

54. At what point on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$?
55. Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?
56. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)
57. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.
58. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.
59. Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$ intersect the paraboloid a second time?
60. At what points does the normal line through the point $(1, 2, 1)$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 12$ intersect the sphere $x^2 + y^2 + z^2 = 102$?
61. Show that the sum of the x -, y -, and z -intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.
62. Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.
63. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.
64. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.
- (b) Graph the cylinder, the plane, and the tangent line on the same screen.
65. Where does the helix $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, t \rangle$ intersect the paraboloid $z = x^2 + y^2$? What is the angle of intersection between the helix and the paraboloid? (This is the angle between the tangent vector to the curve and the tangent plane to the paraboloid.)
66. The helix $\mathbf{r}(t) = \langle \cos(\pi t/2), \sin(\pi t/2), t \rangle$ intersects the sphere $x^2 + y^2 + z^2 = 2$ in two points. Find the angle of intersection at each point.
67. (a) Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if
- $$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{at } P$$
- (b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?
68. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.
- (b) Graph f near the origin and comment on how the graph confirms part (a).
69. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors \mathbf{u} and \mathbf{v} . Is it possible to find ∇f at this point? If so, how would you do it?
70. Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then
- $$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$$
- [Hint: Use Definition 14.4.7 directly.]



(b) Graph the cylinder, the plane, and the tangent line on the same screen.

14.7 Maximum and Minimum Values

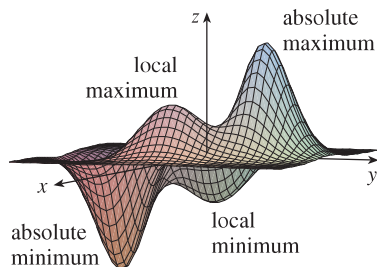


FIGURE 1

As we saw in Chapter 4, one of the main uses of ordinary derivatives is in finding maximum and minimum values (extreme values). In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables. In particular, in Example 6 we will see how to maximize the volume of a box without a lid if we have a fixed amount of cardboard to work with.

Look at the hills and valleys in the graph of f shown in Figure 1. There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$. The larger of these two values is the *absolute maximum*. Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values. The smaller of these two values is the *absolute minimum*.

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) . [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b) .] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) , then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f , then f has an **absolute maximum** (or **absolute minimum**) at (a, b) .

Notice that the conclusion of Theorem 2 can be stated in the notation of gradient vectors as $\nabla f(a, b) = \mathbf{0}$.

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

PROOF Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b) , then g has a local maximum (or minimum) at a , so $g'(a) = 0$ by Fermat's Theorem (see Theorem 4.1.4). But $g'(a) = f_x(a, b)$ (see Equation 14.3.1) and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. ■

If we put $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane (Equation 14.4.2), we get $z = z_0$. Thus the geometric interpretation of Theorem 2 is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b) , then (a, b) is a critical point of f . However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

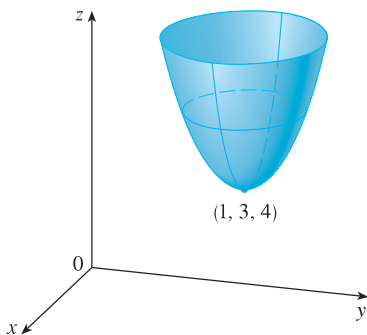


FIGURE 2
 $z = x^2 + y^2 - 2x - 6y + 14$

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \quad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y . Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f . This can be confirmed geometrically from the graph of f , which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in Figure 2. ■

EXAMPLE 2 Find the extreme values of $f(x, y) = y^2 - x^2$.

SOLUTION Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x -axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y -axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f , so f has no extreme value. ■

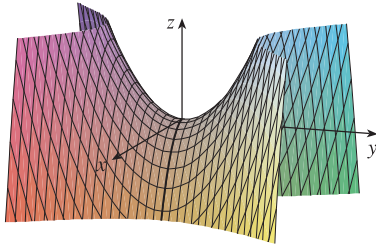


FIGURE 3
 $z = y^2 - x^2$



Photo by Stan Wagon, Macalester College

Example 2 illustrates the fact that a function need not have a maximum or minimum value at a critical point. Figure 3 shows how this is possible. The graph of f is the hyperbolic paraboloid $z = y^2 - x^2$, which has a horizontal tangent plane ($z = 0$) at the origin. You can see that $f(0, 0) = 0$ is a maximum in the direction of the x -axis but a minimum in the direction of the y -axis. Near the origin the graph has the shape of a saddle and so $(0, 0)$ is called a *saddle point* of f .

A mountain pass also has the shape of a saddle. As the photograph of the geological formation illustrates, for people hiking in one direction the saddle point is the lowest point on their route, while for those traveling in a different direction the saddle point is the highest point.

We need to be able to determine whether or not a function has an extreme value at a critical point. The following test, which is proved at the end of this section, is analogous to the Second Derivative Test for functions of one variable.

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

NOTE 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b) .

NOTE 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b) , or (a, b) could be a saddle point of f .

NOTE 3 To remember the formula for D , it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

SOLUTION We first locate the critical points:

$$f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

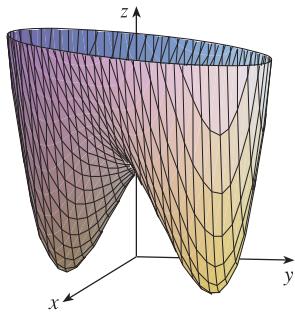


FIGURE 4
 $z = x^4 + y^4 - 4xy + 1$

A contour map of the function f in Example 3 is shown in Figure 5. The level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape and indicate that as we move away from $(1, 1)$ or $(-1, -1)$ in any direction the values of f are increasing. The level curves near $(0, 0)$, on the other hand, resemble hyperbolas. They reveal that as we move away from the origin (where the value of f is 1), the values of f decrease in some directions but increase in other directions. Thus the contour map suggests the presence of the minima and saddle point that we found in Example 3.

