

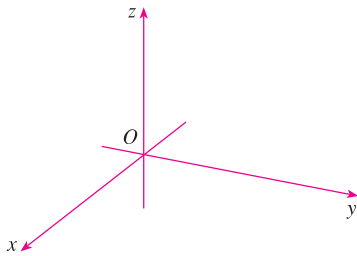
## 12.1 Three-Dimensional Coordinate Systems

### 3D Space

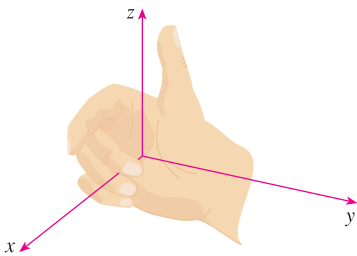
To locate a point in a plane, we need two numbers. We know that any point in the plane can be represented as an ordered pair  $(a, b)$  of real numbers, where  $a$  is the  $x$ -coordinate and  $b$  is the  $y$ -coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple  $(a, b, c)$  of real numbers.

In order to represent points in space, we first choose a fixed point  $O$  (the origin) and three directed lines through  $O$  that are perpendicular to each other, called the **coordinate axes** and labeled the  $x$ -axis,  $y$ -axis, and  $z$ -axis. Usually we think of the  $x$ - and  $y$ -axes as being horizontal and the  $z$ -axis as being vertical, and we draw the orientation of the axes as in Figure 1. The direction of the  $z$ -axis is determined by the **right-hand rule** as illustrated in Figure 2: If you curl the fingers of your right hand around the  $z$ -axis in the direction of a  $90^\circ$  counterclockwise rotation from the positive  $x$ -axis to the positive  $y$ -axis, then your thumb points in the positive direction of the  $z$ -axis.

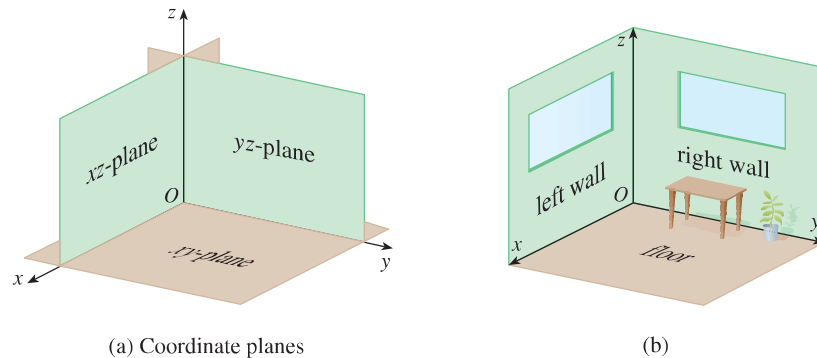
The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a). The  $xy$ -plane is the plane that contains the  $x$ - and  $y$ -axes; the  $yz$ -plane contains the  $y$ - and  $z$ -axes; the  $xz$ -plane contains the  $x$ - and  $z$ -axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.



**FIGURE 1**  
Coordinate axes



**FIGURE 2**  
Right-hand rule



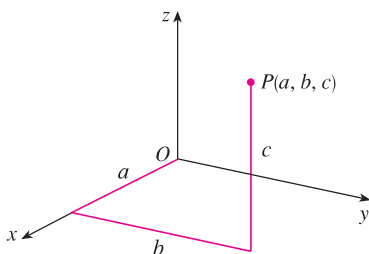
**FIGURE 3**

(a) Coordinate planes

(b)

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the  $xz$ -plane, the wall on your right is in the  $yz$ -plane, and the floor is in the  $xy$ -plane. The  $x$ -axis runs along the intersection of the floor and the left wall. The  $y$ -axis runs along the intersection of the floor and the right wall. The  $z$ -axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point  $O$ .

Now if  $P$  is any point in space, let  $a$  be the (directed) distance from the  $yz$ -plane to  $P$ , let  $b$  be the distance from the  $xz$ -plane to  $P$ , and let  $c$  be the distance from the  $xy$ -plane to  $P$ . We represent the point  $P$  by the ordered triple  $(a, b, c)$  of real numbers and we call  $a$ ,  $b$ , and  $c$  the **coordinates** of  $P$ ;  $a$  is the  $x$ -coordinate,  $b$  is the  $y$ -coordinate, and  $c$  is the  $z$ -coordinate. Thus, to locate the point  $(a, b, c)$ , we can start at the origin  $O$  and move  $a$  units along the  $x$ -axis, then  $b$  units parallel to the  $y$ -axis, and then  $c$  units parallel to the  $z$ -axis as in Figure 4.



**FIGURE 4**

The point  $P(a, b, c)$  determines a rectangular box as in Figure 5. If we drop a perpendicular from  $P$  to the  $xy$ -plane, we get a point  $Q$  with coordinates  $(a, b, 0)$  called the **projection** of  $P$  onto the  $xy$ -plane. Similarly,  $R(0, b, c)$  and  $S(a, 0, c)$  are the projections of  $P$  onto the  $yz$ -plane and  $xz$ -plane, respectively.

As numerical illustrations, the points  $(-4, 3, -5)$  and  $(3, -2, -6)$  are plotted in Figure 6.

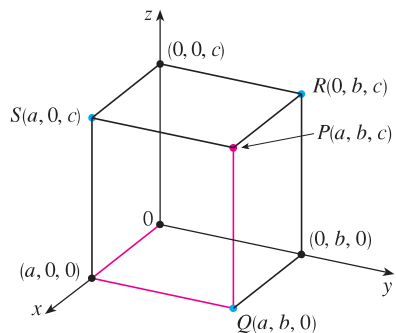


FIGURE 5

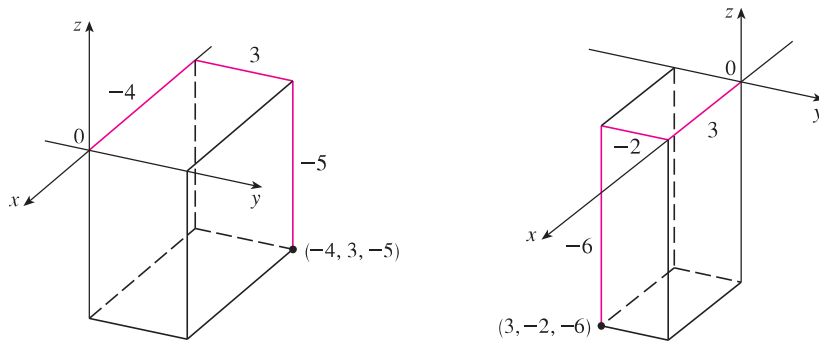


FIGURE 6

The Cartesian product  $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$  is the set of all ordered triples of real numbers and is denoted by  $\mathbb{R}^3$ . We have given a one-to-one correspondence between points  $P$  in space and ordered triples  $(a, b, c)$  in  $\mathbb{R}^3$ . It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

■ **Surfaces**

In two-dimensional analytic geometry, the graph of an equation involving  $x$  and  $y$  is a curve in  $\mathbb{R}^2$ . In three-dimensional analytic geometry, an equation in  $x, y,$  and  $z$  represents a *surface* in  $\mathbb{R}^3$ .

