

16.1 Vector Fields

The vectors in Figure 1 are air velocity vectors that indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area. We see at a glance from the largest arrows in part (a) that the greatest wind speeds at that time occurred as the winds entered the bay across the Golden Gate Bridge. Part (b) shows the very different wind pattern 12 hours earlier. Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field*.

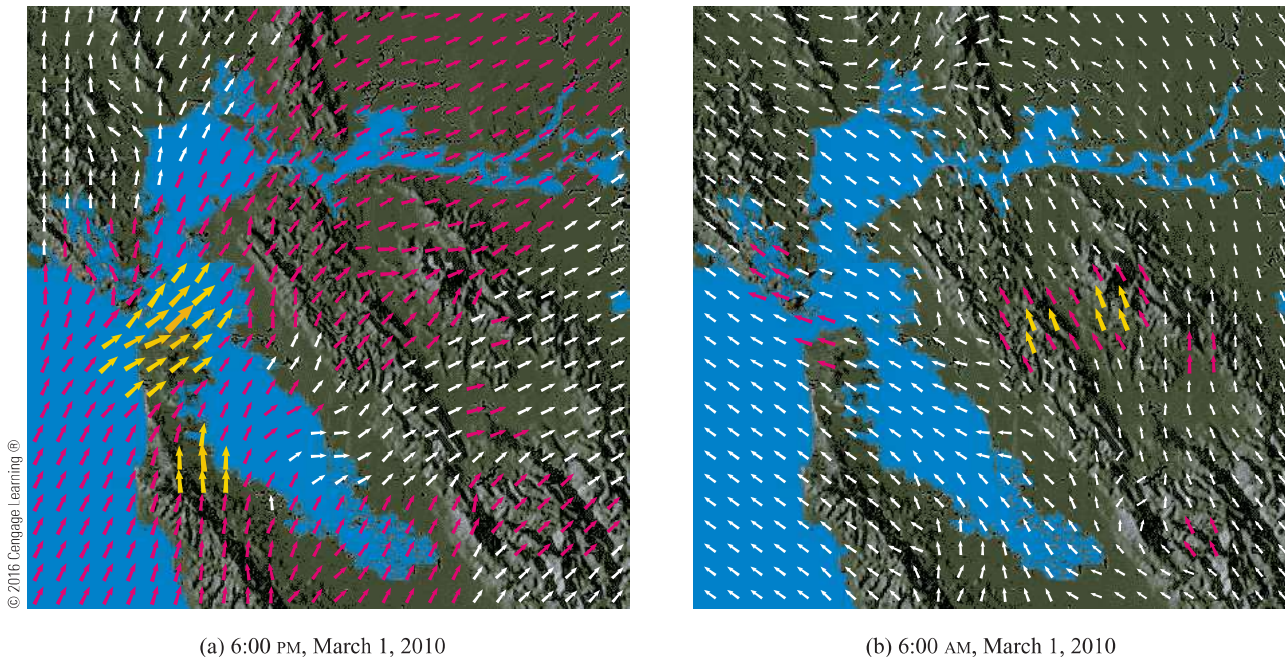


FIGURE 1 Velocity vector fields showing San Francisco Bay wind patterns

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.