FIGURE 5

TEC In Module 14.7 you can use contour maps to estimate the locations of critical points.

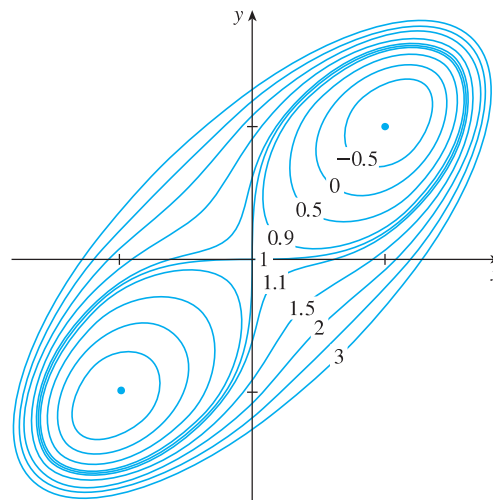
Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown in Figure 4. ■



EXAMPLE 4 Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f .

SOLUTION The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \quad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

4 $2x(10y - 5 - 2x^2) = 0$

5 $5x^2 - 4y - 4y^3 = 0$

From Equation 4 we see that either

$$x = 0 \quad \text{or} \quad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

In the second case ($10y - 5 - 2x^2 = 0$), we get

$$\boxed{6} \quad x^2 = 5y - 2.5$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$\boxed{7} \quad 4y^3 - 21y + 12.5 = 0$$

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$

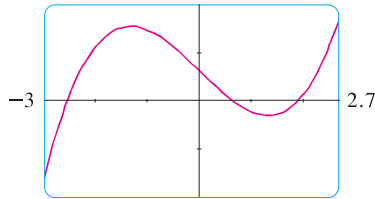


FIGURE 6

as in Figure 6, we see that Equation 7 has three real roots. By zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452 \quad y \approx 0.6468 \quad y \approx 1.8984$$

(Alternatively, we could have used Newton's method or solved numerically using a calculator or computer to locate these roots.) From Equation 6, the corresponding x -values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

| Critical point | Value of f | f_{xx} | D | Conclusion |
|--------------------|--------------|----------|---------|---------------|
| $(0, 0)$ | 0.00 | -10.00 | 80.00 | local maximum |
| $(\pm 2.64, 1.90)$ | 8.50 | -55.93 | 2488.72 | local maximum |
| $(\pm 0.86, 0.65)$ | -1.48 | -5.87 | -187.64 | saddle point |

Figures 7 and 8 give two views of the graph of f and we see that the surface opens downward. [This can also be seen from the expression for $f(x, y)$: the dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.] Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is $f(\pm 2.64, 1.90) \approx 8.50$. In other words, the highest points on the graph of f are $(\pm 2.64, 1.90, 8.50)$.

TEC Visual 14.7 shows several families of surfaces. The surface in Figures 7 and 8 is a member of one of these families.

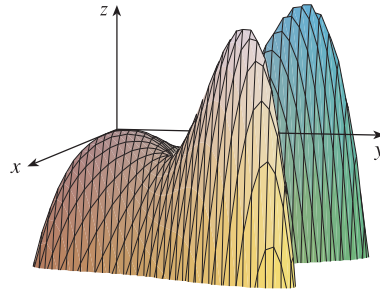


FIGURE 7

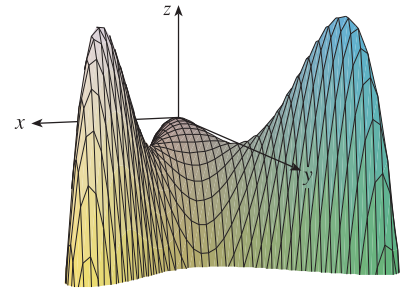


FIGURE 8

The five critical points of the function f in Example 4 are shown in red in the contour map of f in Figure 9.

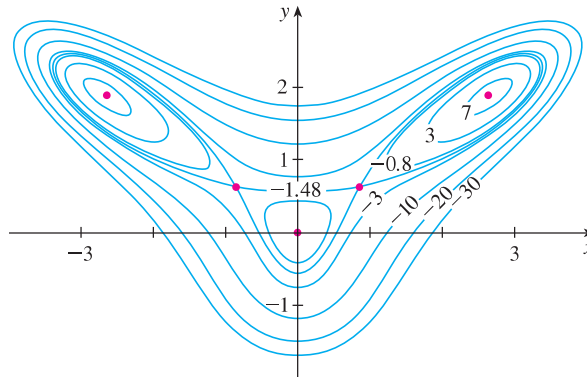


FIGURE 9

EXAMPLE 5 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

SOLUTION The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x - 1)^2 + y^2 + (z + 2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x - 1)^2 + y^2 + (6 - x - 2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x - 1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equations

$$f_x = 2(x - 1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $(\frac{11}{6}, \frac{5}{3})$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $(\frac{11}{6}, \frac{5}{3})$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$d = \sqrt{(x - 1)^2 + y^2 + (6 - x - 2y)^2} = \sqrt{(\frac{5}{6})^2 + (\frac{5}{3})^2 + (\frac{5}{6})^2} = \frac{5}{6}\sqrt{6}$$

Example 5 could also be solved using vectors. Compare with the methods of Section 12.5.