**EXAMPLE 1** What surfaces in  $\mathbb{R}^3$  are represented by the following equations?

- (a)  $z = 3$
- (b)  $y = 5$

**SOLUTION**

(a) The equation  $z = 3$  represents the set  $\{(x, y, z) \mid z = 3\}$ , which is the set of all points in  $\mathbb{R}^3$  whose  $z$ -coordinate is 3 ( $x$  and  $y$  can each be any value). This is the horizontal plane that is parallel to the  $xy$ -plane and three units above it as in Figure 7(a).

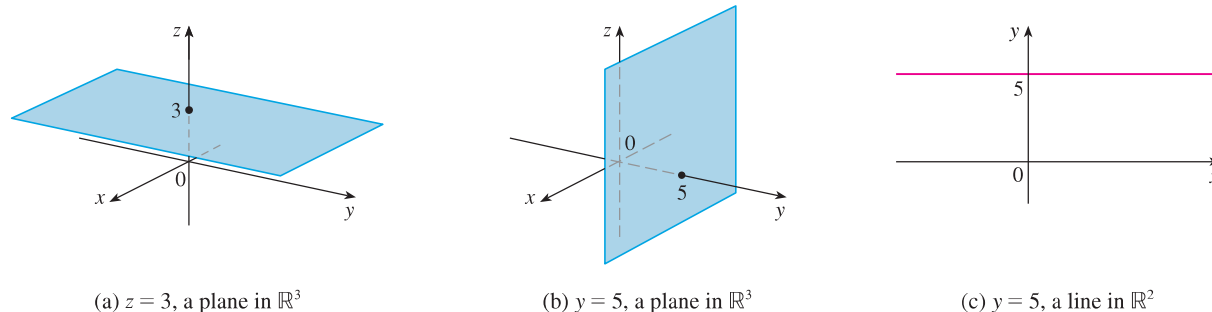


FIGURE 7

(b) The equation  $y = 5$  represents the set of all points in  $\mathbb{R}^3$  whose  $y$ -coordinate is 5. This is the vertical plane that is parallel to the  $xz$ -plane and five units to the right of it as in Figure 7(b).

**NOTE** When an equation is given, we must understand from the context whether it represents a curve in  $\mathbb{R}^2$  or a surface in  $\mathbb{R}^3$ . In Example 1,  $y = 5$  represents a plane in  $\mathbb{R}^3$ , but of course  $y = 5$  can also represent a line in  $\mathbb{R}^2$  if we are dealing with two-dimensional analytic geometry. See Figure 7(b) and (c).

In general, if  $k$  is a constant, then  $x = k$  represents a plane parallel to the  $yz$ -plane,  $y = k$  is a plane parallel to the  $xz$ -plane, and  $z = k$  is a plane parallel to the  $xy$ -plane. In Figure 5, the faces of the rectangular box are formed by the three coordinate planes  $x = 0$  (the  $yz$ -plane),  $y = 0$  (the  $xz$ -plane), and  $z = 0$  (the  $xy$ -plane), and the planes  $x = a$ ,  $y = b$ , and  $z = c$ .

### EXAMPLE 2

(a) Which points  $(x, y, z)$  satisfy the equations

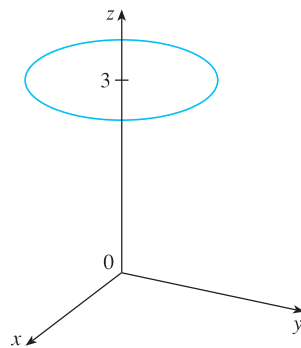
$$x^2 + y^2 = 1 \quad \text{and} \quad z = 3$$

(b) What does the equation  $x^2 + y^2 = 1$  represent as a surface in  $\mathbb{R}^3$ ?

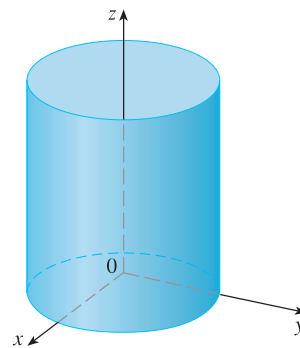
#### SOLUTION

(a) Because  $z = 3$ , the points lie in the horizontal plane  $z = 3$  from Example 1(a). Because  $x^2 + y^2 = 1$ , the points lie on the circle with radius 1 and center on the  $z$ -axis. See Figure 8.

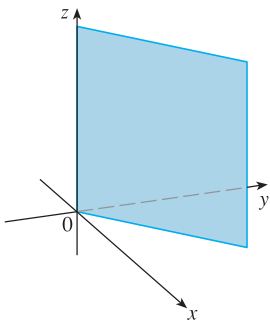
(b) Given that  $x^2 + y^2 = 1$ , with no restrictions on  $z$ , we see that the point  $(x, y, z)$  could lie on a circle in any horizontal plane  $z = k$ . So the surface  $x^2 + y^2 = 1$  in  $\mathbb{R}^3$  consists of all possible horizontal circles  $x^2 + y^2 = 1$ ,  $z = k$ , and is therefore the circular cylinder with radius 1 whose axis is the  $z$ -axis. See Figure 9.



**FIGURE 8**  
The circle  $x^2 + y^2 = 1$ ,  $z = 3$



**FIGURE 9**  
The cylinder  $x^2 + y^2 = 1$



**FIGURE 10**  
The plane  $y = x$

**EXAMPLE 3** Describe and sketch the surface in  $\mathbb{R}^3$  represented by the equation  $y = x$ .

**SOLUTION** The equation represents the set of all points in  $\mathbb{R}^3$  whose  $x$ - and  $y$ -coordinates are equal, that is,  $\{(x, x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$ . This is a vertical plane that intersects the  $xy$ -plane in the line  $y = x$ ,  $z = 0$ . The portion of this plane that lies in the first octant is sketched in Figure 10.

### Distance and Spheres

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

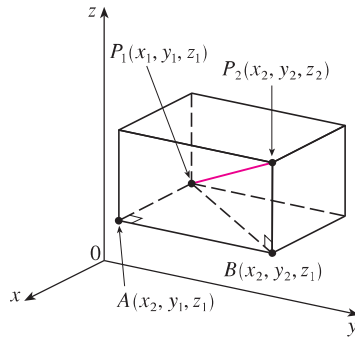


FIGURE 11

**Distance Formula in Three Dimensions** The distance  $|P_1P_2|$  between the points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

To see why this formula is true, we construct a rectangular box as in Figure 11, where  $P_1$  and  $P_2$  are opposite vertices and the faces of the box are parallel to the coordinate planes. If  $A(x_2, y_1, z_1)$  and  $B(x_2, y_2, z_1)$  are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \quad |AB| = |y_2 - y_1| \quad |BP_2| = |z_2 - z_1|$$

Because triangles  $P_1BP_2$  and  $P_1AB$  are both right-angled, two applications of the Pythagorean Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

and

$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these equations, we get

$$\begin{aligned} |P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Therefore  $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

**EXAMPLE 4** The distance from the point  $P(2, -1, 7)$  to the point  $Q(1, -3, 5)$  is

$$|PQ| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = 3 \quad \blacksquare$$

**EXAMPLE 5** Find an equation of a sphere with radius  $r$  and center  $C(h, k, l)$ .

**SOLUTION** By definition, a sphere is the set of all points  $P(x, y, z)$  whose distance from  $C$  is  $r$ . (See Figure 12.) Thus  $P$  is on the sphere if and only if  $|PC| = r$ . Squaring both sides, we have  $|PC|^2 = r^2$  or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \quad \blacksquare$$

The result of Example 5 is worth remembering.