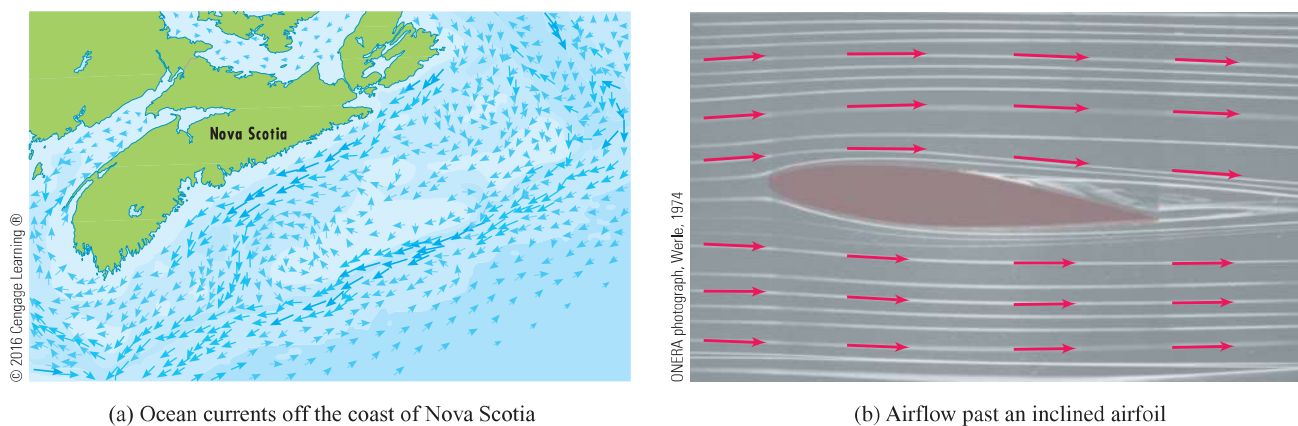


FIGURE 2
Velocity vector fields

Another type of vector field, called a *force field*, associates a force vector with each point in a region. An example is the gravitational force field that we will look at in Example 4.

In general, a vector field is a function whose domain is a set of points in \mathbb{R}^2 (or \mathbb{R}^3) and whose range is a set of vectors in V_2 (or V_3).

1 Definition Let D be a set in \mathbb{R}^2 (a plane region). A **vector field on \mathbb{R}^2** is a function \mathbf{F} that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

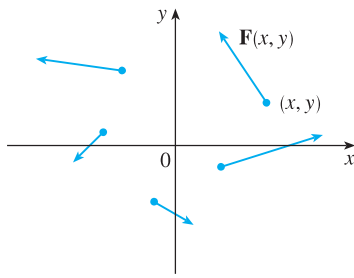


FIGURE 3
Vector field on \mathbb{R}^2

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y) . Of course, it's impossible to do this for all points (x, y) , but we can gain a reasonable impression of \mathbf{F} by doing it for a few representative points in D as in Figure 3. Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

or, for short,

$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

2 Definition Let E be a subset of \mathbb{R}^3 . A **vector field on \mathbb{R}^3** is a function \mathbf{F} that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

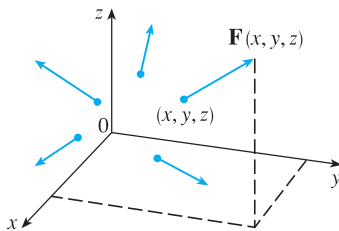


FIGURE 4
Vector field on \mathbb{R}^3

A vector field \mathbf{F} on \mathbb{R}^3 is pictured in Figure 4. We can express it in terms of its component functions P , Q , and R as

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$

As with the vector functions in Section 13.1, we can define continuity of vector fields and show that \mathbf{F} is continuous if and only if its component functions P , Q , and R are continuous.

We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z \rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$. Then \mathbf{F} becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector \mathbf{x} .

EXAMPLE 1 A vector field on \mathbb{R}^2 is defined by $\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$. Describe \mathbf{F} by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

SOLUTION Since $\mathbf{F}(1, 0) = \mathbf{j}$, we draw the vector $\mathbf{j} = \langle 0, 1 \rangle$ starting at the point $(1, 0)$ in Figure 5. Since $\mathbf{F}(0, 1) = -\mathbf{i}$, we draw the vector $\langle -1, 0 \rangle$ with starting point $(0, 1)$. Continuing in this way, we calculate several other representative values of $\mathbf{F}(x, y)$ in the table and draw the corresponding vectors to represent the vector field in Figure 5.

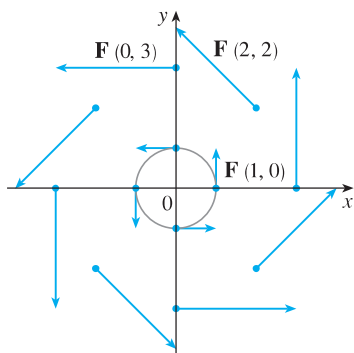


FIGURE 5
 $\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1 \rangle$	$(-1, 0)$	$\langle 0, -1 \rangle$
$(2, 2)$	$\langle -2, 2 \rangle$	$(-2, -2)$	$\langle 2, -2 \rangle$
$(3, 0)$	$\langle 0, 3 \rangle$	$(-3, 0)$	$\langle 0, -3 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$	$(0, -1)$	$\langle 1, 0 \rangle$
$(-2, 2)$	$\langle -2, -2 \rangle$	$(2, -2)$	$\langle 2, 2 \rangle$
$(0, 3)$	$\langle -3, 0 \rangle$	$(0, -3)$	$\langle 3, 0 \rangle$

