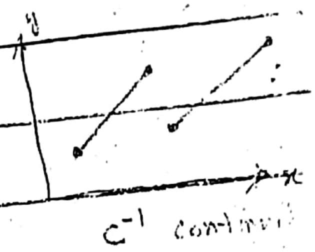
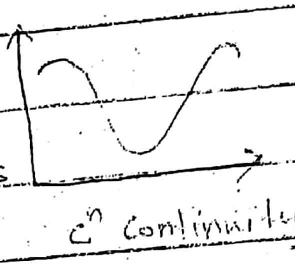
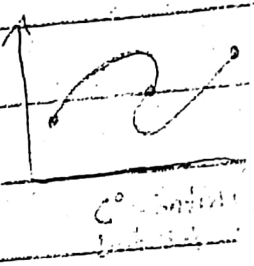
Parametric continuity



The various order of Parametric continuity can be described as follows

- C^{-1}: curves include discontinuities
- C^{0}: curves are joined
- C^{1}: first derivatives are continous
- C^{2}: first and second derivatives are continuous
- C^{n}: first through n^{th} derivatives are continuous





Geometric Continuity

Geometric continuity G^{n} which removes restrictions on the speed with which the parameter traces out the curve. It is the more relaxed form of continuit which requires visual smoothness of curve

"A function $P(t)$ have G^{n} conitinity if $P^{(n)}(t) \neq 0$ and $P^{(n)}(C^{+}) = K\,P^{n}(C^{-})$ where C is joint point"

The various order of Parametric continuity can be described as follows

- G^{0}: The curve touch at the join point or $P(C^{-}) = P(C^{+})$
- G^{1}: The curves share a common tangent direction at the join point. or

$$P(C^{-}) = P(C^{+}) \text{ and } P'(C^{-}) = K\,P'(C^{+})$$

- G^{2}: The curve also share a common center of curvature at the join point

$$P(C^{-}) = P(C^{+}) \text{ and } P'(C^{-}) = K\,P'(C^{+})$$
$$P''(C^{-}) = K_1 P''(C^{+})$$

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Parametric Representation of a curve:-

Parametric representation of a curve represents continuous deformation of line segment. The Coordinates of parametric curve are represented as function of Single variable or parameter taken from an interval.

Example:- In two dimmensions

$$Q(t) = (x(t), y(t)) \quad , \quad t \in I$$

Tangent vector:-

If $Q(t) = (x(t), y(t))$ in parametric equation of the curve then ~~slop of~~ tangent vector is

$$T(t) = \frac{d}{dt}(Q(t)) = \left(\frac{d}{dt}x(t), \frac{d}{dt}y(t)\right)$$

Example:-

Equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In parametric form

$$x = a\cos t \quad , \quad y = b\sin t$$

$$Q(t) = (a\cos(t), b\sin(t))$$

$$T(t) = \frac{d}{dt}(Q(t)) = (-a\sin(t), b\cos(t))$$

at $t = \pi/2$

$$T(t) = (-a, 0)$$

Rational Parametric form:-

Consider the parametrization $x(t), y(t)$ are defined in the form of ratio of two polynomials,

$$P(t) = \frac{P_0(1-t) + 2P_1 w(1-t)t + P_2 t^2}{(1-t) + 2w(1-t)t + t^2}$$

Let $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$

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$$P(t) = \frac{(x_0, y_0)(1-t)^2 + 2(x_1, y_1) w t(1-t) + (x_2, y_2) t^2}{(1-t)^2 + 2wt(1-t) + t^2}$$

$$P(t) = \left(\frac{x_0(1-t)^2 + 2x_1 wt(1-t) + x_2 t^2}{(1-t)^2 + 2wt(1-t) + t^2}, \frac{y_0(1-t)^2 + 2wy_1 t(1-t) + y_2 t^2}{(1-t)^2 + 2wt(1-t) + t^2} \right)$$

Properties:-

1) If $t = 0$, $P(0) = P_0$
2) If $t = 1$, $P(1) = P_2$
3) If $0 < t < 1$, then the shape of the curve $P(t)$ depends upon pts P_0, P_1, P_2
4) If we set

   $w < 1$ then curve is ellips

   $w = 1$ then curve is Parabola

   $w > 1$ then curve is hyperbola

-: Properties of Blending Functions :-

We can Select our blending function such that the curve has any or all of the following properties

- Coordinate system independence / affine invariance.