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$. ■

EXAMPLE 6 A rectangular box without a lid is to be made from 12 m² of cardboard. Find the maximum volume of such a box.

SOLUTION Let the length, width, and height of the box (in meters) be x , y , and z , as shown in Figure 10. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

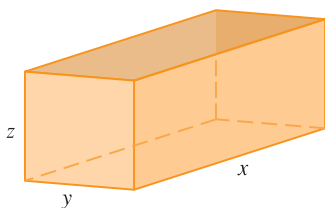


FIGURE 10

Solving this equation for z , we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy \frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} \quad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \quad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V , or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V , so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m^3 . ■

■ Absolute Maximum and Minimum Values

For a function f of one variable, the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. According to the Closed Interval Method in Section 4.1, we found these by evaluating f not only at the critical numbers but also at the endpoints a and b .

There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a **closed set** in \mathbb{R}^2 is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D .] For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

which consists of all points on or inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$). But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)

A **bounded set** in \mathbb{R}^2 is one that is contained within some disk. In other words, it is finite in extent. Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

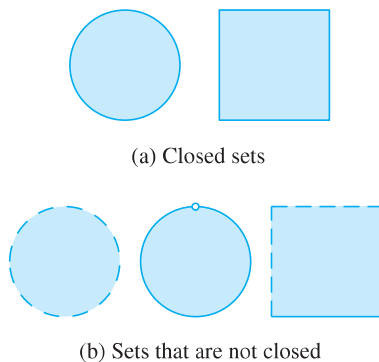


FIGURE 11

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in \mathbb{R}^2 , then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1) , then (x_1, y_1) is either a critical point of f or a boundary point of D . Thus we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .
2. Find the extreme values of f on the boundary of D .
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

SOLUTION Since f is a polynomial, it is continuous on the closed, bounded rectangle D , so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \quad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D , which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \quad 0 \leq x \leq 3$$

This is an increasing function of x , so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \quad 0 \leq y \leq 2$$

This is a decreasing function of y , so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \quad 0 \leq x \leq 3$$

By the methods of Chapter 4, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \quad 0 \leq y \leq 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f . ■

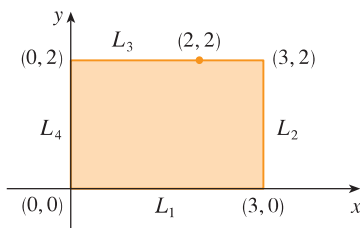


FIGURE 12

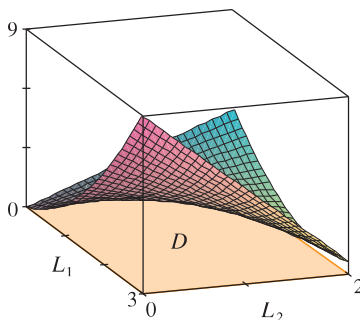


FIGURE 13
 $f(x, y) = x^2 - 2xy + 2y$

We close this section by giving a proof of the first part of the Second Derivatives Test. Part (b) has a similar proof.

PROOF OF THEOREM 3, PART (a) We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by Theorem 14.6.3:

$$D_{\mathbf{u}}f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathbf{u}}^2 f &= D_{\mathbf{u}}(D_{\mathbf{u}}f) = \frac{\partial}{\partial x}(D_{\mathbf{u}}f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}}f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \end{aligned} \quad \text{(by Clairaut's Theorem)}$$

If we complete the square in this expression, we obtain

$$\boxed{10} \quad D_{\mathbf{u}}^2 f = f_{xx} \left(h + \frac{f_{xy}}{f_{xx}} k \right)^2 + \frac{k^2}{f_{xx}} (f_{xx}f_{yy} - f_{xy}^2)$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B . Therefore, by looking at Equation 10, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B . This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of \mathbf{u} , then C is concave upward on an interval of length 2δ . This is true in the direction of every vector \mathbf{u} , so if we restrict (x, y) to lie in B , the graph of f lies above its horizontal tangent plane at P . Thus $f(x, y) \geq f(a, b)$ whenever (x, y) is in B . This shows that $f(a, b)$ is a local minimum. ■

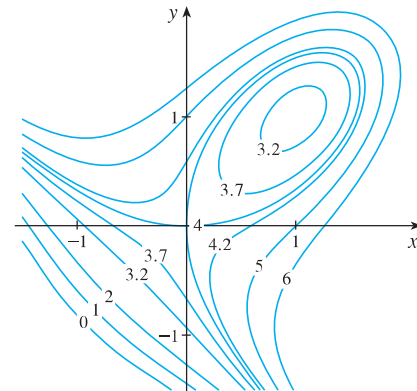
14.7 EXERCISES

- Suppose $(1, 1)$ is a critical point of a function f with continuous second derivatives. In each case, what can you say about f ?
 - $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 1, \quad f_{yy}(1, 1) = 2$
 - $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 3, \quad f_{yy}(1, 1) = 2$
- Suppose $(0, 2)$ is a critical point of a function g with continuous second derivatives. In each case, what can you say about g ?
 - $g_{xx}(0, 2) = -1, \quad g_{xy}(0, 2) = 6, \quad g_{yy}(0, 2) = 1$
 - $g_{xx}(0, 2) = -1, \quad g_{xy}(0, 2) = 2, \quad g_{yy}(0, 2) = -8$
 - $g_{xx}(0, 2) = 4, \quad g_{xy}(0, 2) = 6, \quad g_{yy}(0, 2) = 9$

