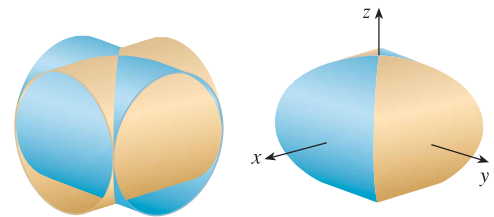


15. (a) Use the Midpoint Rule for double integrals (see Section 15.1) with four squares to estimate the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies above the square $[0, 1] \times [0, 1]$.
- CAS** (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).
16. (a) Use the Midpoint Rule for double integrals with $m = n = 2$ to estimate the area of the surface $z = xy + x^2 + y^2$, $0 \leq x \leq 2$, $0 \leq y \leq 2$.
- CAS** (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).
- CAS** 17. Find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leq x \leq 4$, $0 \leq y \leq 1$.
- CAS** 18. Find the exact area of the surface $z = 1 + x + y + x^2$ $-2 \leq x \leq 1$ $-1 \leq y \leq 1$. Illustrate by graphing the surface.
- CAS** 19. Find, to four decimal places, the area of the part of the surface $z = 1 + x^2y^2$ that lies above the disk $x^2 + y^2 \leq 1$.
- CAS** 20. Find, to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leq 1$. Illustrate by graphing this part of the surface.
21. Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy -plane with area $A(D)$ is $\sqrt{a^2 + b^2 + 1} A(D)$.
22. If you attempt to use Formula 2 to find the area of the top half of the sphere $x^2 + y^2 + z^2 = a^2$, you have a slight problem because the double integral is improper. In fact, the integrand has an infinite discontinuity at every point of the boundary circle $x^2 + y^2 = a^2$. However, the integral can be computed as the limit of the integral over the disk $x^2 + y^2 \leq t^2$ as $t \rightarrow a^-$. Use this method to show that the area of a sphere of radius a is $4\pi a^2$.
23. Find the area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$. [Hint: Project the surface onto the xz -plane.]
24. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



15.6 Triple Integrals

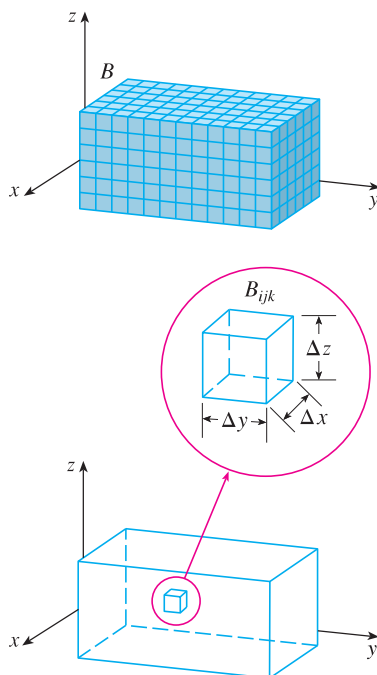


FIGURE 1

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box:

$$\boxed{1} \quad B = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx , dividing $[c, d]$ into m subintervals of width Δy , and dividing $[r, s]$ into n subintervals of width Δz . The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1. Each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$.

Then we form the **triple Riemann sum**

$$\boxed{2} \quad \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk} . By analogy with the definition of a double integral (15.1.5), we define the triple integral as the limit of the triple Riemann sums in (2).

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_i, y_j, z_k) \Delta V$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z . There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y , then z , and then x , we have

$$\iiint_B f(x, y, z) \, dV = \int_a^b \int_r^s \int_c^d f(x, y, z) \, dy \, dz \, dx$$

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2 \, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x , then y , and then z , we obtain

$$\begin{aligned} \iiint_B xyz^2 \, dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx \, dy \, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2 y z^2}{2} \right]_{x=0}^{x=1} dy \, dz \\ &= \int_0^3 \int_{-1}^2 \frac{y z^2}{2} dy \, dz = \int_0^3 \left[\frac{y^2 z^2}{4} \right]_{y=-1}^{y=2} dz \\ &= \int_0^3 \frac{3z^2}{4} dz = \left[\frac{z^3}{4} \right]_0^3 = \frac{27}{4} \end{aligned}$$

Now we define the **triple integral over a general bounded region E** in three-dimensional space (a solid) by much the same procedure that we used for double integrals (15.2.2). We enclose E in a box B of the type given by Equation 1. Then we define F so that it agrees with f on E but is 0 for points in B that are outside E . By definition,

$$\iiint_E f(x, y, z) \, dV = \iiint_B F(x, y, z) \, dV$$

This integral exists if f is continuous and the boundary of E is “reasonably smooth.” The triple integral has essentially the same properties as the double integral (Properties 6–9 in Section 15.2).

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y , that is,

$$\boxed{5} \quad E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of E onto the xy -plane as shown in Figure 2. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument that led to (15.2.3), it can be shown that if E is a type 1 region given by Equation 5, then

$$\boxed{6} \quad \iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right] dA$$

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z .