It appears from Figure 5 that each arrow is tangent to a circle with center the origin. To confirm this, we take the dot product of the position vector $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ with the vector $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x, y)$:

$$\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\mathbf{i} + y\mathbf{j}) \cdot (-y\mathbf{i} + x\mathbf{j}) = -xy + yx = 0$$

This shows that $\mathbf{F}(x, y)$ is perpendicular to the position vector $\langle x, y \rangle$ and is therefore tangent to a circle with center the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$. Notice also that

$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = |\mathbf{x}|$$

so the magnitude of the vector $\mathbf{F}(x, y)$ is equal to the radius of the circle. ■

Some computer algebra systems are capable of plotting vector fields in two or three dimensions. They give a better impression of the vector field than is possible by hand because the computer can plot a large number of representative vectors. Figure 6 shows a computer plot of the vector field in Example 1; Figures 7 and 8 show two other vector fields. Notice that the computer scales the lengths of the vectors so they are not too long and yet are proportional to their true lengths.

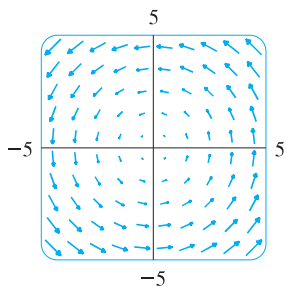


FIGURE 6
 $\mathbf{F}(x, y) = \langle -y, x \rangle$

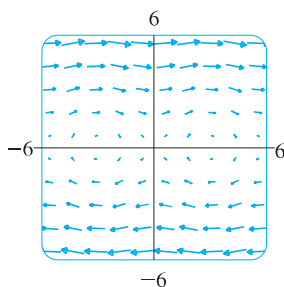


FIGURE 7
 $\mathbf{F}(x, y) = \langle y, \sin x \rangle$

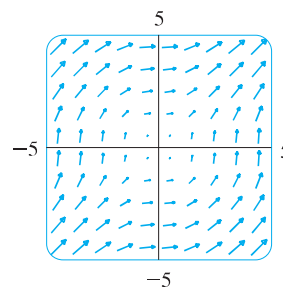


FIGURE 8
 $\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

EXAMPLE 2 Sketch the vector field on \mathbb{R}^3 given by $\mathbf{F}(x, y, z) = z\mathbf{k}$.

SOLUTION The sketch is shown in Figure 9. Notice that all vectors are vertical and point upward above the xy -plane or downward below it. The magnitude increases with the distance from the xy -plane.

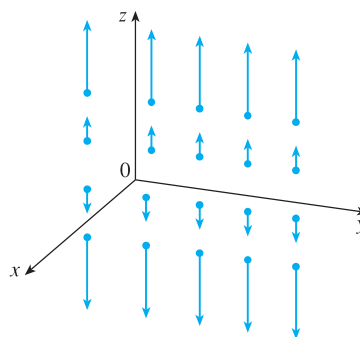


FIGURE 9
 $\mathbf{F}(x, y, z) = z\mathbf{k}$

We were able to draw the vector field in Example 2 by hand because of its particularly simple formula. Most three-dimensional vector fields, however, are virtually impossible

to sketch by hand and so we need to resort to computer software. Examples are shown in Figures 10, 11, and 12. Notice that the vector fields in Figures 10 and 11 have similar formulas, but all the vectors in Figure 11 point in the general direction of the negative y -axis because their y -components are all -2 . If the vector field in Figure 12 represents a velocity field, then a particle would be swept upward and would spiral around the z -axis in the clockwise direction as viewed from above.