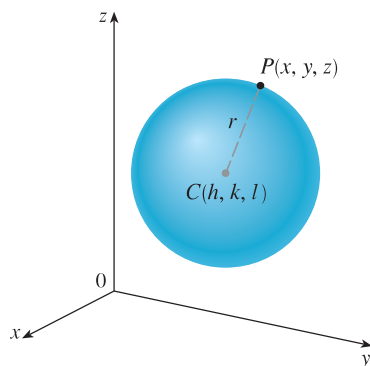


FIGURE 12

**Equation of a Sphere** An equation of a sphere with center  $C(h, k, l)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin  $O$ , then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

**EXAMPLE 6** Show that  $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$  is the equation of a sphere, and find its center and radius.

**SOLUTION** We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$\begin{aligned}(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) &= -6 + 4 + 9 + 1 \\(x + 2)^2 + (y - 3)^2 + (z + 1)^2 &= 8\end{aligned}$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center  $(-2, 3, -1)$  and radius  $\sqrt{8} = 2\sqrt{2}$ . ■

**EXAMPLE 7** What region in  $\mathbb{R}^3$  is represented by the following inequalities?

$$1 \leq x^2 + y^2 + z^2 \leq 4 \quad z \leq 0$$

**SOLUTION** The inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4$$

can be rewritten as

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2$$

so they represent the points  $(x, y, z)$  whose distance from the origin is at least 1 and at most 2. But we are also given that  $z \leq 0$ , so the points lie on or below the  $xy$ -plane. Thus the given inequalities represent the region that lies between (or on) the spheres  $x^2 + y^2 + z^2 = 1$  and  $x^2 + y^2 + z^2 = 4$  and beneath (or on) the  $xy$ -plane. It is sketched in Figure 13. ■

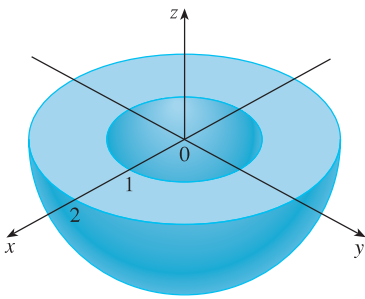


FIGURE 13

## 12.1 EXERCISES

- Suppose you start at the origin, move along the  $x$ -axis a distance of 4 units in the positive direction, and then move downward a distance of 3 units. What are the coordinates of your position?
- Sketch the points  $(1, 5, 3)$ ,  $(0, 2, -3)$ ,  $(-3, 0, 2)$ , and  $(2, -2, -1)$  on a single set of coordinate axes.
- Which of the points  $A(-4, 0, -1)$ ,  $B(3, 1, -5)$ , and  $C(2, 4, 6)$  is closest to the  $yz$ -plane? Which point lies in the  $xz$ -plane?
- What are the projections of the point  $(2, 3, 5)$  on the  $xy$ -,  $yz$ -, and  $xz$ -planes? Draw a rectangular box with the origin and  $(2, 3, 5)$  as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.
- What does the equation  $x = 4$  represent in  $\mathbb{R}^3$ ? What does it represent in  $\mathbb{R}^2$ ? Illustrate with sketches.
- What does the equation  $y = 3$  represent in  $\mathbb{R}^3$ ? What does  $z = 5$  represent? What does the pair of equations  $y = 3, z = 5$  represent? In other words, describe the set of points  $(x, y, z)$  such that  $y = 3$  and  $z = 5$ . Illustrate with a sketch.
- Describe and sketch the surface in  $\mathbb{R}^3$  represented by the equation  $x + y = 2$ .
- Describe and sketch the surface in  $\mathbb{R}^3$  represented by the equation  $x^2 + z^2 = 9$ .
- Find the lengths of the sides of the triangle  $PQR$ . Is it a right triangle? Is it an isosceles triangle?  
9.  $P(3, -2, -3)$ ,  $Q(7, 0, 1)$ ,  $R(1, 2, 1)$
- $P(2, -1, 0)$ ,  $Q(4, 1, 1)$ ,  $R(4, -5, 4)$
- Determine whether the points lie on a straight line.  
(a)  $A(2, 4, 2)$ ,  $B(3, 7, -2)$ ,  $C(1, 3, 3)$   
(b)  $D(0, -5, 5)$ ,  $E(1, -2, 4)$ ,  $F(3, 4, 2)$
- Find the distance from  $(4, -2, 6)$  to each of the following.  
(a) The  $xy$ -plane  
(b) The  $yz$ -plane  
(c) The  $xz$ -plane  
(d) The  $x$ -axis  
(e) The  $y$ -axis  
(f) The  $z$ -axis

13. Find an equation of the sphere with center  $(-3, 2, 5)$  and radius 4. What is the intersection of this sphere with the  $yz$ -plane?
14. Find an equation of the sphere with center  $(2, -6, 4)$  and radius 5. Describe its intersection with each of the coordinate planes.
15. Find an equation of the sphere that passes through the point  $(4, 3, -1)$  and has center  $(3, 8, 1)$ .
16. Find an equation of the sphere that passes through the origin and whose center is  $(1, 2, 3)$ .

17–20 Show that the equation represents a sphere, and find its center and radius.

17.  $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$

18.  $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$

19.  $2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$

20.  $3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$

21. (a) Prove that the midpoint of the line segment from  $P_1(x_1, y_1, z_1)$  to  $P_2(x_2, y_2, z_2)$  is

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

- (b) Find the lengths of the medians of the triangle with vertices  $A(1, 2, 3)$ ,  $B(-2, 0, 5)$ , and  $C(4, 1, 5)$ . (A *median* of a triangle is a line segment that joins a vertex to the midpoint of the opposite side.)

22. Find an equation of a sphere if one of its diameters has endpoints  $(5, 4, 3)$  and  $(1, 6, -9)$ .
23. Find equations of the spheres with center  $(2, -3, 6)$  that touch (a) the  $xy$ -plane, (b) the  $yz$ -plane, (c) the  $xz$ -plane.
24. Find an equation of the largest sphere with center  $(5, 4, 9)$  that is contained in the first octant.

25–38 Describe in words the region of  $\mathbb{R}^3$  represented by the equation(s) or inequality.