In particular, if the projection D of E onto the xy -plane is a type I plane region (as in Figure 3), then

$$E = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$$

and Equation 6 becomes

$$\boxed{7} \quad \iiint_E f(x, y, z) \, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dy \, dx$$

If, on the other hand, D is a type II plane region (as in Figure 4), then

$$E = \{(x, y, z) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$$

and Equation 6 becomes

$$\boxed{8} \quad \iiint_E f(x, y, z) \, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dx \, dy$$

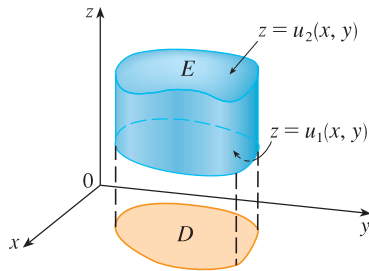


FIGURE 2
A type 1 solid region

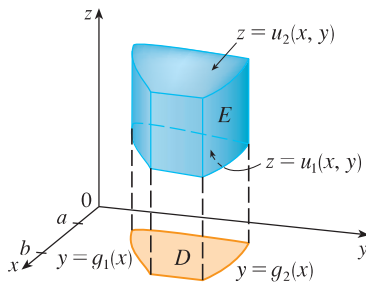


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

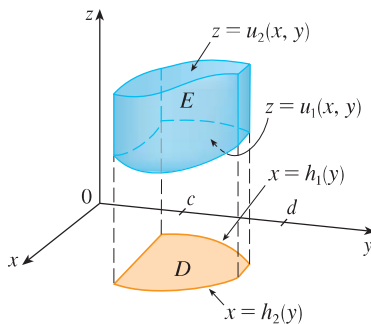


FIGURE 4
A type 1 solid region with a type II projection

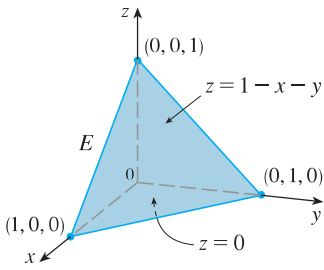


FIGURE 5

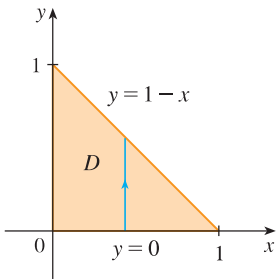


FIGURE 6

EXAMPLE 2 Evaluate $\iiint_E z \, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

SOLUTION When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (see Figure 5) and one of its projection D onto the xy -plane (see Figure 6). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula 7. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy -plane. So the projection of E is the triangular region shown in Figure 6, and we have

$$\textcircled{9} \quad E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$$

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\begin{aligned} \iiint_E z \, dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2} \right]_{z=0}^{z=1-x-y} dy \, dx \\ &= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 dy \, dx = \frac{1}{2} \int_0^1 \left[-\frac{(1-x-y)^3}{3} \right]_{y=0}^{y=1-x} dx \\ &= \frac{1}{6} \int_0^1 (1-x)^3 dx = \frac{1}{6} \left[-\frac{(1-x)^4}{4} \right]_0^1 = \frac{1}{24} \end{aligned}$$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz -plane (see Figure 7). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

$$\textcircled{10} \quad \iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) \, dx \right] dA$$

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xz -plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8). For this type of region we have

$$\textcircled{11} \quad \iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) \, dy \right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 7 and 8).

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2} \, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

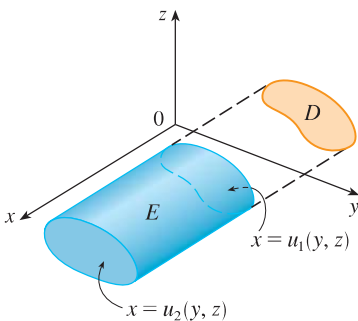


FIGURE 7

A type 2 region

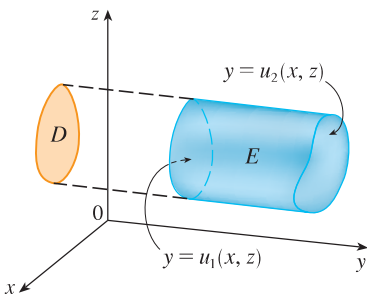


FIGURE 8

A type 3 region

SOLUTION The solid E is shown in Figure 9. If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy -plane, which is the parabolic region in Figure 10. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

TEC Visual 15.6 illustrates how solid regions (including the one in Figure 9) project onto coordinate planes.

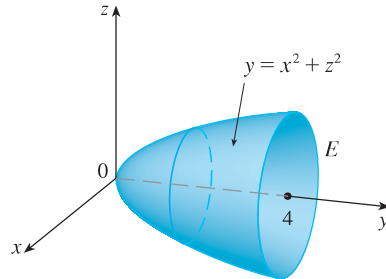


FIGURE 9
Region of integration

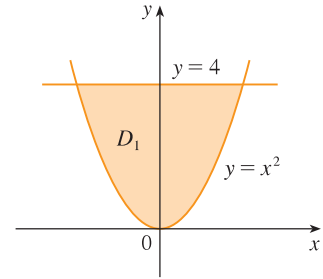


FIGURE 10
Projection onto xy -plane

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore the description of E as a type 1 region is

$$E = \{(x, y, z) \mid -2 \leq x \leq 2, x^2 \leq y \leq 4, -\sqrt{y - x^2} \leq z \leq \sqrt{y - x^2}\}$$

and so we obtain

$$\iiint_E \sqrt{x^2 + z^2} \, dV = \int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2} \, dz \, dy \, dx$$

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider E as a type 3 region. As such, its projection D_3 onto the xz -plane is the disk $x^2 + z^2 \leq 4$ shown in Figure 11.

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

$$\iiint_E \sqrt{x^2 + z^2} \, dV = \iint_{D_3} \left[\int_{x^2+z^2}^4 \sqrt{x^2 + z^2} \, dy \right] dA = \iint_{D_3} (4 - x^2 - z^2) \sqrt{x^2 + z^2} \, dA$$

Although this integral could be written as

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2) \sqrt{x^2 + z^2} \, dz \, dx$$

it's easier to convert to polar coordinates in the xz -plane: $x = r \cos \theta$, $z = r \sin \theta$. This gives

$$\begin{aligned} \iiint_E \sqrt{x^2 + z^2} \, dV &= \iint_{D_3} (4 - x^2 - z^2) \sqrt{x^2 + z^2} \, dA \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4) \, dr \\ &= 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 = \frac{128\pi}{15} \end{aligned}$$

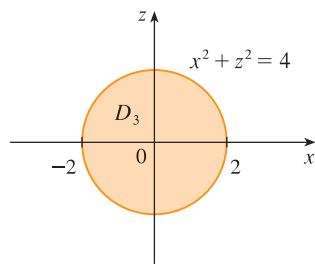


FIGURE 11
Projection onto xz -plane

☞ The most difficult step in evaluating a triple integral is setting up an expression for the region of integration (such as Equation 9 in Example 2). Remember that the limits of integration in the inner integral contain at most two variables, the limits of integration in the middle integral contain at most one variable, and the limits of integration in the outer integral must be constants.

