}$$

(ii) Let $f(r) = e^r$, then $f'(r) = e^r$, $f''(r) = e^r$ and from equation (1), we get

$$\nabla^2(e^r) = \frac{2}{r} e^r + e^r = e^r\left(1 + \frac{2}{r}\right) \qquad (2)$$

(iii) $\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = \nabla^2\left[e^r\left(1 + \frac{2}{r}\right)\right]$ [using equation (2)]

Here $f(r) = e^r\left(1 + \frac{2}{r}\right)$

$$f'(r) = e^r\left(-\frac{2}{r^2}\right) + \left(1 + \frac{2}{r}\right)e^r = e^r\left(1 + \frac{2}{r} - \frac{2}{r^2}\right)$$

$$f''(r) = e^r\left(-\frac{2}{r^2} + \frac{4}{r^3}\right) + \left(1 + \frac{2}{r} - \frac{2}{r^2}\right)e^r = \left(1 + \frac{2}{r} - \frac{4}{r^2} + \frac{4}{r^3}\right)e^r$$

Thus from equation (1), we get

$$\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = \frac{2}{r}\left(1+\frac{2}{r}-\frac{2}{r^2}\right)e^r + \left(1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right)e^r$$

$$= \left(\frac{2}{r}+\frac{4}{r^2}-\frac{4}{r^3}+1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right)e^r$$

$$= e^r\left(1+\frac{4}{r}\right)$$

EXAMPLE (9): Find the function $f(r)$ such that $\nabla^2 f(r) = 0$.

SOLUTION: We have $\nabla^2 f(r) = 0$

or $f''(r) + \frac{2}{r} f'(r) = 0$ [using theorem (4.8)]

or $\frac{f''(r)}{f'(r)} = -\frac{2}{r}$

Integrating, we get $\ln f'(r) = -2 \ln r + \ln C$

$$= \ln \frac{C}{r^2}$$

where C is the constant of integration.

Taking antilog, we have $f'(r) = \frac{C}{r^2}$

Integrating again, $f(r) = -\frac{C}{r} + D = A + \frac{B}{r}$

where $A = D$ and $B = -C$ are arbitrary constants.

4.15 CURL OF A VECTOR POINT FUNCTION

Let $\bar{A}(x, y, z)$ be a differentiable vector point function in a certain region of space. Then the curl or rotation of $\bar{A}$, written as $\nabla \times \bar{A}$ or curl $\bar{A}$, is defined by

$$\nabla \times \bar{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

Note that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must procede A_1, A_2, A_3.

If $\bar{A}$ is a constant vector, then $\nabla \times \bar{A} = \bar{0}$. If $\nabla \times \bar{A} = \bar{0}$ in some region R, then $\bar{A}$ is called an irrotational vector point function in that region.

EXAMPLE (10): If $\overline{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find $\nabla \times \overline{A}$ at the point $(1, -1, 1)$.

SOLUTION: We have

$$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} = (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

$$(\nabla \times \overline{A})_{(1,-1,1)} = 3\hat{j} + 4\hat{k}$$

4.16 PROPERTIES OF THE CURL

THEOREM (4.9): If $\overline{A}$ and $\overline{B}$ are differentiable vector point functions, and ϕ is a differentiable scalar point function, then prove that

(i) $\nabla \times (\overline{A} + \overline{B}) = \nabla \times \overline{A} + \nabla \times \overline{B}$

(ii) $\nabla \times (\phi \overline{A}) = \phi(\nabla \times \overline{A}) + (\nabla \phi) \times \overline{A}$

(iii) $\nabla \times (\nabla \phi) = \overline{0}$ (curl grad $\phi = \overline{0}$)

(iv) $\nabla \cdot (\nabla \times \overline{A}) = 0$ (div curl $\overline{A} = 0$)

PROOF: Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\overline{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\overline{A} + \overline{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$

$$\text{Hence, } \nabla \times (A + \overline{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} = \nabla \times \overline{A} + \nabla \times \overline{B}$$