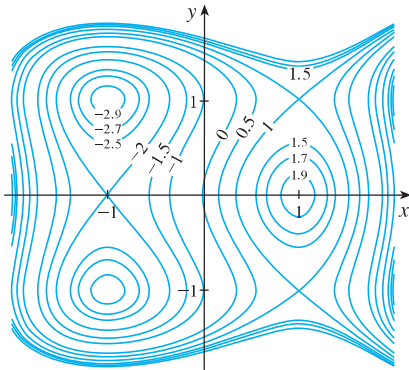
3–4 Use the level curves in the figure to predict the location of the critical points of f and whether f has a saddle point or a local maximum or minimum at each critical point. Explain your

reasoning. Then use the Second Derivatives Test to confirm your predictions.

$$3. \quad f(x, y) = 4 + x^3 + y^3 - 3xy$$



4. $f(x, y) = 3x - x^3 - 2y^2 + y^4$



5–20 Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$
6. $f(x, y) = xy - 2x - 2y - x^2 - y^2$
7. $f(x, y) = (x - y)(1 - xy)$
8. $f(x, y) = y(e^x - 1)$
9. $f(x, y) = x^2 + y^4 + 2xy$
10. $f(x, y) = 2 - x^4 + 2x^2 - y^2$
11. $f(x, y) = x^3 - 3x + 3xy^2$
12. $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$
13. $f(x, y) = x^4 - 2x^2 + y^3 - 3y$
14. $f(x, y) = y \cos x$
15. $f(x, y) = e^x \cos y$
16. $f(x, y) = xy e^{-(x^2+y^2)/2}$
17. $f(x, y) = xy + e^{-xy}$
18. $f(x, y) = (x^2 + y^2)e^{-x}$
19. $f(x, y) = y^2 - 2y \cos x, \quad -1 \leq x \leq 7$
20. $f(x, y) = \sin x \sin y, \quad -\pi < x < \pi, \quad -\pi < y < \pi$

21. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.
22. Show that $f(x, y) = x^2 y e^{-x^2-y^2}$ has maximum values at $(\pm 1, 1/\sqrt{2})$ and minimum values at $(\pm 1, -1/\sqrt{2})$. Show also that f has infinitely many other critical points and $D = 0$ at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

23–26 Use a graph or level curves or both to estimate the local maximum and minimum values and saddle point(s) of the function. Then use calculus to find these values precisely.

23. $f(x, y) = x^2 + y^2 + x^{-2}y^{-2}$
24. $f(x, y) = (x - y)e^{-x^2-y^2}$
25. $f(x, y) = \sin x + \sin y + \sin(x + y),$
 $0 \leq x \leq 2\pi, \quad 0 \leq y \leq 2\pi$
26. $f(x, y) = \sin x + \sin y + \cos(x + y),$
 $0 \leq x \leq \pi/4, \quad 0 \leq y \leq \pi/4$

27–30 Use a graphing device as in Example 4 (or Newton's method or solve numerically using a calculator or computer) to find the critical points of f correct to three decimal places. Then classify the critical points and find the highest or lowest points on the graph, if any.

27. $f(x, y) = x^4 + y^4 - 4x^2y + 2y$
28. $f(x, y) = y^6 - 2y^4 + x^2 - y^2 + y$
29. $f(x, y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1$
30. $f(x, y) = 20e^{-x^2-y^2} \sin 3x \cos 3y, \quad |x| \leq 1, \quad |y| \leq 1$


31–38 Find the absolute maximum and minimum values of f on the set D .

31. $f(x, y) = x^2 + y^2 - 2x, \quad D$ is the closed triangular region with vertices $(2, 0), (0, 2),$ and $(0, -2)$
32. $f(x, y) = x + y - xy, \quad D$ is the closed triangular region with vertices $(0, 0), (0, 2),$ and $(4, 0)$
33. $f(x, y) = x^2 + y^2 + x^2y + 4,$
 $D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$
34. $f(x, y) = x^2 + xy + y^2 - 6y,$
 $D = \{(x, y) \mid -3 \leq x \leq 3, 0 \leq y \leq 5\}$
35. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1,$
 $D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$
36. $f(x, y) = xy^2, \quad D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$
37. $f(x, y) = 2x^3 + y^4, \quad D = \{(x, y) \mid x^2 + y^2 \leq 1\}$
38. $f(x, y) = x^3 - 3x - y^3 + 12y, \quad D$ is the quadrilateral whose vertices are $(-2, 3), (2, 3), (2, 2),$ and $(-2, -2)$

39. For functions of one variable it is impossible for a continuous function to have two local maxima and no local minimum. But for functions of two variables such functions exist. Show that the function

$$f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$$

has only two critical points, but has local maxima at both of them. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

-  40. If a function of one variable is continuous on an interval and has only one critical number, then a local maximum has to be an absolute maximum. But this is not true for functions of two variables. Show that the function

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

has exactly one critical point, and that f has a local maximum there that is not an absolute maximum. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

41. Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$.
42. Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0, 1, 1)$.
43. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.
44. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.
45. Find three positive numbers whose sum is 100 and whose product is a maximum.
46. Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.
47. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r .
48. Find the dimensions of the box with volume 1000 cm^3 that has minimal surface area.
49. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.
50. Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm^2 .
51. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant c .
52. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.
53. A cardboard box without a lid is to have a volume of $32,000 \text{ cm}^3$. Find the dimensions that minimize the amount of cardboard used.
54. A rectangular building is being designed to minimize heat loss. The east and west walls lose heat at a rate of 10 units/m^2 per day, the north and south walls at a rate of 8 units/m^2 per day, the floor at a rate of 1 unit/m^2 per day, and the roof at a rate of 5 units/m^2 per day. Each wall must be at least 30 m long, the height must be at least 4 m , and the volume must be exactly 4000 m^3 .
- (a) Find and sketch the domain of the heat loss as a function of the lengths of the sides.