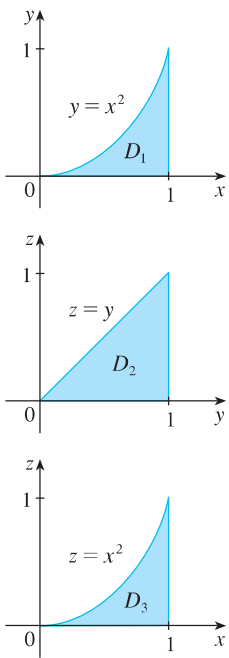


FIGURE 12
Projections of E

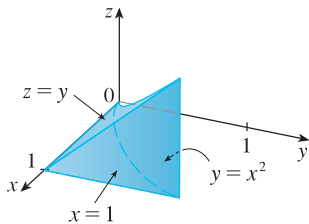


FIGURE 13
The solid E

EXAMPLE 4 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) dz dy dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x , then z , and then y .

SOLUTION We can write

$$\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) dz dy dx = \iiint_E f(x, y, z) dV$$

where $E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2, 0 \leq z \leq y\}$. This description of E enables us to write projections onto the three coordinate planes as follows:

$$\begin{aligned} \text{on the } xy\text{-plane: } D_1 &= \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\} \\ &= \{(x, y) \mid 0 \leq y \leq 1, \sqrt{y} \leq x \leq 1\} \\ \text{on the } yz\text{-plane: } D_2 &= \{(y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y\} \\ \text{on the } xz\text{-plane: } D_3 &= \{(x, z) \mid 0 \leq x \leq 1, 0 \leq z \leq x^2\} \end{aligned}$$

From the resulting sketches of the projections in Figure 12 we sketch the solid E in Figure 13. We see that it is the solid enclosed by the planes $z = 0$, $x = 1$, $y = z$ and the parabolic cylinder $y = x^2$ (or $x = \sqrt{y}$).

If we integrate first with respect to x , then z , and then y , we use an alternate description of E :

$$E = \{(x, y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y, \sqrt{y} \leq x \leq 1\}$$

Thus

$$\iiint_E f(x, y, z) dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x, y, z) dx dz dy$$

Applications of Triple Integrals

Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x) dx$ represents the area under the curve $y = f(x)$ from a to b , and if $f(x, y) \geq 0$, then the double integral $\iint_D f(x, y) dA$ represents the volume under the surface $z = f(x, y)$ and above D . The corresponding interpretation of a triple integral $\iiint_E f(x, y, z) dV$, where $f(x, y, z) \geq 0$, is not very useful because it would be the “hypervolume” of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f ; the graph of f lies in four-dimensional space.) Nonetheless, the triple integral $\iiint_E f(x, y, z) dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x , y , z , and $f(x, y, z)$.

Let’s begin with the special case where $f(x, y, z) = 1$ for all points in E . Then the triple integral does represent the volume of E :

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$$V(E) = \iiint_E dV$$

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1 dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)] dA$$

and from Section 15.2 we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

EXAMPLE 5 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION The tetrahedron T and its projection D onto the xy -plane are shown in Figures 14 and 15. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

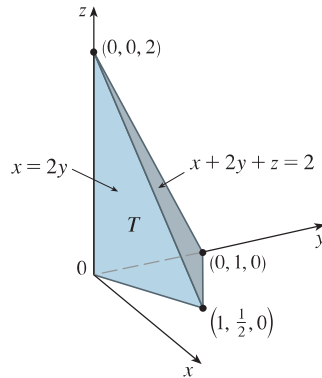


FIGURE 14

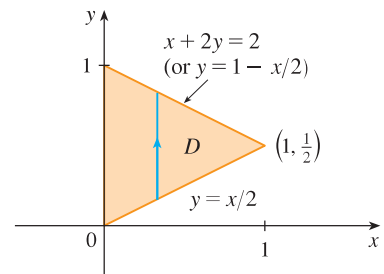


FIGURE 15

Therefore we have

$$\begin{aligned} V(T) &= \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz \, dy \, dx \\ &= \int_0^1 \int_{x/2}^{1-x/2} (2-x-2y) \, dy \, dx = \frac{1}{3} \end{aligned}$$

by the same calculation as in Example 15.2.4.

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.) ■

All the applications of double integrals in Section 15.4 can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z) , then its **mass** is

$$\boxed{13} \quad m = \iiint_E \rho(x, y, z) \, dV$$

and its **moments** about the three coordinate planes are

$$\boxed{14} \quad M_{yz} = \iiint_E x \rho(x, y, z) \, dV \quad M_{xz} = \iiint_E y \rho(x, y, z) \, dV$$

$$M_{xy} = \iiint_E z \rho(x, y, z) \, dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\boxed{15} \quad \bar{x} = \frac{M_{yz}}{m} \quad \bar{y} = \frac{M_{xz}}{m} \quad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of E . The **moments of inertia** about the three coordinate axes are

$$\begin{aligned} \boxed{16} \quad I_x &= \iiint_E (y^2 + z^2) \rho(x, y, z) \, dV & I_y &= \iiint_E (x^2 + z^2) \rho(x, y, z) \, dV \\ I_z &= \iiint_E (x^2 + y^2) \rho(x, y, z) \, dV \end{aligned}$$