(ii) $\phi \overline{A} = \phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k}$, then

$$\nabla \times (\phi \overline{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)\right]\hat{i} + \left[\frac{\partial}{\partial z}(\phi A_1) - \frac{\partial}{\partial x}(\phi A_3)\right]\hat{j} + \left[\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)\right]\hat{k}$$

$$= \left[\phi\frac{\partial A_3}{\partial y}+\frac{\partial \phi}{\partial y}A_3-\phi\frac{\partial A_2}{\partial z}-\frac{\partial \phi}{\partial z}A_2\right]\hat{i}+\left[\phi\frac{\partial A_1}{\partial z}+\frac{\partial \phi}{\partial z}A_1-\phi\frac{\partial A_3}{\partial x}-\frac{\partial \phi}{\partial x}A_3\right]\hat{j}+\left[\phi\frac{\partial A_2}{\partial x}+\frac{\partial \phi}{\partial x}A_2-\phi\frac{\partial A_1}{\partial y}-\frac{\partial \phi}{\partial y}A_1\right]\hat{k}$$

$$= \phi\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\hat{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$
$$+\left[\left(\frac{\partial \phi}{\partial y}A_3-\frac{\partial \phi}{\partial z}A_2\right)\hat{i}+\left(\frac{\partial \phi}{\partial z}A_1-\frac{\partial \phi}{\partial x}A_3\right)\hat{j}+\left(\frac{\partial \phi}{\partial x}A_2-\frac{\partial \phi}{\partial y}A_3\right)\hat{k}\right]$$

$$= \phi(\nabla\times\bar{A})+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix} = \phi(\nabla\times\bar{A})+(\nabla\phi)\times\bar{A}$$

(iii) $$\nabla\times(\nabla\phi) = \nabla\times\left(\frac{\partial \phi}{\partial x}\hat{i}+\frac{\partial \phi}{\partial y}\hat{j}+\frac{\partial \phi}{\partial z}\hat{k}\right) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\end{vmatrix}$$

$$= \left(\frac{\partial^2 \phi}{\partial y\partial z}-\frac{\partial^2 \phi}{\partial z\partial y}\right)\hat{i}+\left(\frac{\partial^2 \phi}{\partial z\partial x}-\frac{\partial^2 \phi}{\partial x\partial z}\right)\hat{j}+\left(\frac{\partial^2 \phi}{\partial x\partial y}-\frac{\partial^2 \phi}{\partial y\partial x}\right)\hat{k}$$

$$= \bar{0}$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial ,

i.e. $$\frac{\partial^2 \phi}{\partial y\partial z} = \frac{\partial^2 \phi}{\partial z\partial y} , \quad \frac{\partial^2 \phi}{\partial z\partial x} = \frac{\partial^2 \phi}{\partial x\partial z} , \quad \frac{\partial^2 \phi}{\partial x\partial y} = \frac{\partial^2 \phi}{\partial y\partial x}$$

(iv) Since $$\nabla\times\bar{A} = \left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\hat{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$

Hence $$\nabla.(\nabla\times\bar{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}+\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}+\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y} = 0$$

assuming that $\bar{A}$ has continuous second partial derivatives .

THEOREM (4.10): Prove that

(i) $\nabla\times\bar{r} = \bar{0}$ (ii) $\nabla\times[f(r)\bar{r}] = \bar{0}$

(iii) $\nabla\times(r^n\bar{r}) = \bar{0}$ (iv) $\nabla\times\left(\frac{\bar{r}}{r^3}\right) = \bar{0}$

where $\bar{r}$ is the position vector .

PROOF: Since $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, therefore

(i) $$\nabla \times \overline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \overline{0}$$

(ii) Using theorem (4.9) part (ii), we have

$$\nabla \times (\phi \overline{A}) = \phi (\nabla \times \overline{A}) + \nabla \phi \times \overline{A} \text{ , therefore}$$

$$\nabla \times [f(r)\overline{r}] = f(r)(\nabla \times \overline{r}) + [\nabla f(r)] \times \overline{r}$$

$$= f(r)\overline{0} + \frac{f'(r)}{r}\overline{r} \times \overline{r} = \overline{0} \quad (\text{since } \overline{r} \times \overline{r} = \overline{0}).$$

(iii) Setting $f(r) = r^n$ in part (ii), we get

$$\nabla \times (r^n \overline{r}) = \overline{0}$$

(iv) Let $n = -2$ in part (iii), we get

$$\nabla \times \left(\frac{\overline{r}}{r^2}\right) = \overline{0} .$$

4.17 GEOMETRICAL INTERPRETATION OF THE CURL

To find a possible interpretation of the curl , let us consider a body rotating with uniform angular speed ω about an axis ℓ . Let us define the angular velocity vector $\overline{\omega}$ to be a vector of length ω extending along ℓ in the direction in which a right-handed screw would move if given the same rotation as the body . Finally , let $\overline{r}$ be the vector drawn from any point O on the axis ℓ to an arbitrary point P (x , y , z) on the body as shown in figure (4.9) .