- (b) Find the dimensions that minimize heat loss. (Check both the critical points and the points on the boundary of the domain.)
- (c) Could you design a building with even less heat loss if the restrictions on the lengths of the walls were removed?

55. If the length of the diagonal of a rectangular box must be L , what is the largest possible volume?
56. A model for the yield Y of an agricultural crop as a function of the nitrogen level N and phosphorus level P in the soil (measured in appropriate units) is

$$Y(N, P) = kNP e^{-N-P}$$

where k is a positive constant. What levels of nitrogen and phosphorus result in the best yield?

57. The Shannon index (sometimes called the Shannon-Wiener index or Shannon-Weaver index) is a measure of diversity in an ecosystem. For the case of three species, it is defined as

$$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3$$

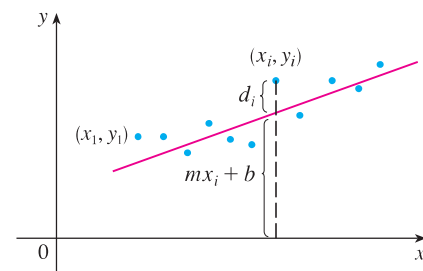
where p_i is the proportion of species i in the ecosystem.

- (a) Express H as a function of two variables using the fact that $p_1 + p_2 + p_3 = 1$.
- (b) What is the domain of H ?
- (c) Find the maximum value of H . For what values of p_1, p_2, p_3 does it occur?
58. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where $p, q,$ and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most $\frac{2}{3}$.

59. Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately, for some values of m and b . The scientist performs an experiment and collects data in the form of points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, and then plots these points. The points don't lie exactly on a straight line, so the scientist wants to find constants m and b so that the line $y = mx + b$ "fits" the points as well as possible (see the figure).



Let $d_i = y_i - (mx_i + b)$ be the vertical deviation of the point (x_i, y_i) from the line. The **method of least squares** determines m and b so as to minimize $\sum_{i=1}^n d_i^2$, the sum of the squares of these deviations. Show that, according to this method, the line of best fit is obtained when

$$m \sum_{i=1}^n x_i + bn = \sum_{i=1}^n y_i$$

and

$$m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i$$

Thus the line is found by solving these two equations in the two unknowns m and b . (See Section 1.2 for a further discussion and applications of the method of least squares.)

60. Find an equation of the plane that passes through the point $(1, 2, 3)$ and cuts off the smallest volume in the first octant.

APPLIED PROJECT

DESIGNING A DUMPSTER

For this project we locate a rectangular trash Dumpster in order to study its shape and construction. We then attempt to determine the dimensions of a container of similar design that minimize construction cost.

1. First locate a trash Dumpster in your area. Carefully study and describe all details of its construction, and determine its volume. Include a sketch of the container.
2. While maintaining the general shape and method of construction, determine the dimensions such a container of the same volume should have in order to minimize the cost of construction. Use the following assumptions in your analysis:
 - The sides, back, and front are to be made from 12-gauge (0.1046 inch thick) steel sheets, which cost \$0.70 per square foot (including any required cuts or bends).
 - The base is to be made from a 10-gauge (0.1345 inch thick) steel sheet, which costs \$0.90 per square foot.
 - Lids cost approximately \$50.00 each, regardless of dimensions.
 - Welding costs approximately \$0.18 per foot for material and labor combined.

Give justification of any further assumptions or simplifications made of the details of construction.

3. Describe how any of your assumptions or simplifications may affect the final result.
4. If you were hired as a consultant on this investigation, what would your conclusions be? Would you recommend altering the design of the Dumpster? If so, describe the savings that would result.

DISCOVERY PROJECT

QUADRATIC APPROXIMATIONS AND CRITICAL POINTS

The Taylor polynomial approximation to functions of one variable that we discussed in Chapter 11 can be extended to functions of two or more variables. Here we investigate quadratic approximations to functions of two variables and use them to give insight into the Second Derivatives Test for classifying critical points.

In Section 14.4 we discussed the linearization of a function f of two variables at a point (a, b) :

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Recall that the graph of L is the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$ and the corresponding linear approximation is $f(x, y) \approx L(x, y)$. The linearization L is also called the **first-degree Taylor polynomial** of f at (a, b) .

1. If f has continuous second-order partial derivatives at (a, b) , then the **second-degree Taylor polynomial** of f at (a, b) is

$$Q(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{1}{2}f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2}f_{yy}(a, b)(y - b)^2$$

and the approximation $f(x, y) \approx Q(x, y)$ is called the **quadratic approximation** to f at (a, b) . Verify that Q has the same first- and second-order partial derivatives as f at (a, b) .