As in Section 15.4, the total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z) \, dV$$

If we have three continuous random variables $X, Y,$ and $Z,$ their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z) \, dV$$

In particular,

$$P(a \leq X \leq b, c \leq Y \leq d, r \leq Z \leq s) = \int_a^b \int_c^d \int_r^s f(x, y, z) \, dz \, dy \, dx$$

The joint density function satisfies

$$f(x, y, z) \geq 0 \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) \, dz \, dy \, dx = 1$$

EXAMPLE 6 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0,$ and $x = 1.$

SOLUTION The solid E and its projection onto the xy -plane are shown in Figure 16. The lower and upper surfaces of E are the planes $z = 0$ and $z = x,$ so we describe E as a type 1 region:

$$E = \{(x, y, z) \mid -1 \leq y \leq 1, y^2 \leq x \leq 1, 0 \leq z \leq x\}$$

Then, if the density is $\rho(x, y, z) = \rho,$ the mass is

$$\begin{aligned} m &= \iiint_E \rho \, dV = \int_{-1}^1 \int_{y^2}^1 \int_0^x \rho \, dz \, dx \, dy \\ &= \rho \int_{-1}^1 \int_{y^2}^1 x \, dx \, dy = \rho \int_{-1}^1 \left[\frac{x^2}{2} \right]_{x=y^2}^{x=1} dy \\ &= \frac{\rho}{2} \int_{-1}^1 (1 - y^4) \, dy = \rho \int_0^1 (1 - y^4) \, dy \\ &= \rho \left[y - \frac{y^5}{5} \right]_0^1 = \frac{4\rho}{5} \end{aligned}$$

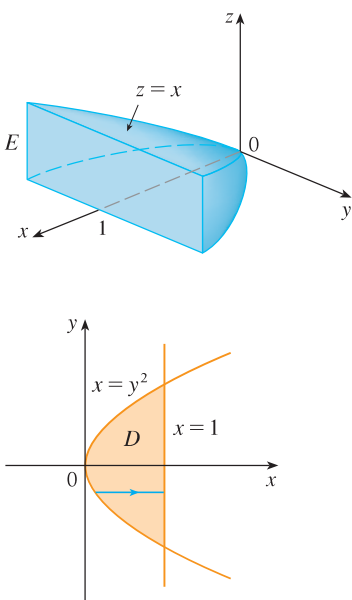


FIGURE 16

Because of the symmetry of E and ρ about the xz -plane, we can immediately say that $M_{xz} = 0$ and therefore $\bar{y} = 0$. The other moments are

$$\begin{aligned} M_{yz} &= \iiint_E x\rho \, dV = \int_{-1}^1 \int_{y^2}^1 \int_0^x x\rho \, dz \, dx \, dy \\ &= \rho \int_{-1}^1 \int_{y^2}^1 x^2 \, dx \, dy = \rho \int_{-1}^1 \left[\frac{x^3}{3} \right]_{x=y^2}^{x=1} dy \\ &= \frac{2\rho}{3} \int_0^1 (1 - y^6) \, dy = \frac{2\rho}{3} \left[y - \frac{y^7}{7} \right]_0^1 = \frac{4\rho}{7} \end{aligned}$$

$$\begin{aligned} M_{xy} &= \iiint_E z\rho \, dV = \int_{-1}^1 \int_{y^2}^1 \int_0^x z\rho \, dz \, dx \, dy \\ &= \rho \int_{-1}^1 \int_{y^2}^1 \left[\frac{z^2}{2} \right]_{z=0}^{z=x} dx \, dy = \frac{\rho}{2} \int_{-1}^1 \int_{y^2}^1 x^2 \, dx \, dy \\ &= \frac{\rho}{3} \int_0^1 (1 - y^6) \, dy = \frac{2\rho}{7} \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(\frac{5}{7}, 0, \frac{5}{14} \right)$$

15.6 EXERCISES

- Evaluate the integral in Example 1, integrating first with respect to y , then z , and then x .
- Evaluate the integral $\iiint_E (xy + z^2) \, dV$, where $E = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 3\}$ using three different orders of integration.

3–8 Evaluate the iterated integral.

- $\int_0^2 \int_0^{z^2} \int_0^{y-z} (2x - y) \, dx \, dy \, dz$
- $\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy \, dz \, dx \, dy$
- $\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y} \, dy \, dx \, dz$
- $\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1} \, dx \, dz \, dy$
- $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-z^2}} z \sin x \, dy \, dz \, dx$
- $\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z \, dz \, dy \, dx$

9–18 Evaluate the triple integral.

- $\iiint_E y \, dV$, where $E = \{(x, y, z) \mid 0 \leq x \leq 3, 0 \leq y \leq x, x - y \leq z \leq x + y\}$
- $\iiint_E e^{z/y} \, dV$, where $E = \{(x, y, z) \mid 0 \leq y \leq 1, y \leq x \leq 1, 0 \leq z \leq xy\}$
- $\iiint_E \frac{z}{x^2 + z^2} \, dV$, where $E = \{(x, y, z) \mid 1 \leq y \leq 4, y \leq z \leq 4, 0 \leq x \leq z\}$
- $\iiint_E \sin y \, dV$, where E lies below the plane $z = x$ and above the triangular region with vertices $(0, 0, 0)$, $(\pi, 0, 0)$, and $(0, \pi, 0)$
- $\iiint_E 6xy \, dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy -plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$
- $\iiint_E (x - y) \, dV$, where E is enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$, and $y = 2$
- $\iiint_T y^2 \, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

16. $\iiint_T xz \, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$
17. $\iiint_E x \, dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$
18. $\iiint_E z \, dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

19–22 Use a triple integral to find the volume of the given solid.

19. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$
20. The solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$
21. The solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$
22. The solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$

23. (a) Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral.