This means that the curve will not change if the Coordinate system is changed. It means that if the control ~~poin~~ mesh is rotated, translated, sheared or scaled then corresponding curve goes same rotation, translation, shear or Scale.

In order to provide coordinate system independence, the blending functions must form a partition of unity i-e

$$\sum_{i=0}^{n} f_i(t) = 1$$

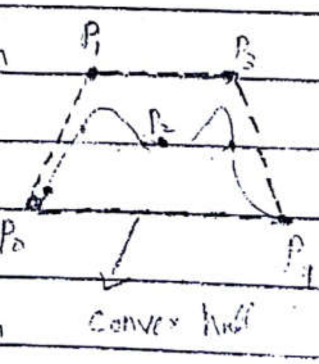
★ If $P_1, P_2, \ldots, P_n$ are points and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are Scalar s.t $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ then the expression

$\alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_n P_n$ is called "**affine Combination**"

) Convex hull property

This means that each line segment joining two points on curve lie within the convex hull of control points.

This property exists in curve which are coordinate system independent and for which blending functions are all non negative. i-e



$$\sum_{i=0}^{n} f_i(t) = 1 \quad ; \quad f_i(t) \geq 0 \quad i = 0, \ldots, n$$

★ If $P_1, P_2, \ldots, P_n$ are control points and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are Scalar s.t $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and $\alpha_i \geq 0$ then the expression

$\alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_n P_n$ is called "**Convex Combination**"

3.) Symmetry

Curve which are Symmetric do not change if the Contral points are order in reverse sequence.

For a curve whose domain is $[t_0, t_1]$ is symmetry if

$$\sum_{i=0}^{n} f_i(t) P_i = \sum_{i=0}^{n} f_{n-i}(t_0+t_1-t) P_{n-i}$$

where $f_i(t)$ are Blending function and P_i are contral points where $i = 0, 1, \ldots, n$

This holds iff

$$f_i(t) = f_{n-i}(t_0+t_1-t).$$

For a curve whose domain is $[0,1]$ we have.

$$\sum_{i=0}^{n} f_i(t) P_i = \sum_{i=0}^{n} f_{n-i}(1-t) P_{n-i}$$

This holds if

$$f_i(t) = f_{n-i}(1-t)$$

4) Linear independence.

A set of blending functions is linearly independent if $\sum_{i=0}^{n} C_i f_i(t) = 0 \implies C_i = 0 \quad \forall i$

If a set of blending functions are linearly independent they can caled **basis Function**

5) End point interpolation:

If a curve over the domain $[t_0, t_1]$ is to Pass through the first and last contral points, the following conditions must be met

$f_0(t_0) = 1 \qquad f_i(t_0) = 0 \qquad i = 1, \ldots, n$

$f_n(t_1) = 1 \qquad f_i(t_1) = 0 \qquad i = 0, \ldots, n-1$

Any set of blending functions can be analyzed in terms of the above properties

Bernstein Polynomial

Polynomials in Bernstein form were first used by "Sergei Natanovich Bernstein" in a constructive proof for the "stone-Weierstrass approximation theorem".

The Bernstein Basic Polynomials of degree n are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

for $i = 0, 1, \dots, n$ where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$

* There are n+1 Bernstein polynomials of degree n.
* The Bernstein polynomials of degree 1 are

$B_{0,1}(t) = 1-t$

$B_{1,1}(t) = t$

for $0 \le t \le 1$ we can draw as fig 1.

