

34. At what point do the curves $\mathbf{r}_1(t) = \langle t, 1 - t, 3 + t^2 \rangle$ and $\mathbf{r}_2(s) = \langle 3 - s, s - 2, s^2 \rangle$ intersect? Find their angle of intersection correct to the nearest degree.

35–40 Evaluate the integral.

35. $\int_0^2 (t\mathbf{i} - t^3\mathbf{j} + 3t^5\mathbf{k}) dt$

36. $\int_1^4 (2t^{3/2}\mathbf{i} + (t + 1)\sqrt{t}\mathbf{k}) dt$

37. $\int_0^1 \left(\frac{1}{t+1}\mathbf{i} + \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k} \right) dt$

38. $\int_0^{\pi/4} (\sec t \tan t \mathbf{i} + t \cos 2t \mathbf{j} + \sin^2 2t \cos 2t \mathbf{k}) dt$

39. $\int (\sec^2 t \mathbf{i} + t(t^2 + 1)^3 \mathbf{j} + t^2 \ln t \mathbf{k}) dt$

40. $\int \left(te^{2t}\mathbf{i} + \frac{t}{1-t}\mathbf{j} + \frac{1}{\sqrt{1-t^2}}\mathbf{k} \right) dt$

41. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

42. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\mathbf{i} + e^t\mathbf{j} + te^t\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

43. Prove Formula 1 of Theorem 3.

44. Prove Formula 3 of Theorem 3.

45. Prove Formula 5 of Theorem 3.

46. Prove Formula 6 of Theorem 3.

47. If $\mathbf{u}(t) = \langle \sin t, \cos t, t \rangle$ and $\mathbf{v}(t) = \langle t, \cos t, \sin t \rangle$, use Formula 4 of Theorem 3 to find

$$\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)]$$

48. If \mathbf{u} and \mathbf{v} are the vector functions in Exercise 47, use Formula 5 of Theorem 3 to find

$$\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)]$$

49. Find $f'(2)$, where $f(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$, $\mathbf{u}(2) = \langle 1, 2, -1 \rangle$, $\mathbf{u}'(2) = \langle 3, 0, 4 \rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$.

50. If $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$, where \mathbf{u} and \mathbf{v} are the vector functions in Exercise 49, find $\mathbf{r}'(2)$.

51. If $\mathbf{r}(t) = \mathbf{a} \cos \omega t + \mathbf{b} \sin \omega t$, where \mathbf{a} and \mathbf{b} are constant vectors, show that $\mathbf{r}(t) \times \mathbf{r}'(t) = \omega \mathbf{a} \times \mathbf{b}$.

52. If \mathbf{r} is the vector function in Exercise 51, show that $\mathbf{r}''(t) + \omega^2 \mathbf{r}(t) = \mathbf{0}$.

53. Show that if \mathbf{r} is a vector function such that \mathbf{r}'' exists, then

$$\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

54. Find an expression for $\frac{d}{dt} [\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))]$.

55. If $\mathbf{r}(t) \neq \mathbf{0}$, show that $\frac{d}{dt} |\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t)$.

$$[\text{Hint: } |\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)]$$

56. If a curve has the property that the position vector $\mathbf{r}(t)$ is always perpendicular to the tangent vector $\mathbf{r}'(t)$, show that the curve lies on a sphere with center the origin.

57. If $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$, show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

58. Show that the tangent vector to a curve defined by a vector function $\mathbf{r}(t)$ points in the direction of increasing t .

[Hint: Refer to Figure 1 and consider the cases $h > 0$ and $h < 0$ separately.]

13.3 Arc Length and Curvature

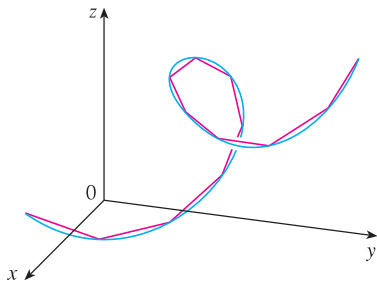


FIGURE 1

The length of a space curve is the limit of lengths of inscribed polygons.