CAS

- (b) Use either the Table of Integrals (on Reference Pages 6–10) or a computer algebra system to find the exact value of the triple integral in part (a).

24. (a) In the **Midpoint Rule for triple integrals** we use a triple Riemann sum to approximate a triple integral over a box B , where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box B_{ijk} . Use the Midpoint Rule to estimate $\iiint_B \sqrt{x^2 + y^2 + z^2} \, dV$, where B is the cube defined by $0 \leq x \leq 4$, $0 \leq y \leq 4$, $0 \leq z \leq 4$. Divide B into eight cubes of equal size.

CAS

- (b) Use a computer algebra system to approximate the integral in part (a) correct to the nearest integer. Compare with the answer to part (a).

25–26 Use the Midpoint Rule for triple integrals (Exercise 24) to estimate the value of the integral. Divide B into eight sub-boxes of equal size.

25. $\iiint_B \cos(xyz) \, dV$, where $B = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$
26. $\iiint_B \sqrt{x} e^{xyz} \, dV$, where $B = \{(x, y, z) \mid 0 \leq x \leq 4, 0 \leq y \leq 1, 0 \leq z \leq 2\}$

27–28 Sketch the solid whose volume is given by the iterated integral.

27. $\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy \, dz \, dx$
28. $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx \, dz \, dy$

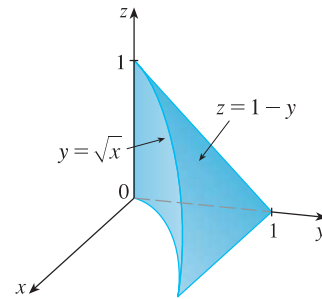
29–32 Express the integral $\iiint_E f(x, y, z) \, dV$ as an iterated integral in six different ways, where E is the solid bounded by the given surfaces.

29. $y = 4 - x^2 - 4z^2$, $y = 0$
30. $y^2 + z^2 = 9$, $x = -2$, $x = 2$
31. $y = x^2$, $z = 0$, $y + 2z = 4$
32. $x = 2$, $y = 2$, $z = 0$, $x + y - 2z = 2$

33. The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx$$

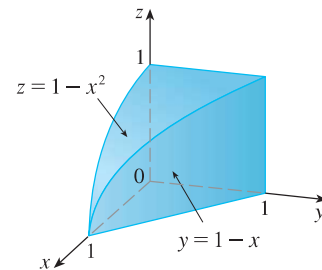
Rewrite this integral as an equivalent iterated integral in the five other orders.



34. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



35–36 Write five other iterated integrals that are equal to the given iterated integral.

35. $\int_0^1 \int_y^1 \int_0^y f(x, y, z) \, dz \, dx \, dy$
36. $\int_0^1 \int_y^1 \int_0^z f(x, y, z) \, dx \, dz \, dy$

37–38 Evaluate the triple integral using only geometric interpretation and symmetry.

37. $\iiint_C (4 + 5x^2yz^2) dV$, where C is the cylindrical region $x^2 + y^2 \leq 4$, $-2 \leq z \leq 2$

38. $\iiint_B (z^3 + \sin y + 3) dV$, where B is the unit ball $x^2 + y^2 + z^2 \leq 1$

39–42 Find the mass and center of mass of the solid E with the given density function ρ .

39. E lies above the xy -plane and below the paraboloid $z = 1 - x^2 - y^2$; $\rho(x, y, z) = 3$

40. E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$; $\rho(x, y, z) = 4$

41. E is the cube given by $0 \leq x \leq a$, $0 \leq y \leq a$, $0 \leq z \leq a$; $\rho(x, y, z) = x^2 + y^2 + z^2$

42. E is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$; $\rho(x, y, z) = y$

43–46 Assume that the solid has constant density k .

43. Find the moments of inertia for a cube with side length L if one vertex is located at the origin and three edges lie along the coordinate axes.

44. Find the moments of inertia for a rectangular brick with dimensions a , b , and c and mass M if the center of the brick is situated at the origin and the edges are parallel to the coordinate axes.

45. Find the moment of inertia about the z -axis of the solid cylinder $x^2 + y^2 \leq a^2$, $0 \leq z \leq h$.

46. Find the moment of inertia about the z -axis of the solid cone $\sqrt{x^2 + y^2} \leq z \leq h$.

47–48 Set up, but do not evaluate, integral expressions for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z -axis.

47. The solid of Exercise 21; $\rho(x, y, z) = \sqrt{x^2 + y^2}$

48. The hemisphere $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$;
 $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

CAS 49. Let E be the solid in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, and $z = 0$ with the density function $\rho(x, y, z) = 1 + x + y + z$. Use a computer algebra system to find the exact values of the following quantities for E .

- The mass
- The center of mass
- The moment of inertia about the z -axis

CAS 50. If E is the solid of Exercise 18 with density function $\rho(x, y, z) = x^2 + y^2$, find the following quantities, correct to three decimal places.

- The mass
- The center of mass
- The moment of inertia about the z -axis

51. The joint density function for random variables X , Y , and Z is $f(x, y, z) = Cxyz$ if $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 2$, and $f(x, y, z) = 0$ otherwise.

- Find the value of the constant C .
- Find $P(X \leq 1, Y \leq 1, Z \leq 1)$.
- Find $P(X + Y + Z \leq 1)$.

52. Suppose X , Y , and Z are random variables with joint density function $f(x, y, z) = Ce^{-(0.5x+0.2y+0.1z)}$ if $x \geq 0$, $y \geq 0$, $z \geq 0$, and $f(x, y, z) = 0$ otherwise.

- Find the value of the constant C .
- Find $P(X \leq 1, Y \leq 1)$.
- Find $P(X \leq 1, Y \leq 1, Z \leq 1)$.