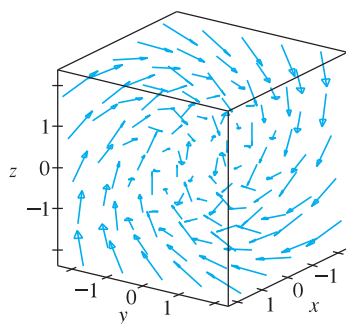


FIGURE 10
 $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

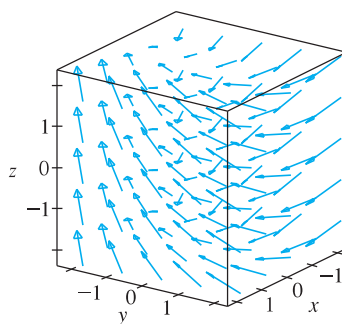


FIGURE 11
 $\mathbf{F}(x, y, z) = y\mathbf{i} - 2\mathbf{j} + x\mathbf{k}$

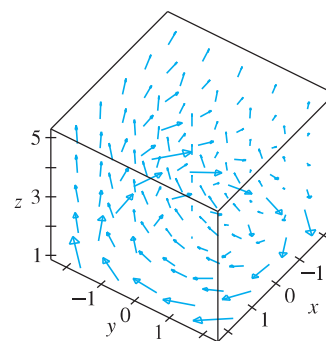


FIGURE 12
 $\mathbf{F}(x, y, z) = \frac{y}{z}\mathbf{i} - \frac{x}{z}\mathbf{j} + \frac{z}{4}\mathbf{k}$

TEC In Visual 16.1 you can rotate the vector fields in Figures 10–12 as well as additional fields.

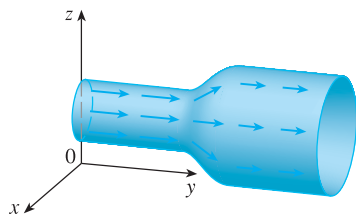


FIGURE 13
Velocity field in fluid flow

EXAMPLE 3 Imagine a fluid flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point (x, y, z) . Then \mathbf{V} assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so \mathbf{V} is a vector field on \mathbb{R}^3 called a **velocity field**. A possible velocity field is illustrated in Figure 13. The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics. For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel. We have seen other examples of velocity fields in Figures 1 and 2. ■

EXAMPLE 4 Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. (This is an example of an inverse square law.) Let's assume that the object with mass M is located at the origin in \mathbb{R}^3 . (For instance, M could be the mass of the earth and the origin would be at its center.) Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$. The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{3} \quad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3} \mathbf{x}$$

[Physicists often use the notation \mathbf{r} instead of \mathbf{x} for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -(mMG/r^3)\mathbf{r}$.] The function given by Equation 3 is

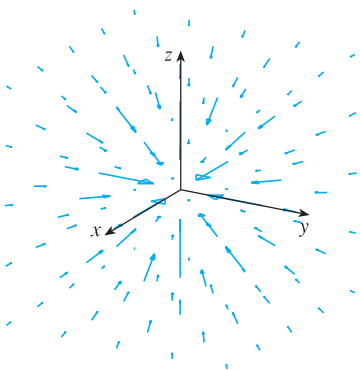


FIGURE 14
Gravitational force field

an example of a vector field, called the **gravitational field**, because it associates a vector [the force $\mathbf{F}(\mathbf{x})$] with every point \mathbf{x} in space.

Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k}$$

The gravitational field \mathbf{F} is pictured in Figure 14. ■

EXAMPLE 5 Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{4} \quad \mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3} \mathbf{x}$$

where ε is a constant (that depends on the units used). For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive. Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force \mathbf{F} , physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q} \mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3} \mathbf{x}$$

Then \mathbf{E} is a vector field on \mathbb{R}^3 called the **electric field** of Q . ■

■ Gradient Fields

If f is a scalar function of two variables, recall from Section 14.6 that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j}$$

Therefore ∇f is really a vector field on \mathbb{R}^2 and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on \mathbb{R}^3 given by

$$\nabla f(x, y, z) = f_x(x, y, z) \mathbf{i} + f_y(x, y, z) \mathbf{j} + f_z(x, y, z) \mathbf{k}$$

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f . How are they related?

SOLUTION The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = 2xy \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$$

Figure 15 shows a contour map of f with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from

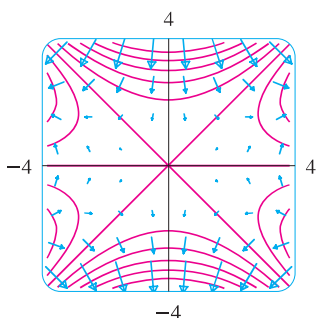


FIGURE 15

Section 14.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph. ■

A vector field \mathbf{F} is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for \mathbf{F} .

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field \mathbf{F} in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} \\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 16.3 and 16.5 we will learn how to tell whether or not a given vector field is conservative.