Length of a Curve

In Section 10.2 we defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula

$$1 \quad L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The length of a space curve is defined in exactly the same way (see Figure 1). Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f' , g' , and h' are continuous. If the curve is traversed exactly once as t increases from a to b , then it can be shown

that its length is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \end{aligned}$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

In the next section we will see that if $\mathbf{r}(t)$ is the position vector of a moving object at time t , then $\mathbf{r}'(t)$ is the velocity vector and $|\mathbf{r}'(t)|$ is the speed. Thus Equation 3 says that to compute distance traveled, we integrate speed.

because, for plane curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Figure 2 shows the arc of the helix whose length is computed in Example 1.

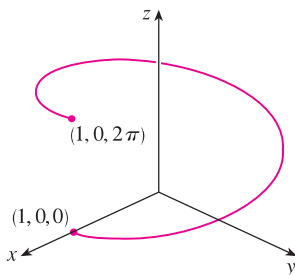


FIGURE 2

EXAMPLE 1 Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

SOLUTION Since $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$, we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from $(1, 0, 0)$ to $(1, 0, 2\pi)$ is described by the parameter interval $0 \leq t \leq 2\pi$ and so, from Formula 3, we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$$

A single curve C can be represented by more than one vector function. For instance, the twisted cubic

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \quad 1 \leq t \leq 2$$

could also be represented by the function

$$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \quad 0 \leq u \leq \ln 2$$

where the connection between the parameters t and u is given by $t = e^u$. We say that Equations 4 and 5 are **parametrizations** of the curve C . If we were to use Equation 3 to compute the length of C using Equations 4 and 5, we would get the same answer. In general, it can be shown that when Equation 3 is used to compute arc length, the answer is independent of the parametrization that is used.

■ The Arc Length Function

Now we suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \leq t \leq b$$

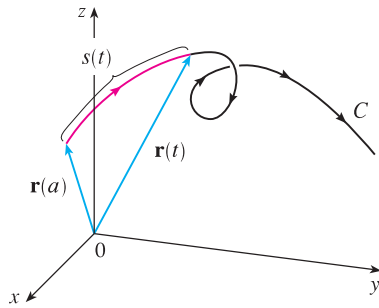


FIGURE 3

where \mathbf{r}' is continuous and C is traversed exactly once as t increases from a to b . We define its **arc length function** s by

$$\boxed{6} \quad s(t) = \int_a^t |\mathbf{r}'(u)| \, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du$$

Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. (See Figure 3.) If we differentiate both sides of Equation 6 using Part 1 of the Fundamental Theorem of Calculus, we obtain

$$\boxed{7} \quad \frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by Equation 6, then we may be able to solve for t as a function of s : $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t : $\mathbf{r} = \mathbf{r}(t(s))$. Thus, if $s = 3$ for instance, $\mathbf{r}(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

EXAMPLE 2 Reparametrize the helix $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t .

SOLUTION The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$. From Example 1 we have

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2}$$

and so
$$s = s(t) = \int_0^t |\mathbf{r}'(u)| \, du = \int_0^t \sqrt{2} \, du = \sqrt{2}t$$

Therefore $t = s/\sqrt{2}$ and the required reparametrization is obtained by substituting for t :

$$\mathbf{r}(t(s)) = \cos(s/\sqrt{2}) \mathbf{i} + \sin(s/\sqrt{2}) \mathbf{j} + (s/\sqrt{2}) \mathbf{k} \quad \blacksquare$$

■ Curvature

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I . A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function \mathbf{r} , recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve. From Figure 4 you can see that $\mathbf{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.) Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of \mathbf{T} .

TEC Visual 13.3A shows animated unit tangent vectors, like those in Figure 4, for a variety of plane curves and space curves.

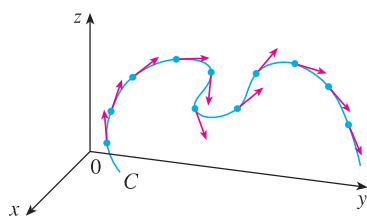


FIGURE 4

Unit tangent vectors at equally spaced points on C

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where \mathbf{T} is the unit tangent vector.