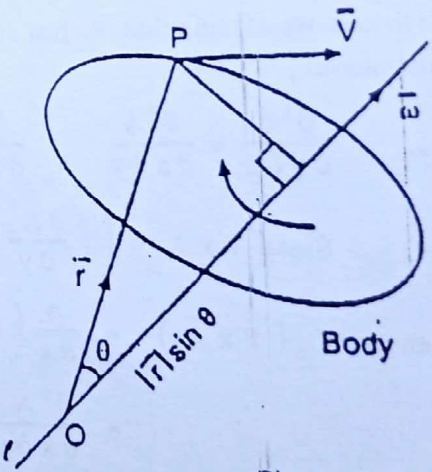

Figure (4.9)

It is clear that the radius at which P rotates is $|\overline{r}|\sin\theta$. Hence , the linear speed of P is

$$|\overline{V}| = \omega |\overline{r}| \sin\theta$$

$$= |\overline{\omega}||\overline{r}|\sin\theta = |\overline{\omega} \times \overline{r}|$$

Moreover , the velocity vector $\overline{V}$ is directed perpendicular to the plane of $\overline{\omega}$ and $\overline{r}$, so that $\overline{\omega}$, $\overline{r}$, and $\overline{V}$ form a right handed system . Hence , the cross product $\overline{\omega} \times \overline{r}$ gives not only the magnitude of $\overline{V}$ but the direction as well , i.e. $\overline{V} = \overline{\omega} \times \overline{r}$

If we now take the point O as the origin of coordinates, we can write

$$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{and} \quad \overline{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$$

Hence, the equation $\overline{V} = \overline{\omega} \times \overline{r}$ can be written as

$$\overline{V} = (\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

If we take the curl of $\overline{V}$, we have

$$\nabla \times \overline{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -(\omega_1 z - \omega_3 x) & \omega_1 y - \omega_2 x \end{vmatrix}$$

Expanding this, remembering that $\overline{\omega}$ is a constant vector, we find

$$\nabla \times \overline{V} = 2\omega_1\hat{i} + 2\omega_2\hat{j} + 2\omega_3\hat{k} = 2\overline{\omega}$$

or $$\overline{\omega} = \frac{1}{2}\nabla \times \overline{V}$$

which says that the angular velocity at any point of a uniformly rotating body is equal to one–half the curl of the linear velocity at that point of the body. This justifies the name rotation used for curl. It is also motivation of the term irrotational for a vector field whose curl is the zero vector. In fluid dynamics, $\nabla \times \overline{V}$ is called vorticity vector and measures the degree to which a fluid swirls, or rotates about a given direction – much as the angular velocity vector measures the rate of rotation of a rigid body.

4.18 OPERATIONS WITH ∇

Here we consider the various combinations of the operator ∇ with vector and scalar functions.

THEOREM (4.11): If $\overline{A}$ and $\overline{B}$ are two vector point functions and ϕ a scalar point function, then show that

(i) $(\overline{A}\cdot\nabla)\phi = \overline{A}\cdot\nabla\phi$

(ii) $(\overline{A}\times\nabla)\phi = \overline{A}\times\nabla\phi$

(iii) $(\overline{A}\cdot\nabla)\overline{B} = A_1\dfrac{\partial \overline{B}}{\partial x} + A_2\dfrac{\partial \overline{B}}{\partial y} + A_3\dfrac{\partial \overline{B}}{\partial z}$

(iv) $(\overline{A}\cdot\nabla)\overline{r} = \overline{A}$

(v) Give possible meaning to $(\overline{A}\times\nabla)\overline{B}$.