16.1 EXERCISES

1–10 Sketch the vector field \mathbf{F} by drawing a diagram like Figure 5 or Figure 9.

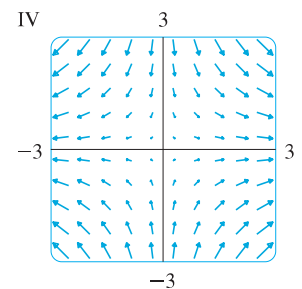
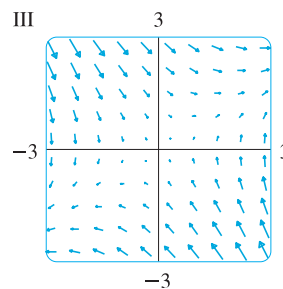
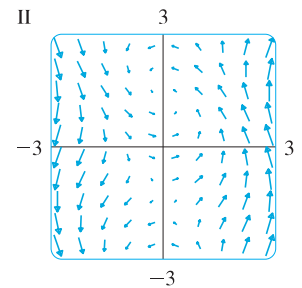
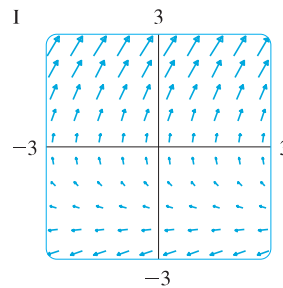
1. $\mathbf{F}(x, y) = 0.3\mathbf{i} - 0.4\mathbf{j}$
2. $\mathbf{F}(x, y) = \frac{1}{2}x\mathbf{i} + y\mathbf{j}$
3. $\mathbf{F}(x, y) = -\frac{1}{2}\mathbf{i} + (y - x)\mathbf{j}$
4. $\mathbf{F}(x, y) = y\mathbf{i} + (x + y)\mathbf{j}$
5. $\mathbf{F}(x, y) = \frac{y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}$
6. $\mathbf{F}(x, y) = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$
7. $\mathbf{F}(x, y, z) = \mathbf{i}$
8. $\mathbf{F}(x, y, z) = z\mathbf{i}$
9. $\mathbf{F}(x, y, z) = -y\mathbf{i}$
10. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{k}$

11–14 Match the vector fields \mathbf{F} with the plots labeled I–IV. Give reasons for your choices.

11. $\mathbf{F}(x, y) = \langle x, -y \rangle$
12. $\mathbf{F}(x, y) = \langle y, x - y \rangle$

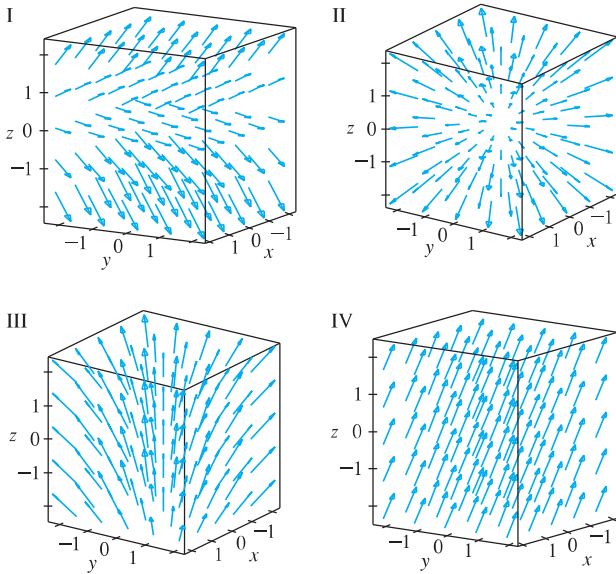
13. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

14. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$



15–18 Match the vector fields \mathbf{F} on \mathbb{R}^3 with the plots labeled I–IV. Give reasons for your choices.

- 15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
- 16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\mathbf{j} + z\mathbf{k}$
- 17. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 3\mathbf{k}$
- 18. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



CAS 19. If you have a CAS that plots vector fields (the command is `fieldplot` in Maple and `PlotVectorField` or `VectorPlot` in Mathematica), use it to plot

$$\mathbf{F}(x, y) = (y^2 - 2xy)\mathbf{i} + (3xy - 6x^2)\mathbf{j}$$

Explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

CAS 20. Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y \rangle$ and $r = |\mathbf{x}|$. Use a CAS to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

21–24 Find the gradient vector field of f .