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s , so we use the Chain Rule (Theorem 13.2.3, Formula 6) to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \quad \text{and} \quad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right|$$

But $ds/dt = |\mathbf{r}'(t)|$ from Equation 7, so

9

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

EXAMPLE 3 Show that the curvature of a circle of radius a is $1/a$.

SOLUTION We can take the circle to have center the origin, and then a parametrization is

$$\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$$

Therefore $\mathbf{r}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j}$ and $|\mathbf{r}'(t)| = a$

so $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t \mathbf{i} + \cos t \mathbf{j}$

and $\mathbf{T}'(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}$

This gives $|\mathbf{T}'(t)| = 1$, so using Formula 9, we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a} \quad \blacksquare$$

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition. We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 Theorem The curvature of the curve given by the vector function \mathbf{r} is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

PROOF Since $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ and $|\mathbf{r}'| = ds/dt$, we have

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$$

so the Product Rule (Theorem 13.2.3, Formula 3) gives

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'$$

Using the fact that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$ (see Example 12.4.2), we have

$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}')$$

Now $|\mathbf{T}(t)| = 1$ for all t , so \mathbf{T} and \mathbf{T}' are orthogonal by Example 13.2.4. Therefore, by Theorem 12.4.9,

$$|\mathbf{r}' \times \mathbf{r}''| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}||\mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'|$$

Thus
$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{(ds/dt)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$

and
$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

EXAMPLE 4 Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0, 0, 0)$.

SOLUTION We first compute the required ingredients:

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \quad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2 \mathbf{i} - 6t \mathbf{j} + 2 \mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

Theorem 10 then gives

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

At the origin, where $t = 0$, the curvature is $\kappa(0) = 2$.

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x \mathbf{i} + f(x) \mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x) \mathbf{j}$ and $\mathbf{r}''(x) = f''(x) \mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x) \mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

11

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

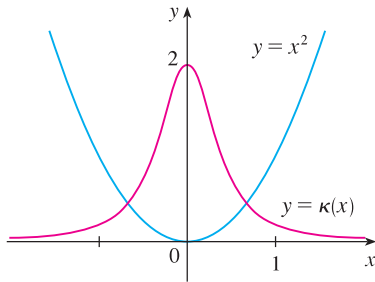


FIGURE 5
The parabola $y = x^2$ and its curvature function

EXAMPLE 5 Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

SOLUTION Since $y' = 2x$ and $y'' = 2$, Formula 11 gives

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

The curvature at $(0, 0)$ is $\kappa(0) = 2$. At $(1, 1)$ it is $\kappa(1) = 2/5^{3/2} \approx 0.18$. At $(2, 4)$ it is $\kappa(2) = 2/17^{3/2} \approx 0.03$. Observe from the expression for $\kappa(x)$ or the graph of κ in Figure 5 that $\kappa(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. This corresponds to the fact that the parabola appears to become flatter as $x \rightarrow \pm\infty$. ■

■ **The Normal and Binormal Vectors**

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t , we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$ by Example 13.2.4, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that, typically, $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where $\kappa \neq 0$ we can define the **principal unit normal vector** $\mathbf{N}(t)$ (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**. It is perpendicular to both \mathbf{T} and \mathbf{N} and is also a unit vector. (See Figure 6.)

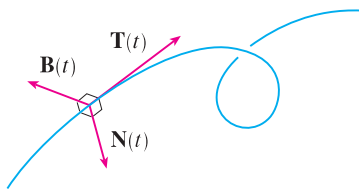


FIGURE 6

Figure 7 illustrates Example 6 by showing the vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} at two locations on the helix. In general, the vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} , starting at the various points on a curve, form a set of orthogonal vectors, called the **TNB frame**, that moves along the curve as t varies. This **TNB frame** plays an important role in the branch of mathematics known as differential geometry and in its applications to the motion of spacecraft.