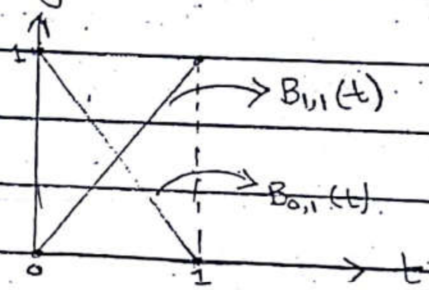


fig 1

* The Bernstein polynomial of degree 2 are

$B_{0,2}(t) = (1-t)^2$

$B_{1,2}(t) = 2t(1-t)$

$B_{2,2}(t) = t^2$

and can be plotted for $0 \le t \le 1$ as in fig (2)

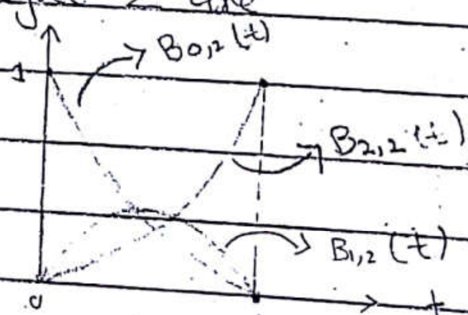


fig (2)

The Bernstein polynomial of degree 3 are

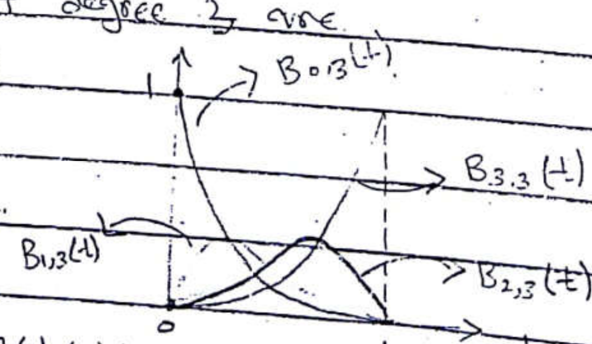
$B_{0,3}(t) = (1-t)^3$

$B_{1,3}(t) = 3t(1-t)^2$

$B_{2,3}(t) = 3t^2(1-t)$

$B_{3,3}(t) = t^3$

and can be plotted for $0 \le t \le 1$ as in fig (3)



A Recursive defination of the Bernstein polynomials

The Bernstein polynomials of degree n Can be defined by blending together two Bernstein polynomials of degree $n-1$ i.e

$$B_{K,n}(t) = (1-t)\, B_{K,n-1}(t) + t\, B_{K-1,n-1}(t)$$

Proof :-

$$\begin{aligned} R.H.S &= (1-t) B_{K,n-1}(t) + t\, B_{K-1,n-1}(t) \\ &= (1-t)\binom{n-1}{K} t^K (1-t)^{n-1-K} + t\binom{n-1}{K-1} t^{K-1} (1-t)^{n-1-K+1} \\ &= \binom{n-1}{K} t^K (1-t)^{n-K} + \binom{n-1}{K-1} t^K (1-t)^{n-K} \\ &= \left\{\binom{n-1}{K} + \binom{n-1}{K-1}\right\} t^K (1-t)^{n-K} \\ &= \binom{n}{K} t^K (1-t)^{n-K} \qquad \because \binom{n-1}{K} + \binom{n-1}{K-1} = \binom{n}{K} \\ &= B_{K,n}(t) \end{aligned}$$