25.  $x = 5$

26.  $y = -2$

27.  $y < 8$

28.  $z \geq -1$

29.  $0 \leq z \leq 6$

30.  $y^2 = 4$

31.  $x^2 + y^2 = 4, z = -1$

32.  $x^2 + y^2 = 4$

33.  $x^2 + y^2 + z^2 = 4$

34.  $x^2 + y^2 + z^2 \leq 4$

35.  $1 \leq x^2 + y^2 + z^2 \leq 5$

36.  $x = z$

37.  $x^2 + z^2 \leq 9$

38.  $x^2 + y^2 + z^2 > 2z$

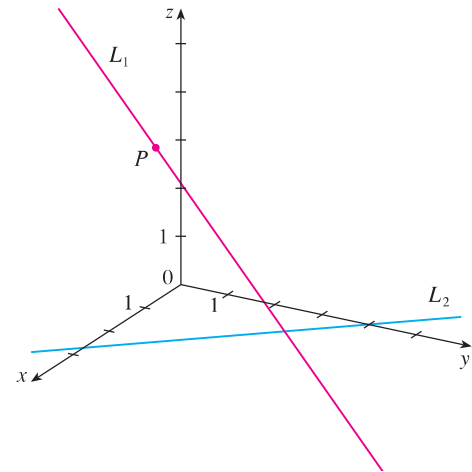
39–42 Write inequalities to describe the region.

39. The region between the  $yz$ -plane and the vertical plane  $x = 5$

40. The solid cylinder that lies on or below the plane  $z = 8$  and on or above the disk in the  $xy$ -plane with center the origin and radius 2
41. The region consisting of all points between (but not on) the spheres of radius  $r$  and  $R$  centered at the origin, where  $r < R$
42. The solid upper hemisphere of the sphere of radius 2 centered at the origin

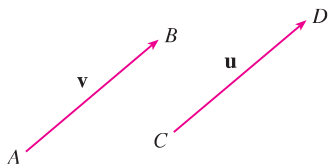
43. The figure shows a line  $L_1$  in space and a second line  $L_2$ , which is the projection of  $L_1$  onto the  $xy$ -plane. (In other words, the points on  $L_2$  are directly beneath, or above, the points on  $L_1$ .)

- (a) Find the coordinates of the point  $P$  on the line  $L_1$ .
- (b) Locate in the diagram the points  $A$ ,  $B$ , and  $C$ , where the line  $L_1$  intersects the  $xy$ -plane, the  $yz$ -plane, and the  $xz$ -plane, respectively.



44. Consider the points  $P$  such that the distance from  $P$  to  $A(-1, 5, 3)$  is twice the distance from  $P$  to  $B(6, 2, -2)$ . Show that the set of all such points is a sphere, and find its center and radius.
45. Find an equation of the set of all points equidistant from the points  $A(-1, 5, 3)$  and  $B(6, 2, -2)$ . Describe the set.
46. Find the volume of the solid that lies inside both of the spheres
- $$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$$
- and
- $$x^2 + y^2 + z^2 = 4$$
47. Find the distance between the spheres  $x^2 + y^2 + z^2 = 4$  and  $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$ .
48. Describe and sketch a solid with the following properties. When illuminated by rays parallel to the  $z$ -axis, its shadow is a circular disk. If the rays are parallel to the  $y$ -axis, its shadow is a square. If the rays are parallel to the  $x$ -axis, its shadow is an isosceles triangle.

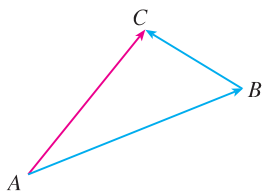
## 12.2 Vectors



**FIGURE 1**  
Equivalent vectors

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface ( $\mathbf{v}$ ) or by putting an arrow above the letter ( $\vec{v}$ ).

For instance, suppose a particle moves along a line segment from point  $A$  to point  $B$ . The corresponding **displacement vector**  $\mathbf{v}$ , shown in Figure 1, has **initial point**  $A$  (the tail) and **terminal point**  $B$  (the tip) and we indicate this by writing  $\mathbf{v} = \vec{AB}$ . Notice that the vector  $\mathbf{u} = \vec{CD}$  has the same length and the same direction as  $\mathbf{v}$  even though it is in a different position. We say that  $\mathbf{u}$  and  $\mathbf{v}$  are **equivalent** (or **equal**) and we write  $\mathbf{u} = \mathbf{v}$ . The **zero vector**, denoted by  $\mathbf{0}$ , has length 0. It is the only vector with no specific direction.



**FIGURE 2**

### Combining Vectors

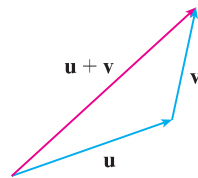
Suppose a particle moves from  $A$  to  $B$ , so its displacement vector is  $\vec{AB}$ . Then the particle changes direction and moves from  $B$  to  $C$ , with displacement vector  $\vec{BC}$  as in Figure 2. The combined effect of these displacements is that the particle has moved from  $A$  to  $C$ . The resulting displacement vector  $\vec{AC}$  is called the *sum* of  $\vec{AB}$  and  $\vec{BC}$  and we write

$$\vec{AC} = \vec{AB} + \vec{BC}$$

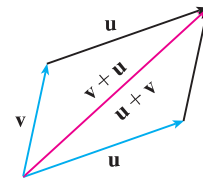
In general, if we start with vectors  $\mathbf{u}$  and  $\mathbf{v}$ , we first move  $\mathbf{v}$  so that its tail coincides with the tip of  $\mathbf{u}$  and define the sum of  $\mathbf{u}$  and  $\mathbf{v}$  as follows.

**Definition of Vector Addition** If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors positioned so the initial point of  $\mathbf{v}$  is at the terminal point of  $\mathbf{u}$ , then the **sum**  $\mathbf{u} + \mathbf{v}$  is the vector from the initial point of  $\mathbf{u}$  to the terminal point of  $\mathbf{v}$ .

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.



**FIGURE 3**  
The Triangle Law



**FIGURE 4**  
The Parallelogram Law

In Figure 4 we start with the same vectors  $\mathbf{u}$  and  $\mathbf{v}$  as in Figure 3 and draw another copy of  $\mathbf{v}$  with the same initial point as  $\mathbf{u}$ . Completing the parallelogram, we see that  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ . This also gives another way to construct the sum: if we place  $\mathbf{u}$  and  $\mathbf{v}$  so they start at the same point, then  $\mathbf{u} + \mathbf{v}$  lies along the diagonal of the parallelogram with  $\mathbf{u}$  and  $\mathbf{v}$  as sides. (This is called the **Parallelogram Law**.)



**FIGURE 5**

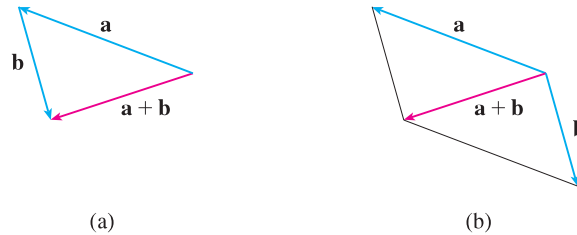
**EXAMPLE 1** Draw the sum of the vectors  $\mathbf{a}$  and  $\mathbf{b}$  shown in Figure 5.

**SOLUTION** First we move  $\mathbf{b}$  and place its tail at the tip of  $\mathbf{a}$ , being careful to draw a copy of  $\mathbf{b}$  that has the same length and direction. Then we draw the vector  $\mathbf{a} + \mathbf{b}$  [see

Figure 6(a)] starting at the initial point of  $\mathbf{a}$  and ending at the terminal point of the copy of  $\mathbf{b}$ .

Alternatively, we could place  $\mathbf{b}$  so it starts where  $\mathbf{a}$  starts and construct  $\mathbf{a} + \mathbf{b}$  by the Parallelogram Law as in Figure 6(b).