- 21. $f(x, y) = y \sin(xy)$
- 22. $f(s, t) = \sqrt{2s + 3t}$
- 23. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
- 24. $f(x, y, z) = x^2ye^{yz}$

25–26 Find the gradient vector field ∇f of f and sketch it.

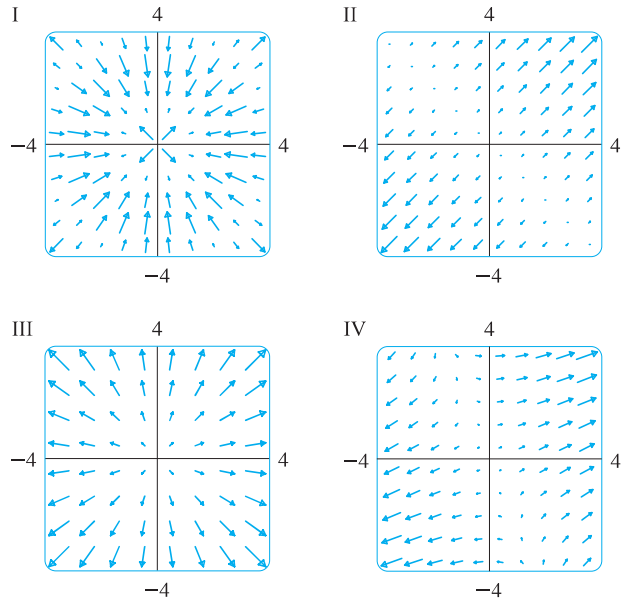
- 25. $f(x, y) = \frac{1}{2}(x - y)^2$
- 26. $f(x, y) = \frac{1}{2}(x^2 - y^2)$

CAS 27–28 Plot the gradient vector field of f together with a contour map of f . Explain how they are related to each other.

- 27. $f(x, y) = \ln(1 + x^2 + 2y^2)$
- 28. $f(x, y) = \cos x - 2 \sin y$

29–32 Match the functions f with the plots of their gradient vector fields labeled I–IV. Give reasons for your choices.

- 29. $f(x, y) = x^2 + y^2$
- 30. $f(x, y) = x(x + y)$
- 31. $f(x, y) = (x + y)^2$
- 32. $f(x, y) = \sin \sqrt{x^2 + y^2}$



33. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

34. At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time $t = 1.05$.

35. The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

- (a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?
- (b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point $(1, 1)$.

- 36. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?
- (b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $dy/dx = x$.
- (c) If a particle starts at the origin in the velocity field given by \mathbf{F} , find an equation of the path it follows.

16.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C . Such integrals are called *line integrals*, although “curve integrals” would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

We start with a plane curve C given by the parametric equations

$$\boxed{1} \quad x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

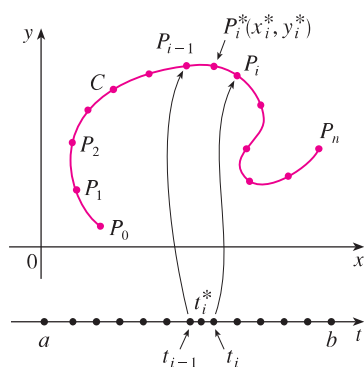


FIGURE 1

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, and we assume that C is a smooth curve. [This means that \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. See Section 13.3.] If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$. (See Figure 1.) We choose any point $P_i^*(x_i^*, y_i^*)$ in the i th subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if f is any function of two variables whose domain includes the curve C , we evaluate f at the point (x_i^*, y_i^*) , multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

if this limit exists.

In Section 10.2 we found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

A similar type of argument can be used to show that if f is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

$$\boxed{3} \quad \int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b .

The arc length function s is discussed in Section 13.3.

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So the way to remember Formula 3 is to express everything in terms of the parameter t : Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \leq x \leq b$. Formula 3 then becomes

$$\int_C f(x, y) ds = \int_a^b f(x, 0) dx$$

and so the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if $f(x, y) \geq 0$, $\int_C f(x, y) ds$ represents the area of one side of the “fence” or “curtain” in Figure 2, whose base is C and whose height above the point (x, y) is $f(x, y)$.