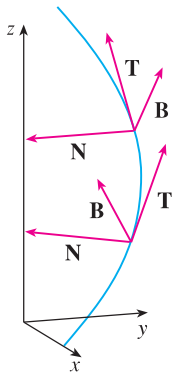


FIGURE 7

EXAMPLE 6 Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$$

SOLUTION We first compute the ingredients needed for the unit normal vector:

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \quad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}} (-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}} (-\cos t \mathbf{i} - \sin t \mathbf{j}) \quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t \mathbf{i} - \sin t \mathbf{j} = \langle -\cos t, -\sin t, 0 \rangle$$

This shows that the normal vector at any point on the helix is horizontal and points toward the z -axis. The binormal vector is

$$\begin{aligned} \mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle \end{aligned}$$

TEC Visual 13.3B shows how the **TNB** frame moves along several curves.

Figure 8 shows the helix and the osculating plane in Example 7.

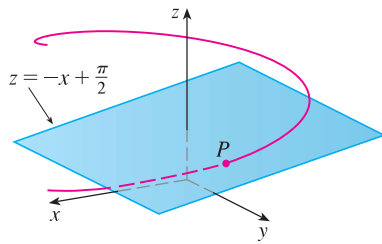


FIGURE 8

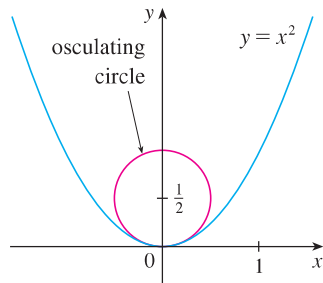


FIGURE 9

Notice that the circle and the parabola appear to bend similarly at the origin.

TEC Visual 13.3C shows how the osculating circle changes as a point moves along a curve.

The plane determined by the normal and binormal vectors \mathbf{N} and \mathbf{B} at a point P on a curve C is called the **normal plane** of C at P . It consists of all lines that are orthogonal to the tangent vector \mathbf{T} . The plane determined by the vectors \mathbf{T} and \mathbf{N} is called the **osculating plane** of C at P . The name comes from the Latin *osculum*, meaning “kiss.” It is the plane that comes closest to containing the part of the curve near P . (For a plane curve, the osculating plane is simply the plane that contains the curve.)

The circle that lies in the osculating plane of C at P , has the same tangent as C at P , lies on the concave side of C (toward which \mathbf{N} points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the **osculating circle** (or the **circle of curvature**) of C at P . It is the circle that best describes how C behaves near P ; it shares the same tangent, normal, and curvature at P .

EXAMPLE 7 Find equations of the normal plane and osculating plane of the helix in Example 6 at the point $P(0, 1, \pi/2)$.

SOLUTION The point P corresponds to $t = \pi/2$ and the normal plane there has normal vector $\mathbf{r}'(\pi/2) = \langle -1, 0, 1 \rangle$, so an equation is

$$-1(x - 0) + 0(y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \quad \text{or} \quad z = x + \frac{\pi}{2}$$

The osculating plane at P contains the vectors \mathbf{T} and \mathbf{N} , so its normal vector is $\mathbf{T} \times \mathbf{N} = \mathbf{B}$. From Example 6 we have

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle \quad \mathbf{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

A simpler normal vector is $\langle 1, 0, 1 \rangle$, so an equation of the osculating plane is

$$1(x - 0) + 0(y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \quad \text{or} \quad z = -x + \frac{\pi}{2}$$

EXAMPLE 8 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

SOLUTION From Example 5, the curvature of the parabola at the origin is $\kappa(0) = 2$. So the radius of the osculating circle at the origin is $1/\kappa = \frac{1}{2}$ and its center is $(0, \frac{1}{2})$. Its equation is therefore

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

For the graph in Figure 9 we use parametric equations of this circle:

$$x = \frac{1}{2} \cos t \quad y = \frac{1}{2} + \frac{1}{2} \sin t$$

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$


13.3 EXERCISES

1–6 Find the length of the curve.