**TEC** Visual 12.2 shows how the Triangle and Parallelogram Laws work for various vectors  $\mathbf{a}$  and  $\mathbf{b}$ .



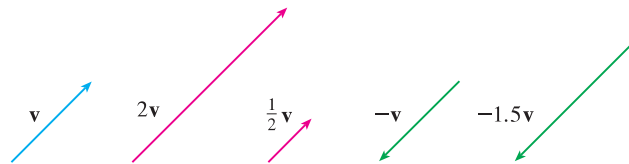
**FIGURE 6**

It is possible to multiply a vector by a real number  $c$ . (In this context we call the real number  $c$  a **scalar** to distinguish it from a vector.) For instance, we want  $2\mathbf{v}$  to be the same vector as  $\mathbf{v} + \mathbf{v}$ , which has the same direction as  $\mathbf{v}$  but is twice as long. In general, we multiply a vector by a scalar as follows.

**Definition of Scalar Multiplication** If  $c$  is a scalar and  $\mathbf{v}$  is a vector, then the **scalar multiple**  $c\mathbf{v}$  is the vector whose length is  $|c|$  times the length of  $\mathbf{v}$  and whose direction is the same as  $\mathbf{v}$  if  $c > 0$  and is opposite to  $\mathbf{v}$  if  $c < 0$ . If  $c = 0$  or  $\mathbf{v} = \mathbf{0}$ , then  $c\mathbf{v} = \mathbf{0}$ .

This definition is illustrated in Figure 7. We see that real numbers work like scaling factors here; that's why we call them scalars. Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector  $-\mathbf{v} = (-1)\mathbf{v}$  has the same length as  $\mathbf{v}$  but points in the opposite direction. We call it the **negative** of  $\mathbf{v}$ .

**FIGURE 7**  
Scalar multiples of  $\mathbf{v}$



By the **difference**  $\mathbf{u} - \mathbf{v}$  of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

So we can construct  $\mathbf{u} - \mathbf{v}$  by first drawing the negative of  $\mathbf{v}$ ,  $-\mathbf{v}$ , and then adding it to  $\mathbf{u}$  by the Parallelogram Law as in Figure 8(a). Alternatively, since  $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$ , the vector  $\mathbf{u} - \mathbf{v}$ , when added to  $\mathbf{v}$ , gives  $\mathbf{u}$ . So we could construct  $\mathbf{u} - \mathbf{v}$  as in Figure 8(b) by means of the Triangle Law. Notice that if  $\mathbf{u}$  and  $\mathbf{v}$  both start from the same initial point, then  $\mathbf{u} - \mathbf{v}$  connects the tip of  $\mathbf{v}$  to the tip of  $\mathbf{u}$ .

**FIGURE 8**  
Drawing  $\mathbf{u} - \mathbf{v}$

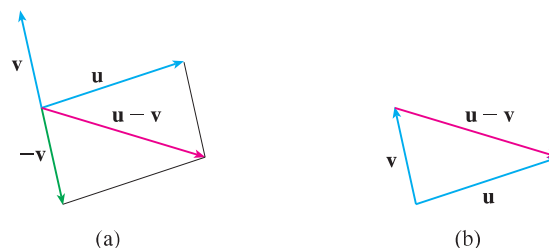






FIGURE 9

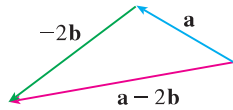


FIGURE 10

**EXAMPLE 2** If  $\mathbf{a}$  and  $\mathbf{b}$  are the vectors shown in Figure 9, draw  $\mathbf{a} - 2\mathbf{b}$ .

**SOLUTION** We first draw the vector  $-2\mathbf{b}$  pointing in the direction opposite to  $\mathbf{b}$  and twice as long. We place it with its tail at the tip of  $\mathbf{a}$  and then use the Triangle Law to draw  $\mathbf{a} + (-2\mathbf{b})$  as in Figure 10. ■

■ **Components**

For some purposes it's best to introduce a coordinate system and treat vectors algebraically. If we place the initial point of a vector  $\mathbf{a}$  at the origin of a rectangular coordinate system, then the terminal point of  $\mathbf{a}$  has coordinates of the form  $(a_1, a_2)$  or  $(a_1, a_2, a_3)$ , depending on whether our coordinate system is two- or three-dimensional (see Figure 11). These coordinates are called the **components** of  $\mathbf{a}$  and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation  $\langle a_1, a_2 \rangle$  for the ordered pair that refers to a vector so as not to confuse it with the ordered pair  $(a_1, a_2)$  that refers to a point in the plane.

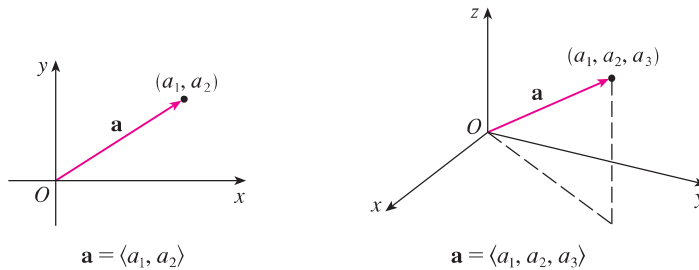


FIGURE 11

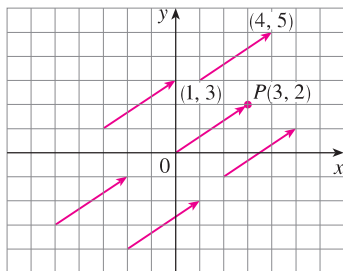


FIGURE 12  
Representations of  $\mathbf{a} = \langle 3, 2 \rangle$

For instance, the vectors shown in Figure 12 are all equivalent to the vector  $\vec{OP} = \langle 3, 2 \rangle$  whose terminal point is  $P(3, 2)$ . What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We can think of all these geometric vectors as **representations** of the algebraic vector  $\mathbf{a} = \langle 3, 2 \rangle$ . The particular representation  $\vec{OP}$  from the origin to the point  $P(3, 2)$  is called the **position vector** of the point  $P$ .

In three dimensions, the vector  $\mathbf{a} = \vec{OP} = \langle a_1, a_2, a_3 \rangle$  is the **position vector** of the point  $P(a_1, a_2, a_3)$ . (See Figure 13.) Let's consider any other representation  $\vec{AB}$  of  $\mathbf{a}$ , where the initial point is  $A(x_1, y_1, z_1)$  and the terminal point is  $B(x_2, y_2, z_2)$ . Then we must have  $x_1 + a_1 = x_2$ ,  $y_1 + a_2 = y_2$ , and  $z_1 + a_3 = z_2$  and so  $a_1 = x_2 - x_1$ ,  $a_2 = y_2 - y_1$ , and  $a_3 = z_2 - z_1$ . Thus we have the following result.

**1** Given the points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$ , the vector  $\mathbf{a}$  with representation  $\vec{AB}$  is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**EXAMPLE 3** Find the vector represented by the directed line segment with initial point  $A(2, -3, 4)$  and terminal point  $B(-2, 1, 1)$ .