Properties of Bernstein Polynomial:-

1> Bernstine polynomial forms partition of unity i.e

$$\sum_{i=0}^{n} B_{i,n}(t) = 1$$

Proof:-

$$\sum_{i=0}^{n} B_{i,n}(t) = B_{0,n}(t) + B_{1,n}(t) + \cdots + B_{n,n}(t)$$

$$= \binom{n}{0}(1-t)^n + \binom{n}{1}(1-t)^{n-1} t + \binom{n}{2}(1-t)^{n-2} t^2 + \cdots$$

$$\cdots + \binom{n}{n} t^n \quad \text{——— (1)}$$

By Binomial expension

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} b^n$$

put $a = 1-t$, $b = t$

$$(1-\cancel{t}+\cancel{t})^n = \binom{n}{0}(1-t)^n + \binom{n}{1}(1-t)^{n-1} t + \binom{n}{2}(1-t)^{n-2} t^2 + \cdots$$

$$\cdots + \binom{n}{n} t^n$$

$$1 = \binom{n}{0}(1-t)^n + \binom{n}{1}(1-t)^{n-1} t + \binom{n}{2}(1-t)^{n-2} t^2 + \cdots + \binom{n}{n} t^n$$

Put this in eq ① we got

$$\sum_{i=0}^{n} B_{i,n}(t) = 1$$

Hence proved

$\Rightarrow$ $B_{i,n}(t) \geq 0 \quad \forall \; t \in [0,1]$

proof:- we know that $B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

As we know that

$0 \leq t \leq 1$

$\Rightarrow t \geq 0$ and $1 \geq t$

$\Rightarrow t \geq 0$ and $1-t \geq 0$

$\Rightarrow t^i \geq 0$ and $(1-t)^{n-i} \geq 0$

$\Rightarrow t^i (1-t)^{n-i} \geq 0$

$\Rightarrow \binom{n}{i} t^i (1-t)^{n-i} \geq 0 \quad \because \binom{n}{i} \geq 0 \;\; \forall \; i = 0, 1, \ldots, n$

$\Rightarrow B_{i,n}(t) \geq 0 \quad \forall \; t \in [0,1]$

Hence proved.

$\Rightarrow$ Symmetric property $B_{i,n}(t) = B_{n-i,n}(1-t)$

proof:-

$$B_{n-i,n}(1-t) = \binom{n}{n-i} (1-t)^{n-i} \{1-(1-t)\}^{n-(n-i)}$$

$$= \frac{n!}{(n-i)!\,(n-(n-i))!} (1-t)^{n-i} (\cancel{1} - \cancel{1} + t)^{\cancel{n}-\cancel{n}+i}$$

$$= \frac{n!}{i!\,(n-i)!} (1-t)^{n-i} t^i$$

$$= B_{i,n}(t)$$

4) Bernstein polynomials are linearly independent.

Proof :-

Consider the Bernstein polynomials of degree n

~~we~~ To prove that ~~those~~ are linearly independent

Set

$$\sum_{i=0}^{n} C_i B_{i,n}(t) = 0$$

$$C_0 B_{0,n}(t) + C_1 B_{1,n}(t) + C_2 B_{2,n}(t) + \cdots + C_n B_{n,n}(t) = 0$$

$$\Rightarrow C_0 \binom{n}{0}(1-t)^n + C_1\binom{n}{1} t (1-t)^{n-1} + C_2\binom{n}{2} t^2 (1-t)^{n-2} + \cdots + C_n\binom{n}{n} t^n = 0$$

By using Binomial expension to evalute $(1-t)^{n-i}$

$$(1-t)^{n-i} = \binom{n-i}{0} - \binom{n-i}{1} t + \binom{n-i}{2} t^2 + \cdots + (-1)^{n-i}\binom{n-i}{n-i} t^{n-i}$$

$\forall \; i = 0, 1, 2, \ldots, n$

$$\Rightarrow C_0\binom{n}{0}\left\{\binom{n}{0} - \binom{n}{1} t + \binom{n}{2} t^2 + \cdots + (-1)^n\binom{n}{n} t^n\right\} + C_1\binom{n}{1} t\left\{\binom{n-1}{0} - \binom{n-1}{1} t + \cdots + \binom{n-1}{n-1}(-1)^{n-1}\right\}$$