- $\mathbf{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle, \quad -5 \leq t \leq 5$
- $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle, \quad 0 \leq t \leq 1$
- $\mathbf{r}(t) = \sqrt{2}t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}, \quad 0 \leq t \leq 1$
- $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \ln \cos t \mathbf{k}, \quad 0 \leq t \leq \pi/4$
- $\mathbf{r}(t) = \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}, \quad 0 \leq t \leq 1$
- $\mathbf{r}(t) = t^2 \mathbf{i} + 9t \mathbf{j} + 4t^{3/2} \mathbf{k}, \quad 1 \leq t \leq 4$

7–9 Find the length of the curve correct to four decimal places. (Use a calculator to approximate the integral.)

- $\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle, \quad 0 \leq t \leq 2$
- $\mathbf{r}(t) = \langle t, e^{-t}, te^{-t} \rangle, \quad 1 \leq t \leq 3$
- $\mathbf{r}(t) = \langle \cos \pi t, 2t, \sin 2\pi t \rangle, \quad \text{from } (1, 0, 0) \text{ to } (1, 4, 0)$

-  10. Graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$. Find the total length of this curve correct to four decimal places.
11. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$.
12. Find, correct to four decimal places, the length of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

13–14 (a) Find the arc length function for the curve measured from the point P in the direction of increasing t and then reparametrize the curve with respect to arc length starting from P , and (b) find the point 4 units along the curve (in the direction of increasing t) from P .

- $\mathbf{r}(t) = (5 - t) \mathbf{i} + (4t - 3) \mathbf{j} + 3t \mathbf{k}, \quad P(4, 1, 3)$
- $\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} + \sqrt{2}e^t \mathbf{k}, \quad P(0, 1, \sqrt{2})$

15. Suppose you start at the point $(0, 0, 3)$ and move 5 units along the curve $x = 3 \sin t$, $y = 4t$, $z = 3 \cos t$ in the positive direction. Where are you now?

16. Reparametrize the curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2 + 1} - 1 \right) \mathbf{i} + \frac{2t}{t^2 + 1} \mathbf{j}$$

with respect to arc length measured from the point $(1, 0)$ in the direction of increasing t . Express the reparametrization in its simplest form. What can you conclude about the curve?

17–20


- Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.
- Use Formula 9 to find the curvature.

- $\mathbf{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle$
- $\mathbf{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle, \quad t > 0$
- $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$
- $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$

21–23 Use Theorem 10 to find the curvature.

- $\mathbf{r}(t) = t^3 \mathbf{j} + t^2 \mathbf{k}$
- $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + e^t \mathbf{k}$
- $\mathbf{r}(t) = \sqrt{6}t^2 \mathbf{i} + 2t \mathbf{j} + 2t^3 \mathbf{k}$

- Find the curvature of $\mathbf{r}(t) = \langle t^2, \ln t, t \ln t \rangle$ at the point $(1, 0, 0)$.
- Find the curvature of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$.

-  26. Graph the curve with parametric equations $x = \cos t$, $y = \sin t$, $z = \sin 5t$ and find the curvature at the point $(1, 0, 0)$.

27–29 Use Formula 11 to find the curvature.

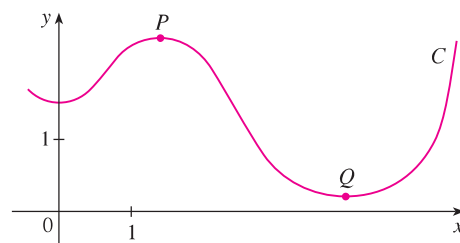
- $y = x^4$
- $y = \tan x$
- $y = xe^x$


30–31 At what point does the curve have maximum curvature? What happens to the curvature as $x \rightarrow \infty$?

- $y = \ln x$
- $y = e^x$

32. Find an equation of a parabola that has curvature 4 at the origin.


- Is the curvature of the curve C shown in the figure greater at P or at Q ? Explain.
- Estimate the curvature at P and at Q by sketching the osculating circles at those points.



 **34–35** Use a graphing calculator or computer to graph both the curve and its curvature function $\kappa(x)$ on the same screen. Is the graph of κ what you would expect?