**SOLUTION** By (1), the vector corresponding to  $\vec{AB}$  is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle = \langle -4, 4, -3 \rangle$$

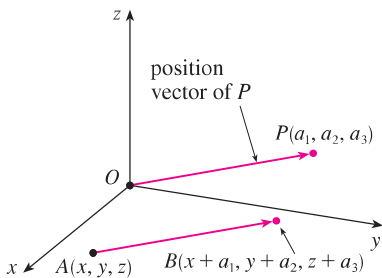


FIGURE 13  
Representations of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

The **magnitude** or **length** of the vector  $\mathbf{v}$  is the length of any of its representations and is denoted by the symbol  $|\mathbf{v}|$  or  $\|\mathbf{v}\|$ . By using the distance formula to compute the length of a segment  $OP$ , we obtain the following formulas.

The length of the two-dimensional vector  $\mathbf{a} = \langle a_1, a_2 \rangle$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

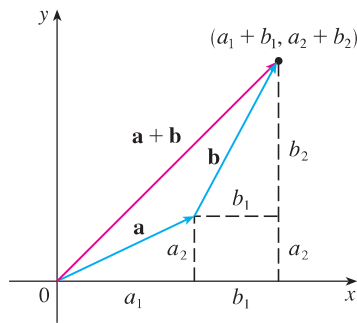


FIGURE 14

How do we add vectors algebraically? Figure 14 shows that if  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$ , then the sum is  $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ , at least for the case where the components are positive. In other words, *to add algebraic vectors we add corresponding components*. Similarly, *to subtract vectors we subtract corresponding components*. From the similar triangles in Figure 15 we see that the components of  $c\mathbf{a}$  are  $ca_1$  and  $ca_2$ . So *to multiply a vector by a scalar we multiply each component by that scalar*.

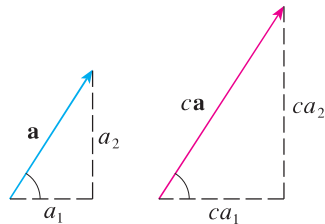


FIGURE 15

If  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$ , then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \quad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

**EXAMPLE 4** If  $\mathbf{a} = \langle 4, 0, 3 \rangle$  and  $\mathbf{b} = \langle -2, 1, 5 \rangle$ , find  $|\mathbf{a}|$  and the vectors  $\mathbf{a} + \mathbf{b}$ ,  $\mathbf{a} - \mathbf{b}$ ,  $3\mathbf{b}$ , and  $2\mathbf{a} + 5\mathbf{b}$ .

**SOLUTION**  $|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle \\ &= \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle \end{aligned}$$

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle \\ &= \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle \end{aligned}$$

$$3\mathbf{b} = 3\langle -2, 1, 5 \rangle = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$$

$$\begin{aligned} 2\mathbf{a} + 5\mathbf{b} &= 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle \\ &= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle \end{aligned}$$

Vectors in  $n$  dimensions are used to list various quantities in an organized way. For instance, the components of a six-dimensional vector

$$\mathbf{p} = \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle$$

might represent the prices of six different ingredients required to make a particular product. Four-dimensional vectors  $\langle x, y, z, t \rangle$  are used in relativity theory, where the first three components specify a position in space and the fourth represents time.

We denote by  $V_2$  the set of all two-dimensional vectors and by  $V_3$  the set of all three-dimensional vectors. More generally, we will later need to consider the set  $V_n$  of all  $n$ -dimensional vectors. An  $n$ -dimensional vector is an ordered  $n$ -tuple:

$$\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$$

where  $a_1, a_2, \dots, a_n$  are real numbers that are called the components of  $\mathbf{a}$ . Addition and scalar multiplication are defined in terms of components just as for the cases  $n = 2$  and  $n = 3$ .

**Properties of Vectors** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_n$  and  $c$  and  $d$  are scalars, then

- |   |  |
|---|--|
| 1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$      | 2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ |
| 3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$                   | 4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$   |
| 5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ | 6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$                                   |
| 7. $(cd)\mathbf{a} = c(d\mathbf{a})$                        | 8. $1\mathbf{a} = \mathbf{a}$  |

These eight properties of vectors can be readily verified either geometrically or algebraically. For instance, Property 1 can be seen from Figure 4 (it's equivalent to the Parallelogram Law) or as follows for the case  $n = 2$ :

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \\ &= \langle b_1 + a_1, b_2 + a_2 \rangle = \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle \\ &= \mathbf{b} + \mathbf{a} \end{aligned}$$

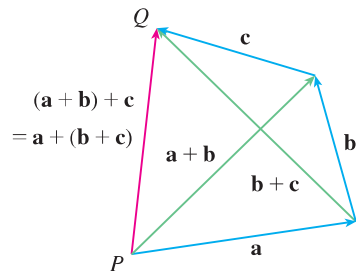


FIGURE 16

We can see why Property 2 (the associative law) is true by looking at Figure 16 and applying the Triangle Law several times: the vector  $\overrightarrow{PQ}$  is obtained either by first constructing  $\mathbf{a} + \mathbf{b}$  and then adding  $\mathbf{c}$  or by adding  $\mathbf{a}$  to the vector  $\mathbf{b} + \mathbf{c}$ .

Three vectors in  $V_3$  play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

These vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are called the **standard basis vectors**. They have length 1 and point in the directions of the positive  $x$ -,  $y$ -, and  $z$ -axes. Similarly, in two dimensions we define  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$ . (See Figure 17.)

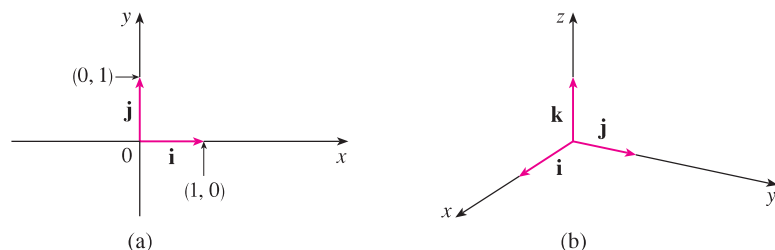


FIGURE 17

Standard basis vectors in  $V_2$  and  $V_3$

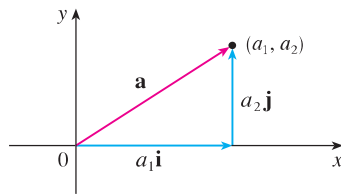
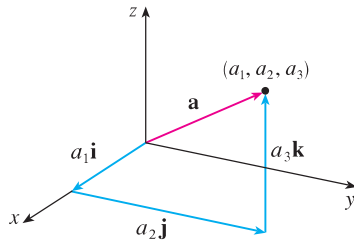
(a)  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ (b)  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ 

FIGURE 18

**Gibbs**

Josiah Willard Gibbs (1839–1903), a professor of mathematical physics at Yale College, published the first book on vectors, *Vector Analysis*, in 1881. More complicated objects, called quaternions, had earlier been invented by Hamilton as mathematical tools for describing space, but they weren't easy for scientists to use. Quaternions have a scalar part and a vector part. Gibbs's idea was to use the vector part separately. Maxwell and Heaviside had similar ideas, but Gibbs's approach has proved to be the most convenient way to study space.