53–54 The **average value** of a function $f(x, y, z)$ over a solid region E is defined to be

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z) dV$$

where $V(E)$ is the volume of E . For instance, if ρ is a density function, then ρ_{ave} is the average density of E .

53. Find the average value of the function $f(x, y, z) = xyz$ over the cube with side length L that lies in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

54. Find the average height of the points in the solid hemisphere $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$.

55. (a) Find the region E for which the triple integral

$$\iiint_E (1 - x^2 - 2y^2 - 3z^2) dV$$

is a maximum.

CAS (b) Use a computer algebra system to calculate the exact maximum value of the triple integral in part (a).

DISCOVERY PROJECT VOLUMES OF HYPERSPHERES

In this project we find formulas for the volume enclosed by a hypersphere in n -dimensional space.

1. Use a double integral and trigonometric substitution, together with Formula 64 in the Table of Integrals, to find the area of a circle with radius r .
2. Use a triple integral and trigonometric substitution to find the volume of a sphere with radius r .
3. Use a quadruple integral to find the (4-dimensional) volume enclosed by the hypersphere $x^2 + y^2 + z^2 + w^2 = r^2$ in \mathbb{R}^4 . (Use only trigonometric substitution and the reduction formulas for $\int \sin^n x \, dx$ or $\int \cos^n x \, dx$.)
4. Use an n -tuple integral to find the volume enclosed by a hypersphere of radius r in n -dimensional space \mathbb{R}^n . [Hint: The formulas are different for n even and n odd.]

15.7 Triple Integrals in Cylindrical Coordinates

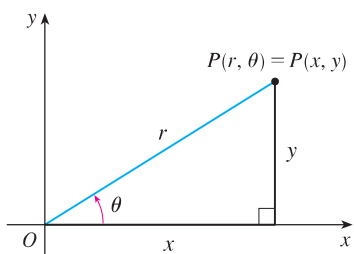


FIGURE 1

In plane geometry the polar coordinate system is used to give a convenient description of certain curves and regions. (See Section 10.3.) Figure 1 enables us to recall the connection between polar and Cartesian coordinates. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ) , then, from the figure,

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta \\ r^2 &= x^2 + y^2 & \tan \theta &= \frac{y}{x} \end{aligned}$$

In three dimensions there is a coordinate system, called *cylindrical coordinates*, that is similar to polar coordinates and gives convenient descriptions of some commonly occurring surfaces and solids. As we will see, some triple integrals are much easier to evaluate in cylindrical coordinates.

Cylindrical Coordinates

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z) , where r and θ are polar coordinates of the projection of P onto the xy -plane and z is the directed distance from the xy -plane to P . (See Figure 2.)

To convert from cylindrical to rectangular coordinates, we use the equations

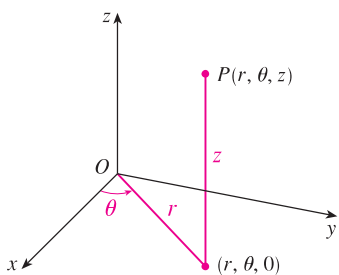


FIGURE 2

The cylindrical coordinates of a point

$$\boxed{1} \quad x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$\boxed{2} \quad r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad z = z$$

EXAMPLE 1

- (a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
 (b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

SOLUTION

(a) The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in Figure 3. From Equations 1, its rectangular coordinates are

$$x = 2 \cos \frac{2\pi}{3} = 2 \left(-\frac{1}{2} \right) = -1$$

$$y = 2 \sin \frac{2\pi}{3} = 2 \left(\frac{\sqrt{3}}{2} \right) = \sqrt{3}$$

$$z = 1$$

So the point is $(-1, \sqrt{3}, 1)$ in rectangular coordinates.

(b) From Equations 2 we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan \theta = \frac{-3}{3} = -1 \quad \text{so} \quad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $(3\sqrt{2}, 7\pi/4, -7)$. Another is $(3\sqrt{2}, -\pi/4, -7)$. As with polar coordinates, there are infinitely many choices. ■

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z -axis is chosen to coincide with this axis of symmetry. For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z -axis. In cylindrical coordinates this cylinder has the very simple equation $r = c$. (See Figure 4.) This is the reason for the name “cylindrical” coordinates.

EXAMPLE 2 Describe the surface whose equation in cylindrical coordinates is $z = r$.

SOLUTION The equation says that the z -value, or height, of each point on the surface is the same as r , the distance from the point to the z -axis. Because θ doesn't appear, it can vary. So any horizontal trace in the plane $z = k$ ($k > 0$) is a circle of radius k . These traces suggest that the surface is a cone. This prediction can be confirmed by converting the equation into rectangular coordinates. From the first equation in (2) we have

$$z^2 = r^2 = x^2 + y^2$$

We recognize the equation $z^2 = x^2 + y^2$ (by comparison with Table 1 in Section 12.6) as being a circular cone whose axis is the z -axis (see Figure 5). ■