2. (a) Find the first- and second-degree Taylor polynomials L and Q of $f(x, y) = e^{-x^2-y^2}$ at $(0, 0)$.
 (b) Graph f , L , and Q . Comment on how well L and Q approximate f .
3. (a) Find the first- and second-degree Taylor polynomials L and Q for $f(x, y) = xe^y$ at $(1, 0)$.
 (b) Compare the values of L , Q , and f at $(0.9, 0.1)$.
 (c) Graph f , L , and Q . Comment on how well L and Q approximate f .
4. In this problem we analyze the behavior of the polynomial $f(x, y) = ax^2 + bxy + cy^2$ (without using the Second Derivatives Test) by identifying the graph as a paraboloid.
 (a) By completing the square, show that if $a \neq 0$, then

$$f(x, y) = ax^2 + bxy + cy^2 = a \left[\left(x + \frac{b}{2a}y \right)^2 + \left(\frac{4ac - b^2}{4a^2} \right) y^2 \right]$$

- (b) Let $D = 4ac - b^2$. Show that if $D > 0$ and $a > 0$, then f has a local minimum at $(0, 0)$.
 (c) Show that if $D > 0$ and $a < 0$, then f has a local maximum at $(0, 0)$.
 (d) Show that if $D < 0$, then $(0, 0)$ is a saddle point.
5. (a) Suppose f is any function with continuous second-order partial derivatives such that $f(0, 0) = 0$ and $(0, 0)$ is a critical point of f . Write an expression for the second-degree Taylor polynomial, Q , of f at $(0, 0)$.
 (b) What can you conclude about Q from Problem 4?
 (c) In view of the quadratic approximation $f(x, y) \approx Q(x, y)$, what does part (b) suggest about f ?

14.8 Lagrange Multipliers

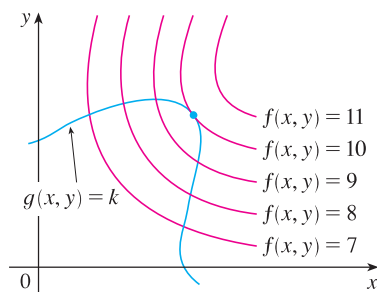


FIGURE 1

TEC Visual 14.8 animates Figure 1 for both level curves and level surfaces.

In Example 14.7.6 we maximized a volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$, which expressed the side condition that the surface area was 12 m^2 . In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 1 shows this curve together with several level curves of f . These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$. To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This

means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ .

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P . If t_0 is the parameter value corresponding to the point P , then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C . Since f has an extreme value at (x_0, y_0, z_0) , it follows that h has an extreme value at t_0 , so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C . But we already know from Section 14.6 that the gradient vector of g , $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve. (See Equation 14.6.18.) This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

1

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The number λ in Equation 1 is called a **Lagrange multiplier**. The procedure based on Equation 1 is as follows.

Lagrange multipliers are named after the French-Italian mathematician Joseph-Louis Lagrange (1736–1813). See page 289 for a biographical sketch of Lagrange.

In deriving Lagrange’s method we assumed that $\nabla g \neq \mathbf{0}$. In each of our examples you can check that $\nabla g \neq \mathbf{0}$ at all points where $g(x, y, z) = k$. See Exercise 25 for what can go wrong if $\nabla g = \mathbf{0}$.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z , and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f ; the smallest is the minimum value of f .

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \quad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x , y , z , and λ , but it is not necessary to find explicit values for λ .

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x , y , and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad g(x, y) = k$$

Our first illustration of Lagrange's method is to reconsider the problem given in Example 14.7.6.

EXAMPLE 1 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION As in Example 14.7.6, we let x , y , and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x , y , z , and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$\boxed{2} \quad yz = \lambda(2z + y)$$

$$\boxed{3} \quad xz = \lambda(2z + x)$$

$$\boxed{4} \quad xy = \lambda(2x + 2y)$$

$$\boxed{5} \quad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (2) by x , (3) by y , and (4) by z , then the left sides of these equations will be identical. Doing this, we have

$$\boxed{6} \quad xyz = \lambda(2xz + xy)$$

$$\boxed{7} \quad xyz = \lambda(2yz + xy)$$

$$\boxed{8} \quad xyz = \lambda(2xz + 2yz)$$

Another method for solving the system of equations (2–5) is to solve each of Equations 2, 3, and 4 for λ and then to equate the resulting expressions.

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from (2), (3),

and (4) and this would contradict (5). Therefore, from (6) and (7), we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From (7) and (8) we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in (5), we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since $x, y,$ and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. This agrees with our answer in Section 14.7. ■

EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

SOLUTION We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad g(x, y) = 1$$

or as

$$\text{9} \quad 2x = 2x\lambda$$

$$2y = 2y\lambda$$

$$\text{10} \quad 4y = 2y\lambda$$

$$x^2 + y^2 = 1$$

$$\text{11} \quad x^2 + y^2 = 1$$

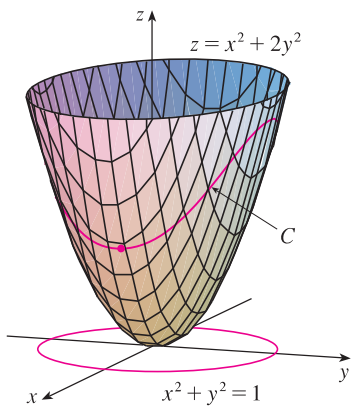


FIGURE 2

From (9) we have $x = 0$ or $\lambda = 1$. If $x = 0$, then (11) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (10), so then (11) gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1), (0, -1), (1, 0),$ and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \quad f(0, -1) = 2 \quad f(1, 0) = 1 \quad f(-1, 0) = 1$$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. In geometric terms, these correspond to the highest and lowest points on the curve C in Figure 2, where C consists of those points on the paraboloid $z = x^2 + 2y^2$ that are directly above the constraint circle $x^2 + y^2 = 1$. ■

The geometry behind the use of Lagrange multipliers in Example 2 is shown in Figure 3. The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$.