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ , then we can write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle\end{aligned}$$

$$\boxed{2} \quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Thus any vector in  $V_3$  can be expressed in terms of  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ . For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

$$\boxed{3} \quad \mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.

**EXAMPLE 5** If  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$  and  $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$ , express the vector  $2\mathbf{a} + 3\mathbf{b}$  in terms of  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ .

**SOLUTION** Using Properties 1, 2, 5, 6, and 7 of vectors, we have

$$\begin{aligned}2\mathbf{a} + 3\mathbf{b} &= 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k}) \\ &= 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} = 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}\end{aligned}$$

A **unit vector** is a vector whose length is 1. For instance,  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are all unit vectors. In general, if  $\mathbf{a} \neq \mathbf{0}$ , then the unit vector that has the same direction as  $\mathbf{a}$  is

$$\boxed{4} \quad \mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let  $c = 1/|\mathbf{a}|$ . Then  $\mathbf{u} = c\mathbf{a}$  and  $c$  is a positive scalar, so  $\mathbf{u}$  has the same direction as  $\mathbf{a}$ . Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

**EXAMPLE 6** Find the unit vector in the direction of the vector  $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ .

**SOLUTION** The given vector has length

$$|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

so, by Equation 4, the unit vector with the same direction is

$$\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

### Applications

Vectors are useful in many aspects of physics and engineering. In Chapter 13 we will see how they describe the velocity and acceleration of objects moving in space. Here we look at forces.

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

**EXAMPLE 7** A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces)  $\mathbf{T}_1$  and  $\mathbf{T}_2$  in both wires and the magnitudes of the tensions.

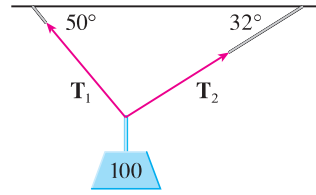


FIGURE 19

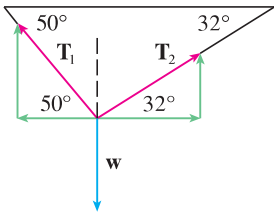


FIGURE 20

**SOLUTION** We first express  $\mathbf{T}_1$  and  $\mathbf{T}_2$  in terms of their horizontal and vertical components. From Figure 20 we see that

$$\boxed{5} \quad \mathbf{T}_1 = -|\mathbf{T}_1| \cos 50^\circ \mathbf{i} + |\mathbf{T}_1| \sin 50^\circ \mathbf{j}$$

$$\boxed{6} \quad \mathbf{T}_2 = |\mathbf{T}_2| \cos 32^\circ \mathbf{i} + |\mathbf{T}_2| \sin 32^\circ \mathbf{j}$$

The resultant  $\mathbf{T}_1 + \mathbf{T}_2$  of the tensions counterbalances the weight  $\mathbf{w} = -100\mathbf{j}$  and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\mathbf{j}$$

Thus

$$(-|\mathbf{T}_1| \cos 50^\circ + |\mathbf{T}_2| \cos 32^\circ) \mathbf{i} + (|\mathbf{T}_1| \sin 50^\circ + |\mathbf{T}_2| \sin 32^\circ) \mathbf{j} = 100\mathbf{j}$$

Equating components, we get

$$-|\mathbf{T}_1| \cos 50^\circ + |\mathbf{T}_2| \cos 32^\circ = 0$$

$$|\mathbf{T}_1| \sin 50^\circ + |\mathbf{T}_2| \sin 32^\circ = 100$$

Solving the first of these equations for  $|\mathbf{T}_2|$  and substituting into the second, we get

$$|\mathbf{T}_1| \sin 50^\circ + \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ} \sin 32^\circ = 100$$

$$|\mathbf{T}_1| \left( \sin 50^\circ + \cos 50^\circ \frac{\sin 32^\circ}{\cos 32^\circ} \right) = 100$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb}$$

and

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb}$$

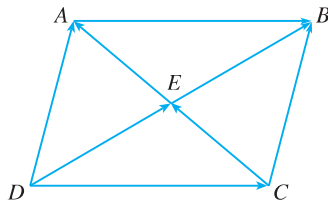
Substituting these values in (5) and (6), we obtain the tension vectors

$$\mathbf{T}_1 \approx -55.05 \mathbf{i} + 65.60 \mathbf{j}$$

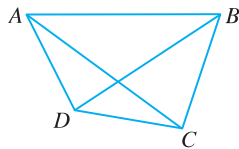
$$\mathbf{T}_2 \approx 55.05 \mathbf{i} + 34.40 \mathbf{j}$$

## 12.2 EXERCISES

- Are the following quantities vectors or scalars? Explain.
  - The cost of a theater ticket
  - The current in a river
  - The initial flight path from Houston to Dallas
  - The population of the world
- What is the relationship between the point  $(4, 7)$  and the vector  $\langle 4, 7 \rangle$ ? Illustrate with a sketch.
- Name all the equal vectors in the parallelogram shown.



- Write each combination of vectors as a single vector.
  - $\vec{AB} + \vec{BC}$
  - $\vec{CD} + \vec{DB}$
  - $\vec{DB} - \vec{AB}$
  - $\vec{DC} + \vec{CA} + \vec{AB}$



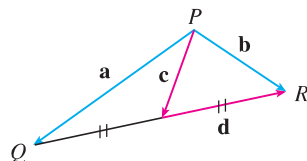
- Copy the vectors in the figure and use them to draw the following vectors.
  - $\mathbf{u} + \mathbf{v}$
  - $\mathbf{u} + \mathbf{w}$
  - $\mathbf{v} + \mathbf{w}$
  - $\mathbf{u} - \mathbf{v}$
  - $\mathbf{v} + \mathbf{u} + \mathbf{w}$
  - $\mathbf{u} - \mathbf{w} - \mathbf{v}$



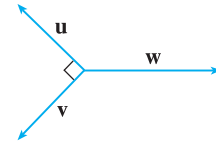
- Copy the vectors in the figure and use them to draw the following vectors.
  - $\mathbf{a} + \mathbf{b}$
  - $\mathbf{a} - \mathbf{b}$
  - $\frac{1}{2}\mathbf{a}$
  - $-3\mathbf{b}$
  - $\mathbf{a} + 2\mathbf{b}$
  - $2\mathbf{b} - \mathbf{a}$



- In the figure, the tip of  $\mathbf{c}$  and the tail of  $\mathbf{d}$  are both the midpoint of  $\overline{QR}$ . Express  $\mathbf{c}$  and  $\mathbf{d}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .



- If the vectors in the figure satisfy  $|\mathbf{u}| = |\mathbf{v}| = 1$  and  $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$ , what is  $|\mathbf{w}|$ ?



- Find a vector  $\mathbf{a}$  with representation given by the directed line segment  $\overline{AB}$ . Draw  $\overline{AB}$  and the equivalent representation starting at the origin.

- $A(-2, 1), B(1, 2)$
- $A(-5, -1), B(-3, 3)$
- $A(3, -1), B(2, 3)$
- $A(3, 2), B(1, 0)$
- $A(0, 3, 1), B(2, 3, -1)$
- $A(0, 6, -1), B(3, 4, 4)$

- Find the sum of the given vectors and illustrate geometrically.