34. $y = x^4 - 2x^2$

35. $y = x^{-2}$

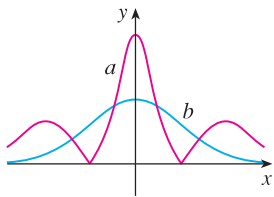
 **36–37** Plot the space curve and its curvature function $\kappa(t)$. Comment on how the curvature reflects the shape of the curve.

36. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, 4 \cos(t/2) \rangle, \quad 0 \leq t \leq 8\pi$

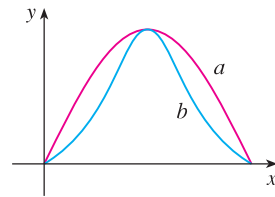
37. $\mathbf{r}(t) = \langle te^t, e^{-t}, \sqrt{2}t \rangle, \quad -5 \leq t \leq 5$


38–39 Two graphs, a and b , are shown. One is a curve $y = f(x)$ and the other is the graph of its curvature function $y = \kappa(x)$. Identify each curve and explain your choices.


38.



39.



 **40.** (a) Graph the curve $\mathbf{r}(t) = \langle \sin 3t, \sin 2t, \sin 3t \rangle$. At how many points on the curve does it appear that the curvature has a local or absolute maximum?
(b) Use a CAS to find and graph the curvature function. Does this graph confirm your conclusion from part (a)?

 **41.** The graph of $\mathbf{r}(t) = \langle t - \frac{3}{2} \sin t, 1 - \frac{3}{2} \cos t, t \rangle$ is shown in Figure 13.1.12(b). Where do you think the curvature is largest? Use a CAS to find and graph the curvature function. For which values of t is the curvature largest?

42. Use Theorem 10 to show that the curvature of a plane parametric curve $x = f(t)$, $y = g(t)$ is

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t .

43–45 Use the formula in Exercise 42 to find the curvature.

43. $x = t^2, \quad y = t^3$

44. $x = a \cos \omega t, \quad y = b \sin \omega t$

45. $x = e^t \cos t, \quad y = e^t \sin t$

46. Consider the curvature at $x = 0$ for each member of the family of functions $f(x) = e^{cx}$. For which members is $\kappa(0)$ largest?

47–48 Find the vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} at the given point.


47. $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle, \quad (1, \frac{2}{3}, 1)$


48. $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle, \quad (1, 0, 0)$

49–50 Find equations of the normal plane and osculating plane of the curve at the given point.


49. $x = \sin 2t, \quad y = -\cos 2t, \quad z = 4t; \quad (0, 1, 2\pi)$

50. $x = \ln t, \quad y = 2t, \quad z = t^2; \quad (0, 2, 1)$

 **51.** Find equations of the osculating circles of the ellipse $9x^2 + 4y^2 = 36$ at the points $(2, 0)$ and $(0, 3)$. Use a graphing calculator or computer to graph the ellipse and both osculating circles on the same screen.

 **52.** Find equations of the osculating circles of the parabola $y = \frac{1}{2}x^2$ at the points $(0, 0)$ and $(1, \frac{1}{2})$. Graph both osculating circles and the parabola on the same screen.

53. At what point on the curve $x = t^3, \quad y = 3t, \quad z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

 **54.** Is there a point on the curve in Exercise 53 where the osculating plane is parallel to the plane $x + y + z = 1$? [Note: You will need a CAS for differentiating, for simplifying, and for computing a cross product.]

55. Find equations of the normal and osculating planes of the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ at the point $(1, 1, 1)$.

56. Show that the osculating plane at every point on the curve $\mathbf{r}(t) = \langle t + 2, 1 - t, \frac{1}{2}t^2 \rangle$ is the same plane. What can you conclude about the curve?

57. Show that at every point on the curve

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$$

the angle between the unit tangent vector and the z -axis is the same. Then show that the same result holds true for the unit normal and binormal vectors.