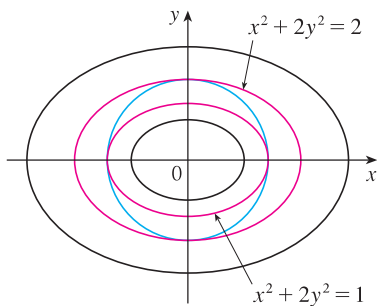


FIGURE 3

EXAMPLE 3 Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

SOLUTION According to the procedure in (14.7.9), we compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0, 0)$. We compare the value of f at that point with the extreme values on the boundary from Example 2:

$$f(0, 0) = 0 \quad f(\pm 1, 0) = 1 \quad f(0, \pm 1) = 2$$

Therefore the maximum value of f on the disk $x^2 + y^2 \leq 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$. ■

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

SOLUTION The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x - 3)^2 + (y - 1)^2 + (z + 1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g, g = 4$. This gives

$$\boxed{12} \quad 2(x - 3) = 2x\lambda$$

$$\boxed{13} \quad 2(y - 1) = 2y\lambda$$

$$\boxed{14} \quad 2(z + 1) = 2z\lambda$$

$$\boxed{15} \quad x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for $x, y,$ and z in terms of λ from (12), (13), and (14), and then substitute these values into (15). From (12) we have

$$x - 3 = x\lambda \quad \text{or} \quad x(1 - \lambda) = 3 \quad \text{or} \quad x = \frac{3}{1 - \lambda}$$

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from (12).] Similarly, (13) and (14) give

$$y = \frac{1}{1 - \lambda} \quad z = -\frac{1}{1 - \lambda}$$

Figure 4 shows the sphere and the nearest point P in Example 4. Can you see how to find the coordinates of P without using calculus?

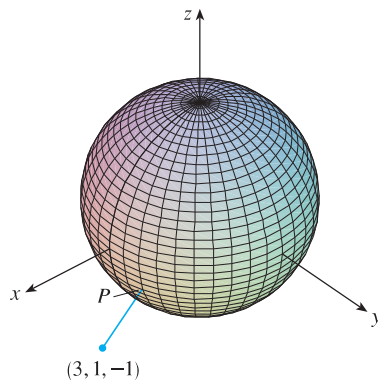


FIGURE 4

Therefore, from (15), we have

$$\frac{3^2}{(1 - \lambda)^2} + \frac{1^2}{(1 - \lambda)^2} + \frac{(-1)^2}{(1 - \lambda)^2} = 4$$

which gives $(1 - \lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z) :

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}} \right) \quad \text{and} \quad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11})$ and the farthest is $(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11})$. ■

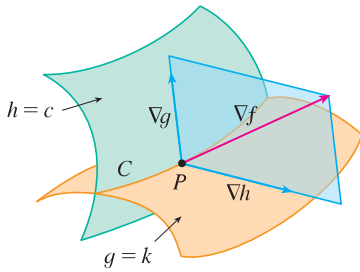


FIGURE 5

Two Constraints

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.) Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P . But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C . This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.) So there are numbers λ and μ (called Lagrange multipliers) such that

16

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In this case Lagrange’s method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ , and μ . These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \\ g(x, y, z) &= k \\ h(x, y, z) &= c \end{aligned}$$

The cylinder $x^2 + y^2 = 1$ intersects the plane $x - y + z = 1$ in an ellipse (Figure 6). Example 5 asks for the maximum value of f when (x, y, z) is restricted to lie on the ellipse.

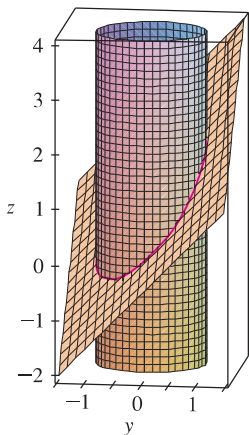


FIGURE 6

EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

SOLUTION We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

17

$$1 = \lambda + 2x\mu$$

18

$$2 = -\lambda + 2y\mu$$

19

$$3 = \lambda$$

20

$$x - y + z = 1$$

21

$$x^2 + y^2 = 1$$

Putting $\lambda = 3$ [from (19)] in (17), we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, (18) gives $y = 5/(2\mu)$. Substitution in (21) then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

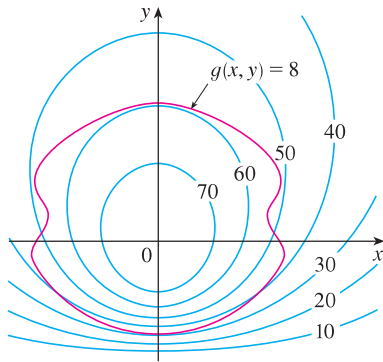
and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from (20), $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp \frac{2}{\sqrt{29}} + 2\left(\pm \frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$. ■

14.8 EXERCISES

1. Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



2. (a) Use a graphing calculator or computer to graph the circle $x^2 + y^2 = 1$. On the same screen, graph several curves of the form $x^2 + y = c$ until you find two that just touch the circle. What is the significance of the values of c for these two curves?
 (b) Use Lagrange multipliers to find the extreme values of $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$. Compare your answers with those in part (a).