$$+ \cdots + C_n t^n = 0$$

$$\Rightarrow C_0\binom{n}{0}\binom{n}{0} + \left\{C_0\binom{n}{0}\binom{n}{1} + C_1\binom{n}{1}\binom{n-1}{0}\right\} t + \left\{C_0\binom{n}{0}\binom{n}{2} + C_1\binom{n}{1}\binom{n-1}{1} + C_2\binom{n}{2}\binom{n-2}{0}\right\} t^2$$

$$+ \cdots + \left\{C_0\binom{n}{0}\binom{n}{n}(-1)^n + C_1\binom{n}{1}\binom{n-1}{n-1}(-1)^{n-1} + C_2\binom{n}{2}\binom{n-2}{n-2}(-1)^{n-2} + \cdots + C_n\binom{n}{n}\right\} t^n$$

Comparing the coefficient we get

$C_0\binom{n}{0}\binom{n}{0} = 0 \quad \Rightarrow \quad C_0 = 0,$

$C_0\binom{n}{0}\binom{n}{1} + C_1\binom{n}{1}\binom{n-1}{0} = 0 \quad \Rightarrow \quad C_1 = 0,$

$C_0\binom{n}{0}\binom{n}{2} + C_1\binom{n}{1}\binom{n-1}{1} + C_2\binom{n}{2}\binom{n-2}{0} = 0 \quad \Rightarrow \quad C_2 = 0,$

$\vdots$

$C_0\binom{n}{0}\binom{n}{n}(-1)^n + C_1\binom{n}{1}\binom{n-1}{n-1}(-1)^{n-1} + \cdots + C_n\binom{n}{n}\binom{n-n}{n-n}(-1)^{n-n} = 0 \Rightarrow C_n = 0.$

As all coefficients are zero So Bernstein polynomials are linearly independent.

(a) $B_{i,n}(0) = \delta_{i0}$ and (b) $B_{i,n}(1) = \delta_{in}$ where δ is Kronceker delta

proof:

We prove that $B_{i,n}(0) = \begin{cases} 1 & i=0 \\ 0 & i\neq 0 \end{cases} = \delta_{i0}$ and

(1) $B_{i,n}(1) = \begin{cases} 1 & i=n \\ 0 & i\neq 0 \end{cases} = \delta_{in}$

Now for this we know that

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

For $i=0$

$$B_{0,n}(t) = \binom{n}{0} t^0 (1-t)^{n-0} = \binom{n}{0}(1-t)^n = (1-t)^N$$

for $t=0$ $\quad B_{0,n}(t) = (1-0)^N = 1$

Now for $i \neq 0$

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$B_{i,n}(0) = \binom{n}{i} (0)^i (1-0)^{n-i} = 0$$

So

$$B_{i,n}(0) = \begin{cases} 1 & i=0 \\ 0 & i\neq 0 \end{cases} \Rightarrow B_{i,n}(0) = \delta_{i0}$$

(2) $B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

for $i=n$

$$B_{n,n}(t) = t^n$$

$$B_{n,n}(1) = 1$$

for $i \neq n$

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$B_{i,n}(1) = \binom{n}{i} (1)^i (1-1)^{n-i} = 0$$

so

$$B_{i,n}(1) = \begin{cases} 1 & i=n \\ 0 & i\neq n \end{cases}$$

$$\Rightarrow B_{i,n}(1) = \delta_{in}$$

6. Derivative of Bernstein polynomial

We know that

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\frac{d}{dt} B_{i,n}(t) = \frac{d}{dt}\left\{\binom{n}{i} t^i (1-t)^{n-i}\right\}$$

$$= \binom{n}{i}\left\{ i t^{i-1}(1-t)^{n-i} - (n-i) t^i (1-t)^{n-i-1}\right\}$$

$$= \frac{n!}{i!(n-i)!}\left\{ i t^{i-1}(1-t)^{n-i} - (n-i) t^i (1-t)^{n-i-1}\right\}$$

$$= \frac{n(n-1)!\, i}{i(i-1)!(n-1-i+1)!} t^{i-1}(1-t)^{n-i+1-1} - \frac{n(n-1)!\,(n-i)}{i!(n-i)(n-i-1)!} t^i (1-t)^{n-1-i}$$