58. The *rectifying plane* of a curve at a point is the plane that contains the vectors \mathbf{T} and \mathbf{B} at that point. Find the rectifying plane of the curve $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + \tan t \mathbf{k}$ at the point $(\sqrt{2}/2, \sqrt{2}/2, 1)$.

59. Show that the curvature κ is related to the tangent and normal vectors by the equation

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

60. Show that the curvature of a plane curve is $\kappa = |d\phi/ds|$, where ϕ is the angle between \mathbf{T} and \mathbf{i} ; that is, ϕ is the angle of inclination of the tangent line. (This shows that the definition of curvature is consistent with the definition for plane curves given in Exercise 10.2.69.)

61. (a) Show that $d\mathbf{B}/ds$ is perpendicular to \mathbf{B} .
 (b) Show that $d\mathbf{B}/ds$ is perpendicular to \mathbf{T} .
 (c) Deduce from parts (a) and (b) that $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$ for some number $\tau(s)$ called the **torsion** of the curve. (The torsion measures the degree of twisting of a curve.)
 (d) Show that for a plane curve the torsion is $\tau(s) = 0$.

62. The following formulas, called the **Frenet-Serret formulas**, are of fundamental importance in differential geometry:

1. $d\mathbf{T}/ds = \kappa\mathbf{N}$
2. $d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B}$
3. $d\mathbf{B}/ds = -\tau\mathbf{N}$

(Formula 1 comes from Exercise 59 and Formula 3 comes from Exercise 61.) Use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ to deduce Formula 2 from Formulas 1 and 3.

63. Use the Frenet-Serret formulas to prove each of the following. (Primes denote derivatives with respect to t . Start as in the proof of Theorem 10.)

- (a) $\mathbf{r}'' = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$
- (b) $\mathbf{r}' \times \mathbf{r}'' = \kappa(s')^3\mathbf{B}$
- (c) $\mathbf{r}''' = [s''' - \kappa^2(s')^3]\mathbf{T} + [3\kappa s' s'' + \kappa'(s')^2]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}$
- (d) $\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$

64. Show that the circular helix $\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$, where a and b are positive constants, has constant curvature and constant torsion. [Use the result of Exercise 63(d).]

65. Use the formula in Exercise 63(d) to find the torsion of the curve $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$.

66. Find the curvature and torsion of the curve $x = \sinh t$, $y = \cosh t$, $z = t$ at the point $(0, 1, 0)$.

67. The DNA molecule has the shape of a double helix (see Figure 3 on page 850). The radius of each helix is about 10 angstroms ($1 \text{ \AA} = 10^{-8} \text{ cm}$). Each helix rises about 34 \AA during each complete turn, and there are about 2.9×10^8 complete turns. Estimate the length of each helix.

68. Let's consider the problem of designing a railroad track to make a smooth transition between sections of straight track. Existing track along the negative x -axis is to be joined smoothly to a track along the line $y = 1$ for $x \geq 1$.

(a) Find a polynomial $P = P(x)$ of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

is continuous and has continuous slope and continuous curvature.

(b) Graph F .

13.4 Motion in Space: Velocity and Acceleration

In this section we show how the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve. In particular, we follow in the footsteps of Newton by using these methods to derive Kepler's First Law of planetary motion.

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Notice from Figure 1 that, for small values of h , the vector

$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{1}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$. Its magnitude measures the size of the displacement vector per unit time. The vector (1) gives the average velocity over a time interval of length h and its limit is the **velocity vector** $\mathbf{v}(t)$ at time t :

$$\mathbf{v}(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t) \tag{2}$$

Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

The **speed** of the particle at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. This is appropriate because, from (2) and from Equation 13.3.7, we have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt} = \text{rate of change of distance with respect to time}$$

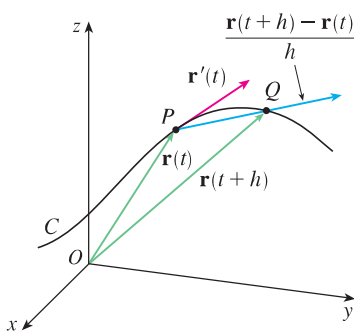


FIGURE 1