PROOF: Let $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then

$$\overline{A}\cdot\nabla = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial y}\right) = A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}$$

(i) $$(\overline{A}\cdot\nabla)\phi = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\phi$$

$$= A_1\frac{\partial\phi}{\partial x}+A_2\frac{\partial\phi}{\partial y}+A_3\frac{\partial\phi}{\partial z}$$

$$= (A_1\hat{i}+A_2\hat{j}+A_3\hat{k})\cdot\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \overline{A}\cdot\nabla\phi$$

(ii) $$(\overline{A}\times\nabla)\phi = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix}\phi$$

$$= \left[\hat{i}\left(A_2\frac{\partial}{\partial z}-A_3\frac{\partial}{\partial y}\right)+\hat{j}\left(A_3\frac{\partial}{\partial x}-A_1\frac{\partial}{\partial z}\right)+\hat{k}\left(A_1\frac{\partial}{\partial y}-A_2\frac{\partial}{\partial x}\right)\right]\phi$$

$$= \left(A_2\frac{\partial\phi}{\partial z}-A_3\frac{\partial\phi}{\partial y}\right)\hat{i}+\left(A_3\frac{\partial\phi}{\partial x}-A_1\frac{\partial\phi}{\partial z}\right)\hat{j}+\left(A_1\frac{\partial\phi}{\partial y}-A_2\frac{\partial\phi}{\partial x}\right)\hat{k}$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix} = \overline{A}\times\nabla\phi$$

(iii) $$(\overline{A}\cdot\nabla)\overline{B} = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\overline{B}$$

$$= A_1\frac{\partial\overline{B}}{\partial x}+A_2\frac{\partial\overline{B}}{\partial y}+A_3\frac{\partial\overline{B}}{\partial z}$$

(iv) $$(\overline{A}\cdot\nabla)\overline{r} = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= A_1\hat{i}+A_2\hat{j}+A_3\hat{k} = \overline{A}$$

(v) No definition or meaning can be assigned to $(\overline{A}\times\nabla)\overline{B}$, because it is a kind of differential operator with vector quantities.

EXAMPLE (11): If $\overline{A} = 2yz\hat{i}-x^2y\hat{j}+xz^2\hat{k}$, $\overline{B} = x^2\hat{i}+yz\hat{j}-xy\hat{k}$ and $\phi = 2x^2yz^3$, find

(i) $(\overline{A}\cdot\nabla)\phi$ (ii) $(\overline{A}\times\nabla)\phi$

(iii) $(\overline{A}\cdot\nabla)\overline{B}$

SOLUTION: Since $\phi = 2x^2yz^3$, we have

$$\nabla\phi = 4xyz^3\hat{i}+2x^2z^3\hat{j}+6x^2yz^2\hat{k}$$

(i) $$(\overline{A}\cdot\nabla)\phi = \overline{A}\cdot\nabla\phi$$
$$= (2yz\,\hat{i} - x^2y\,\hat{j} + xz^2\,\hat{k})\cdot(4xyz^3\,\hat{i} + 2x^2z^3\,\hat{j} + 6x^2yz^2\,\hat{k})$$
$$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$$

(ii) $$(\overline{A}\times\nabla)\phi = \overline{A}\times\nabla\phi$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2 \end{vmatrix}$$
$$= -(6x^4y^2z^2 + 2x^3z^5)\,\hat{i} + (4x^2yz^5 - 12x^2y^2z^3)\,\hat{j}$$
$$+ (4x^2yz^4 + 4x^3y^2z^3)\,\hat{k}$$

(iii) Since $\overline{A}\cdot\nabla = 2yz\dfrac{\partial}{\partial x} - x^2y\dfrac{\partial}{\partial y} + xz^2\dfrac{\partial}{\partial z}$, therefore

$$(\overline{A}\cdot\nabla)\overline{B} = \left(2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right)(x^2\,\hat{i} + yz\,\hat{j} - xy\,\hat{k})$$
$$= 2yz(2x\,\hat{i} - y\,\hat{k}) - x^2y(z\,\hat{j} - x\,\hat{k}) + xz^2(y\,\hat{j})$$
$$= 4xyz\,\hat{i} + (xyz^2 - x^2yz)\,\hat{j} + (x^3y - 2y^2z)\,\hat{k}$$

THEOREM (4.12): If $\overline{A}$ and $\overline{B}$ are two vector functions, prove that

$$(\overline{A}\times\nabla)\cdot\overline{B} = \overline{A}\cdot(\nabla\times\overline{B})$$