$$= n\binom{n-1}{i-1} t^{i-1}(1-t)^{(n-1)-(i-1)} - n\binom{n-1}{i} t^i (1-t)^{n-1-i}$$

$$= n B_{i-1,n-1}(t) - n B_{i,n-1}(t)$$

$$= n\{B_{i-1,n-1}(t) - B_{i,n-1}(t)\}$$

7. Integral of Bernstein polynomial for $t \in [0,1]$

The integral of Bernstein polynomial for $t \in [0,1]$ is

$$\int_0^1 B_{i,n}(t)\,dt = \int_0^1 \binom{n}{i} t^i (1-t)^{n-i}\,dt$$

$$= \binom{n}{i}\left\{ -t^i \frac{(1-t)^{n-i+1}}{n-i+1}\Big|_0^1 + \int_0^1 i t^{i-1} \frac{(1-t)^{n-i+1}}{n-i+1}\,dt\right\}$$

$$= \binom{n}{i}\left\{ 0 + \frac{i}{(n-i+1)} \int_0^1 t^{i-1}(1-t)^{n-(i-1)}\,dt\right\}$$

$$= \frac{n!\, i}{i!(n-i)!(n-i+1)} \int_0^1 t^{i-1}(1-t)^{n-(i-1)}\,dt$$

$$= \frac{n!}{(i-1)!(n-i+1)!} \int_0^1 t^{i-1}(1-t)^{n-(i-1)}\,dt$$

$$= \int_0^1 \binom{n}{i-1} t^{i-1}(1-t)^{n-(i-1)}\,dt$$

$$= \int_0^1 B_{i-1,n}(t)\,dt$$

$$\Rightarrow \int_0^1 B_{i,n}(t)\,dt = \int_0^1 B_{i-1,n}(t)\,dt$$
$$= \int_0^1 B_{i-2,n}(t)\,dt = \cdots = \int_0^1 B_{0,n}(t)\,dt$$

$$\Rightarrow \int_0^1 B_{i,n}(t)\,dt = \int_0^1 (1-t)^n\,dt$$
$$= \left.\frac{-(1-t)^{n+1}}{n+1}\right|_0^1$$
$$= \frac{-(1-1)^{n+1}}{n+1} + \frac{(1-0)^{n+1}}{n+1} = \frac{1}{n+1}$$

So
$$\int_0^1 B_{i,n}(t)\,dt = \frac{1}{n+1}$$

so integral of Bernstien polynomial of degree n is constant for $i = 0, 1, 2, ---, n$.

3) $B_{i,n}(t) = 0$ for $i > n$ and $i < 0$

proof:- we prove that $B_{i,n}(t) = 0$ for $i = -1, -2, ---$ and $i = n+1, n+2, ---$

we know that

$$B_{i,n}(t) = (1-t)B_{i,n-1}(t) + t\,B_{i-1,n-1}(t)$$
$$\Rightarrow B_{i,n}(t) = \frac{1}{1-t}\left[B_{i,n+1}(t) - t\,B_{i-1,n}(t)\right]$$

for $i = n+1$

$$B_{n+1,n}(t) = \frac{1}{1-t}\left[B_{n+1,n+1}(t) - t\,B_{n+1-1,n}(t)\right]$$
$$= \frac{1}{(1-t)}\left[\binom{N+1}{N+1}t^{n+1}(1-t)^0 - t\binom{n}{n}t^n(1-t)^0\right]$$
$$= \frac{1}{1-t}\left[t^{n+1} - t^{n+1}\right] = 0$$

Now suppose that it is true for $i = n+K$ i.e

$$B_{n+K,n}(t) = 0$$

We prove it is true for $i = n+K+1$

For this

$$B_{n+K+1,n}(t) = \binom{n}{n+K+1} t^{n+K+1}(1-t)^{n-(n+K+1)}$$