- $\langle -1, 4 \rangle, \langle 6, -2 \rangle$
- $\langle 3, -1 \rangle, \langle -1, 5 \rangle$
- $\langle 3, 0, 1 \rangle, \langle 0, 8, 0 \rangle$
- $\langle 1, 3, -2 \rangle, \langle 0, 0, 6 \rangle$

- Find  $\mathbf{a} + \mathbf{b}$ ,  $4\mathbf{a} + 2\mathbf{b}$ ,  $|\mathbf{a}|$ , and  $|\mathbf{a} - \mathbf{b}|$ .

- $\mathbf{a} = \langle -3, 4 \rangle, \mathbf{b} = \langle 9, -1 \rangle$
- $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}, \mathbf{b} = -\mathbf{i} - 2\mathbf{j}$
- $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$
- $\mathbf{a} = \langle 8, 1, -4 \rangle, \mathbf{b} = \langle 5, -2, 1 \rangle$

- Find a unit vector that has the same direction as the given vector.

- $\langle 6, -2 \rangle$
- $-5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
- $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

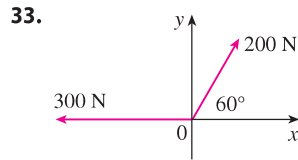
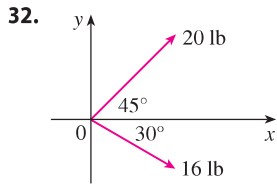
- Find the vector that has the same direction as  $\langle 6, 2, -3 \rangle$  but has length 4.

- What is the angle between the given vector and the positive direction of the  $x$ -axis?

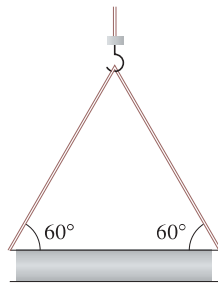
- $\mathbf{i} + \sqrt{3}\mathbf{j}$
- $8\mathbf{i} + 6\mathbf{j}$

- If  $\mathbf{v}$  lies in the first quadrant and makes an angle  $\pi/3$  with the positive  $x$ -axis and  $|\mathbf{v}| = 4$ , find  $\mathbf{v}$  in component form.
- If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of  $38^\circ$  above the horizontal, find the horizontal and vertical components of the force.
- A quarterback throws a football with angle of elevation  $40^\circ$  and speed 60 ft/s. Find the horizontal and vertical components of the velocity vector.

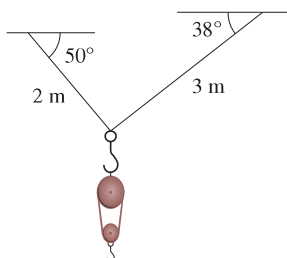
**32–33** Find the magnitude of the resultant force and the angle it makes with the positive  $x$ -axis.



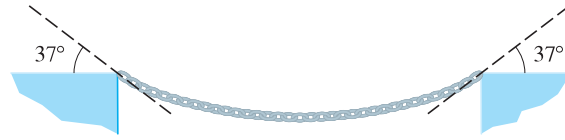
- 34.** The magnitude of a velocity vector is called *speed*. Suppose that a wind is blowing *from* the direction  $N45^\circ W$  at a speed of 50 km/h. (This means that the direction from which the wind blows is  $45^\circ$  west of the northerly direction.) A pilot is steering a plane in the direction  $N60^\circ E$  at an airspeed (speed in still air) of 250 km/h. The *true course*, or *track*, of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane.
- 35.** A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the woman relative to the surface of the water.
- 36.** A crane suspends a 500-lb steel beam horizontally by support cables (with negligible weight) attached from a hook to each end of the beam. The support cables each make an angle of  $60^\circ$  with the beam. Find the tension vector in each support cable and the magnitude of each tension.



- 37.** A block-and-tackle pulley hoist is suspended in a warehouse by ropes of lengths 2 m and 3 m. The hoist weighs 350 N. The ropes, fastened at different heights, make angles of  $50^\circ$  and  $38^\circ$  with the horizontal. Find the tension in each rope and the magnitude of each tension.



- 38.** The tension  $T$  at each end of a chain has magnitude 25 N (see the figure). What is the weight of the chain?



- 39.** A boatman wants to cross a canal that is 3 km wide and wants to land at a point 2 km upstream from his starting point. The current in the canal flows at 3.5 km/h and the speed of his boat is 13 km/h.
- In what direction should he steer?
  - How long will the trip take?
- 40.** Three forces act on an object. Two of the forces are at an angle of  $100^\circ$  to each other and have magnitudes 25 N and 12 N. The third is perpendicular to the plane of these two forces and has magnitude 4 N. Calculate the magnitude of the force that would exactly counterbalance these three forces.
- 41.** Find the unit vectors that are parallel to the tangent line to the parabola  $y = x^2$  at the point  $(2, 4)$ .
- 42.** (a) Find the unit vectors that are parallel to the tangent line to the curve  $y = 2 \sin x$  at the point  $(\pi/6, 1)$ .  
 (b) Find the unit vectors that are perpendicular to the tangent line.  
 (c) Sketch the curve  $y = 2 \sin x$  and the vectors in parts (a) and (b), all starting at  $(\pi/6, 1)$ .
- 43.** If  $A$ ,  $B$ , and  $C$  are the vertices of a triangle, find
- $$\vec{AB} + \vec{BC} + \vec{CA}$$
- 44.** Let  $C$  be the point on the line segment  $AB$  that is twice as far from  $B$  as it is from  $A$ . If  $\mathbf{a} = \vec{OA}$ ,  $\mathbf{b} = \vec{OB}$ , and  $\mathbf{c} = \vec{OC}$ , show that  $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ .
- 45.** (a) Draw the vectors  $\mathbf{a} = \langle 3, 2 \rangle$ ,  $\mathbf{b} = \langle 2, -1 \rangle$ , and  $\mathbf{c} = \langle 7, 1 \rangle$ .  
 (b) Show, by means of a sketch, that there are scalars  $s$  and  $t$  such that  $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ .  
 (c) Use the sketch to estimate the values of  $s$  and  $t$ .  
 (d) Find the exact values of  $s$  and  $t$ .
- 46.** Suppose that  $\mathbf{a}$  and  $\mathbf{b}$  are nonzero vectors that are not parallel and  $\mathbf{c}$  is any vector in the plane determined by  $\mathbf{a}$  and  $\mathbf{b}$ . Give a geometric argument to show that  $\mathbf{c}$  can be written as  $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$  for suitable scalars  $s$  and  $t$ . Then give an argument using components.
- 47.** If  $\mathbf{r} = \langle x, y, z \rangle$  and  $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ , describe the set of all points  $(x, y, z)$  such that  $|\mathbf{r} - \mathbf{r}_0| = 1$ .
- 48.** If  $\mathbf{r} = \langle x, y \rangle$ ,  $\mathbf{r}_1 = \langle x_1, y_1 \rangle$ , and  $\mathbf{r}_2 = \langle x_2, y_2 \rangle$ , describe the set of all points  $(x, y)$  such that  $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$ , where  $k > |\mathbf{r}_1 - \mathbf{r}_2|$ .
- 49.** Figure 16 gives a geometric demonstration of Property 2 of vectors. Use components to give an algebraic proof of this fact for the case  $n = 2$ .