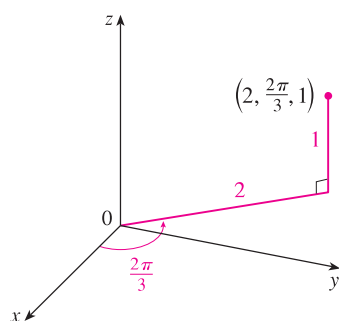


FIGURE 3

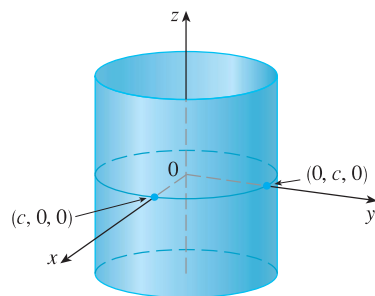


FIGURE 4

$r = c$, a cylinder

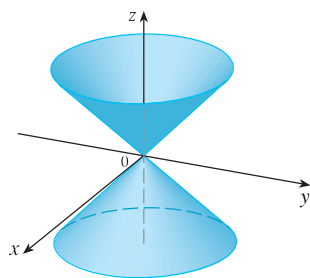


FIGURE 5

$z = r$, a cone

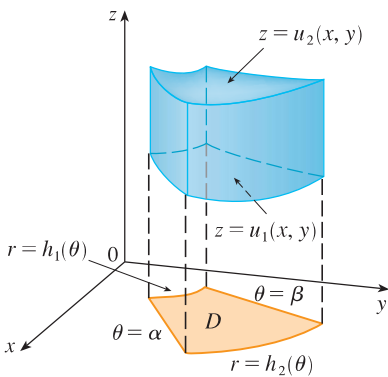


FIGURE 6

Evaluating Triple Integrals with Cylindrical Coordinates

Suppose that E is a type 1 region whose projection D onto the xy -plane is conveniently described in polar coordinates (see Figure 6). In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

We know from Equation 15.6.6 that

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right] \, dA$$

But we also know how to evaluate double integrals in polar coordinates. In fact, combining Equation 3 with Equation 15.3.3, we obtain

$$\iiint_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta$$

Formula 4 is the **formula for triple integration in cylindrical coordinates**. It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r \cos \theta$, $y = r \sin \theta$, leaving z as it is, using the appropriate limits of integration for z , r , and θ , and replacing dV by $r \, dz \, dr \, d\theta$. (Figure 7 shows how to remember this.) It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

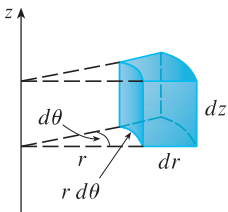


FIGURE 7
Volume element in cylindrical coordinates: $dV = r \, dz \, dr \, d\theta$

EXAMPLE 3 A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E .

SOLUTION In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z -axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant. Therefore, from Formula 15.6.13, the mass of E is

$$\begin{aligned} m &= \iiint_E K\sqrt{x^2 + y^2} \, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr) \, r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 Kr^2 [4 - (1 - r^2)] \, dr \, d\theta = K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4) \, dr \\ &= 2\pi K \left[r^3 + \frac{r^5}{5} \right]_0^1 = \frac{12\pi K}{5} \end{aligned}$$

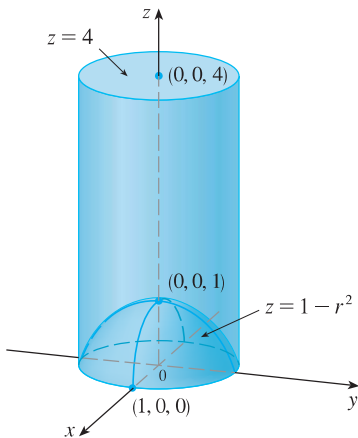


FIGURE 8

EXAMPLE 4 Evaluate $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) dz dy dx$.

SOLUTION This iterated integral is a triple integral over the solid region

$$E = \{(x, y, z) \mid -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}, \sqrt{x^2+y^2} \leq z \leq 2\}$$

and the projection of E onto the xy -plane is the disk $x^2 + y^2 \leq 4$. The lower surface of E is the cone $z = \sqrt{x^2 + y^2}$ and its upper surface is the plane $z = 2$. (See Figure 9.)

This region has a much simpler description in cylindrical coordinates:

$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, r \leq z \leq 2\}$$

Therefore we have

$$\begin{aligned} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) dz dy dx &= \iiint_E (x^2 + y^2) dV \\ &= \int_0^{2\pi} \int_0^2 \int_r^2 r^2 r dz dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 r^3(2-r) dr \\ &= 2\pi \left[\frac{1}{2}r^4 - \frac{1}{5}r^5 \right]_0^2 = \frac{16}{5}\pi \end{aligned}$$

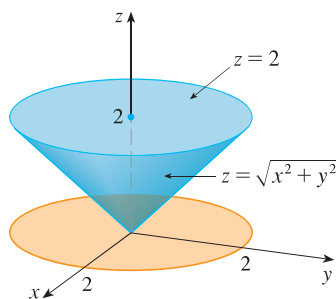


FIGURE 9

15.7 EXERCISES

1–2 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(4, \pi/3, -2)$ (b) $(2, -\pi/2, 1)$
2. (a) $(\sqrt{2}, 3\pi/4, 2)$ (b) $(1, 1, 1)$

3–4 Change from rectangular to cylindrical coordinates.

3. (a) $(-1, 1, 1)$ (b) $(-2, 2\sqrt{3}, 3)$
4. (a) $(-\sqrt{2}, \sqrt{2}, 1)$ (b) $(2, 2, 2)$

5–6 Describe in words the surface whose equation is given.

5. $r = 2$ (b) $\theta = \pi/6$

7–8 Identify the surface whose equation is given.

7. $r^2 + z^2 = 4$ (b) $r = 2 \sin \theta$


9–10 Write the equations in cylindrical coordinates.

9. (a) $x^2 - x + y^2 + z^2 = 1$ (b) $z = x^2 - y^2$
10. (a) $2x^2 + 2y^2 - z^2 = 4$ (b) $2x - y + z = 1$

11–12 Sketch the solid described by the given inequalities.

11. $r^2 \leq z \leq 8 - r^2$
12. $0 \leq \theta \leq \pi/2, r \leq z \leq 2$

13. A cylindrical shell is 20 cm long, with inner radius 6 cm and outer radius 7 cm. Write inequalities that describe the shell in an appropriate coordinate system. Explain how you have positioned the coordinate system with respect to the shell.

 **14.** Use a graphing device to draw the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$.

15–16 Sketch the solid whose volume is given by the integral and evaluate the integral.