3–14 Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2$; $x^2 + y^2 = 1$
4. $f(x, y) = 3x + y$; $x^2 + y^2 = 10$
5. $f(x, y) = xy$; $4x^2 + y^2 = 8$
6. $f(x, y) = xe^{xy}$; $x^2 + y^2 = 2$
7. $f(x, y, z) = 2x + 2y + z$; $x^2 + y^2 + z^2 = 9$
8. $f(x, y, z) = e^{xyz}$; $2x^2 + y^2 + z^2 = 24$
9. $f(x, y, z) = xy^2z$; $x^2 + y^2 + z^2 = 4$
10. $f(x, y, z) = \ln(x^2 + 1) + \ln(y^2 + 1) + \ln(z^2 + 1)$;
 $x^2 + y^2 + z^2 = 12$

11. $f(x, y, z) = x^2 + y^2 + z^2$; $x^4 + y^4 + z^4 = 1$
12. $f(x, y, z) = x^4 + y^4 + z^4$; $x^2 + y^2 + z^2 = 1$
13. $f(x, y, z, t) = x + y + z + t$; $x^2 + y^2 + z^2 + t^2 = 1$
14. $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$;
 $x_1^2 + x_2^2 + \dots + x_n^2 = 1$

15. The method of Lagrange multipliers assumes that the extreme values exist, but that is not always the case. Show that the problem of finding the minimum value of $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 1$ can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint.
16. Find the minimum value of $f(x, y, z) = x^2 + 2y^2 + 3z^2$ subject to the constraint $x + 2y + 3z = 10$. Show that f has no maximum value with this constraint.

17–20 Find the extreme values of f subject to both constraints.

17. $f(x, y, z) = x + y + z$; $x^2 + z^2 = 2$, $x + y = 1$
18. $f(x, y, z) = z$; $x^2 + y^2 = z^2$, $x + y + z = 24$
19. $f(x, y, z) = yz + xy$; $xy = 1$, $y^2 + z^2 = 1$
20. $f(x, y, z) = x^2 + y^2 + z^2$; $x - y = 1$, $y^2 - z^2 = 1$

21–23 Find the extreme values of f on the region described by the inequality.

21. $f(x, y) = x^2 + y^2 + 4x - 4y$, $x^2 + y^2 \leq 9$
22. $f(x, y) = 2x^2 + 3y^2 - 4x - 5$, $x^2 + y^2 \leq 16$
23. $f(x, y) = e^{-xy}$, $x^2 + 4y^2 \leq 1$

24. Consider the problem of maximizing the function $f(x, y) = 2x + 3y$ subject to the constraint $\sqrt{x} + \sqrt{y} = 5$.
- (a) Try using Lagrange multipliers to solve the problem.
 - (b) Does $f(25, 0)$ give a larger value than the one in part (a)?
 - (c) Solve the problem by graphing the constraint equation and several level curves of f .
 - (d) Explain why the method of Lagrange multipliers fails to solve the problem.
 - (e) What is the significance of $f(9, 4)$?

25. Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a piriform).
 (a) Try using Lagrange multipliers to solve the problem.
 (b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ .
 (c) Explain why Lagrange multipliers fail to find the minimum value in this case.
- CAS** 26. (a) If your computer algebra system plots implicitly defined curves, use it to estimate the minimum and maximum values of $f(x, y) = x^3 + y^3 + 3xy$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$ by graphical methods.
 (b) Solve the problem in part (a) with the aid of Lagrange multipliers. Use your CAS to solve the equations numerically. Compare your answers with those in part (a).
27. The total production P of a certain product depends on the amount L of labor used and the amount K of capital investment. In Sections 14.1 and 14.3 we discussed how the Cobb-Douglas model $P = bL^\alpha K^{1-\alpha}$ follows from certain economic assumptions, where b and α are positive constants and $\alpha < 1$. If the cost of a unit of labor is m and the cost of a unit of capital is n , and the company can spend only p dollars as its total budget, then maximizing the production P is subject to the constraint $mL + nK = p$. Show that the maximum production occurs when

$$L = \frac{\alpha p}{m} \quad \text{and} \quad K = \frac{(1 - \alpha)p}{n}$$

28. Referring to Exercise 27, we now suppose that the production is fixed at $bL^\alpha K^{1-\alpha} = Q$, where Q is a constant. What values of L and K minimize the cost function $C(L, K) = mL + nK$?
29. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.
30. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.
Hint: Use Heron's formula for the area:

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where $s = p/2$ and x, y, z are the lengths of the sides.

31–43 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 14.7.

- | | |
|-----------------|-----------------|
| 31. Exercise 41 | 32. Exercise 42 |
| 33. Exercise 43 | 34. Exercise 44 |
| 35. Exercise 45 | 36. Exercise 46 |
| 37. Exercise 47 | 38. Exercise 48 |
| 39. Exercise 49 | 40. Exercise 50 |

41. Exercise 51
 42. Exercise 52
 43. Exercise 55

44. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm .
45. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.
46. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.
 (a) Graph the cone and the plane, and observe the elliptical intersection.
 (b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

CAS 47–48 Find the maximum and minimum values of f subject to the given constraints. Use a computer algebra system to solve the system of equations that arises in using Lagrange multipliers. (If your CAS finds only one solution, you may need to use additional commands.)

47. $f(x, y, z) = ye^{x-z}$; $9x^2 + 4y^2 + 36z^2 = 36, xy + yz = 1$
 48. $f(x, y, z) = x + y + z$; $x^2 - y^2 = z, x^2 + z^2 = 4$

49. (a) Find the maximum value of

$$f(x_1, x_2, \dots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that x_1, x_2, \dots, x_n are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.

- (b) Deduce from part (a) that if x_1, x_2, \dots, x_n are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

50. (a) Maximize $\sum_{i=1}^n x_i y_i$ subject to the constraints $\sum_{i=1}^n x_i^2 = 1$ and $\sum_{i=1}^n y_i^2 = 1$.
 (b) Put

$$x_i = \frac{a_i}{\sqrt{\sum a_j^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$$

to show that

$$\sum a_i b_i \leq \sqrt{\sum a_j^2} \sqrt{\sum b_j^2}$$

for any numbers $a_1, \dots, a_n, b_1, \dots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.