PROOF: We have

$$\overline{A}\times\nabla = \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\hat{i} + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\hat{j} + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\hat{k}, \text{ therefore}$$

$$(\overline{A}\times\nabla)\cdot\overline{B} = \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)(\hat{i}\cdot\overline{B}) + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)(\hat{j}\cdot\overline{B}) + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)(\hat{k}\cdot\overline{B})$$

Since $\hat{i}\cdot\overline{B} = B_1$, $\hat{j}\cdot\overline{B} = B_2$, $\hat{k}\cdot\overline{B} = B_3$, therefore

$$(\overline{A}\times\nabla)\cdot\overline{B} = \left(A_2\frac{\partial B_1}{\partial z} - A_3\frac{\partial B_1}{\partial y}\right) + \left(A_3\frac{\partial B_2}{\partial x} - A_1\frac{\partial B_2}{\partial z}\right) + \left(A_1\frac{\partial B_3}{\partial y} - A_2\frac{\partial B_3}{\partial x}\right)$$
$$= A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) + A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial z}\right) + A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$
$$= (A_1\,\hat{i} + A_2\,\hat{j} + A_3\,\hat{k})\cdot\left[\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\hat{i} + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\hat{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\hat{k}\right]$$
$$= \overline{A}\cdot(\nabla\times\overline{B})$$

4.19 VECTOR IDENTITIES

THEOREM (4.13): If $\bar{A}$ and $\bar{B}$ are two differentiable vector point functions, prove that

(i) $\nabla.(\bar{A}\times\bar{B}) = \bar{B}.(\nabla\times\bar{A}) - \bar{A}.(\nabla\times\bar{B})$

(ii) $\nabla\times(\bar{A}\times\bar{B}) = (\bar{B}.\nabla)\bar{A} - \bar{B}(\nabla.\bar{A}) - (\bar{A}.\nabla)\bar{B} + \bar{A}(\nabla.\bar{B})$

(iii) $\nabla(\bar{A}.\bar{B}) = (\bar{B}.\nabla)\bar{A} + (\bar{A}.\nabla)\bar{B} + \bar{B}\times(\nabla\times\bar{A}) + \bar{A}\times(\nabla\times\bar{B})$

(iv) $\nabla\times(\nabla\times\bar{A}) = \nabla(\nabla.\bar{A}) - \nabla^2\bar{A}$

PROOF: Let $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\bar{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\bar{A}\times\bar{B} = (A_2B_3 - A_3B_2)\hat{i} + (A_3B_1 - A_1B_3)\hat{j} + (A_1B_2 - A_2B_1)\hat{k}$

and $\nabla.(\bar{A}\times\bar{B}) = \frac{\partial}{\partial x}(A_2B_3 - A_3B_2) + \frac{\partial}{\partial y}(A_3B_1 - A_1B_3) + \frac{\partial}{\partial z}(A_1B_2 - A_2B_1)$

$$= A_2\frac{\partial B_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial B_2}{\partial x} - B_2\frac{\partial A_3}{\partial x}$$

$$+ A_3\frac{\partial B_1}{\partial y} + B_1\frac{\partial A_3}{\partial y} - A_1\frac{\partial B_3}{\partial y} - B_3\frac{\partial A_1}{\partial y}$$

$$+ A_1\frac{\partial B_2}{\partial z} + B_2\frac{\partial A_1}{\partial z} - A_2\frac{\partial B_1}{\partial z} - B_1\frac{\partial A_2}{\partial z}$$

$$= B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + B_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + B_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$- A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) - A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right) - A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$

$$= (B_1\hat{i} + B_2\hat{j} + B_3\hat{k}).\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$

$$- (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).\left[\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\hat{i} + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\hat{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\hat{k}\right]$$

$$= \bar{B}.(\nabla\times\bar{A}) - \bar{A}.(\nabla\times\bar{B})$$

(ii) We know that $\nabla\times\bar{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\bar{V}$

$$= \hat{i}\times\frac{\partial\bar{V}}{\partial x} + \hat{j}\times\frac{\partial\bar{V}}{\partial y} + \hat{k}\times\frac{\partial\bar{V}}{\partial z} = \sum\hat{i}\times\frac{\partial\bar{V}}{\partial x}$$

$$\nabla\times(\bar{A}\times\bar{B}) = \sum\left[\hat{i}\times\frac{\partial}{\partial x}(\bar{A}\times\bar{B})\right]$$

$$= \sum\left[\hat{i}\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x} + \frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)\right]$$