15. $\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} r dz dr d\theta$ (b) $\int_0^2 \int_0^{2\pi} \int_0^r r dz d\theta dr$

17–28 Use cylindrical coordinates.

17. Evaluate $\iiint_E \sqrt{x^2 + y^2} dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.
18. Evaluate $\iiint_E z dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

19. Evaluate $\iiint_E (x + y + z) dV$, where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.
20. Evaluate $\iiint_E (x - y) dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy -plane, and below the plane $z = y + 4$.
21. Evaluate $\iiint_E x^2 dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.
22. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.
23. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.
24. Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.
25. (a) Find the volume of the region E that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.
 (b) Find the centroid of E (the center of mass in the case where the density is constant).
26. (a) Find the volume of the solid that the cylinder $r = a \cos \theta$ cuts out of the sphere of radius a centered at the origin.
 (b) Illustrate the solid of part (a) by graphing the sphere and the cylinder on the same screen.
27. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ ($a > 0$) if S has constant density K .
28. Find the mass of a ball B given by $x^2 + y^2 + z^2 \leq a^2$ if the density at any point is proportional to its distance from the z -axis.

29–30 Evaluate the integral by changing to cylindrical coordinates.

29.
$$\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^2 xz \, dz \, dx \, dy$$

30.
$$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2} \, dz \, dy \, dx$$

31. When studying the formation of mountain ranges, geologists estimate the amount of work required to lift a mountain from sea level. Consider a mountain that is essentially in the shape of a right circular cone. Suppose that the weight density of the material in the vicinity of a point P is $g(P)$ and the height is $h(P)$.
- (a) Find a definite integral that represents the total work done in forming the mountain.
- (b) Assume that Mount Fuji in Japan is in the shape of a right circular cone with radius 62,000 ft, height 12,400 ft, and density a constant 200 lb/ft³. How much work was done in forming Mount Fuji if the land was initially at sea level?

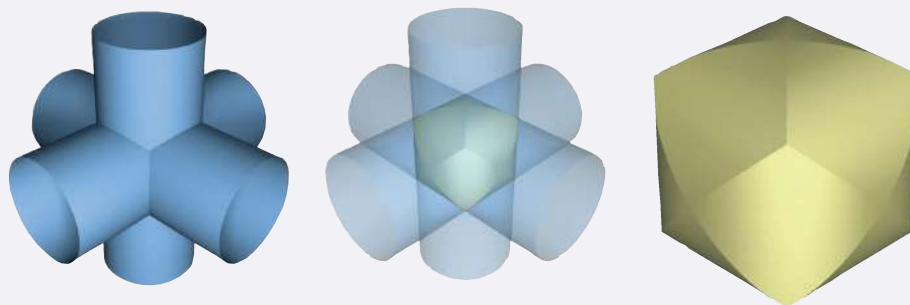


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DISCOVERY PROJECT

THE INTERSECTION OF THREE CYLINDERS

The figure shows the solid enclosed by three circular cylinders with the same diameter that intersect at right angles. In this project we compute its volume and determine how its shape changes if the cylinders have different diameters.



1. Sketch carefully the solid enclosed by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Indicate the positions of the coordinate axes and label the faces with the equations of the corresponding cylinders.
2. Find the volume of the solid in Problem 1.
- CAS** 3. Use a computer algebra system to draw the edges of the solid.
4. What happens to the solid in Problem 1 if the radius of the first cylinder is different from 1? Illustrate with a hand-drawn sketch or a computer graph.
5. If the first cylinder is $x^2 + y^2 = a^2$, where $a < 1$, set up, but do not evaluate, a double integral for the volume of the solid. What if $a > 1$?

15.8 Triple Integrals in Spherical Coordinates

Another useful coordinate system in three dimensions is the *spherical coordinate system*. It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

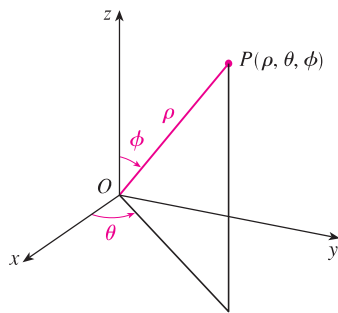


FIGURE 1
The spherical coordinates of a point

■ Spherical Coordinates

The **spherical coordinates** (ρ, θ, ϕ) of a point P in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to P , θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z -axis and the line segment OP . Note that

$$\rho \geq 0 \quad 0 \leq \phi \leq \pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point. For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 2); this is the reason for the name “spherical” coordinates. The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\phi = c$ represents a half-cone with the z -axis as its axis (see Figure 4).

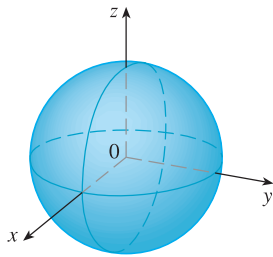


FIGURE 2 $\rho = c$, a sphere

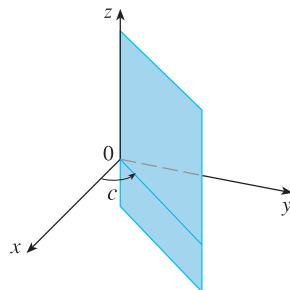


FIGURE 3 $\theta = c$, a half-plane

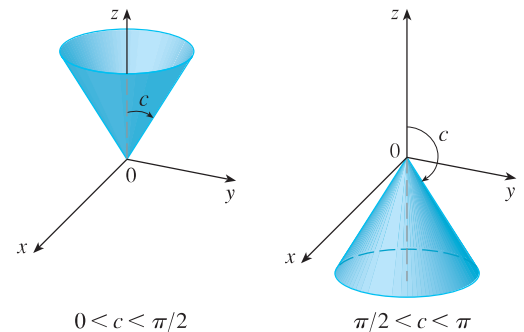


FIGURE 4 $\phi = c$, a half-cone