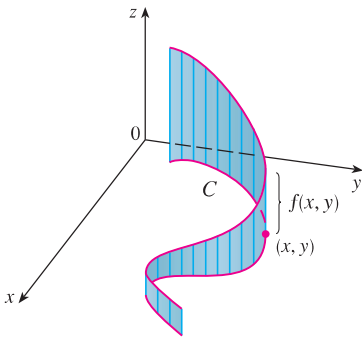


FIGURE 2

EXAMPLE 1 Evaluate $\int_C (2 + x^2y) ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

SOLUTION In order to use Formula 3, we first need parametric equations to represent C . Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \quad y = \sin t$$

and the upper half of the circle is described by the parameter interval $0 \leq t \leq \pi$. (See Figure 3.) Therefore Formula 3 gives

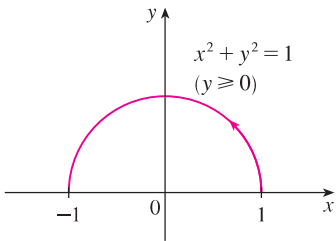


FIGURE 3

$$\begin{aligned} \int_C (2 + x^2y) ds &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t} dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) dt = \left[2t - \frac{\cos^3 t}{3} \right]_0^\pi \\ &= 2\pi + \frac{2}{3} \end{aligned}$$

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves C_1, C_2, \dots, C_n , where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i . Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C :

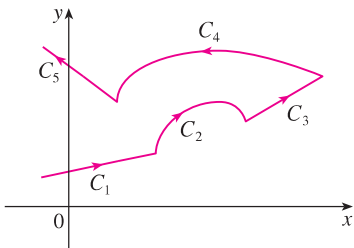


FIGURE 4
A piecewise-smooth curve

$$\int_C f(x, y) ds = \int_{C_1} f(x, y) ds + \int_{C_2} f(x, y) ds + \dots + \int_{C_n} f(x, y) ds$$

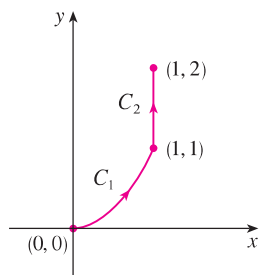


FIGURE 5

 $C = C_1 \cup C_2$

EXAMPLE 2 Evaluate $\int_C 2x \, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

SOLUTION The curve C is shown in Figure 5. C_1 is the graph of a function of x , so we can choose x as the parameter and the equations for C_1 become

$$x = x \quad y = x^2 \quad 0 \leq x \leq 1$$

Therefore

$$\begin{aligned} \int_{C_1} 2x \, ds &= \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^1 2x \sqrt{1 + 4x^2} \, dx \\ &= \frac{1}{4} \cdot \frac{2}{3} (1 + 4x^2)^{3/2} \Big|_0^1 = \frac{5\sqrt{5} - 1}{6} \end{aligned}$$

On C_2 we choose y as the parameter, so the equations of C_2 are

$$x = 1 \quad y = y \quad 1 \leq y \leq 2$$

and
$$\int_{C_2} 2x \, ds = \int_1^2 2(1) \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2} \, dy = \int_1^2 2 \, dy = 2$$

Thus
$$\int_C 2x \, ds = \int_{C_1} 2x \, ds + \int_{C_2} 2x \, ds = \frac{5\sqrt{5} - 1}{6} + 2$$
 ■

Any physical interpretation of a line integral $\int_C f(x, y) \, ds$ depends on the physical interpretation of the function f . Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C . Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*) \Delta s_i$ and so the total mass of the wire is approximately $\sum \rho(x_i^*, y_i^*) \Delta s_i$. By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(x_i^*, y_i^*) \Delta s_i = \int_C \rho(x, y) \, ds$$

[For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.] The **center of mass** of the wire with density function ρ is located at the point (\bar{x}, \bar{y}) , where

$$\boxed{4} \quad \bar{x} = \frac{1}{m} \int_C x \rho(x, y) \, ds \quad \bar{y} = \frac{1}{m} \int_C y \rho(x, y) \, ds$$

Other physical interpretations of line integrals will be discussed later in this chapter.

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geq 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

SOLUTION As in Example 1 we use the parametrization $x = \cos t$, $y = \sin t$, $0 \leq t \leq \pi$, and find that $ds = dt$. The linear density is

$$\rho(x, y) = k(1 - y)$$

where k is a constant, and so the mass of the wire is

$$m = \int_C k(1 - y) \, ds = \int_0^\pi k(1 - \sin t) \, dt = k[t + \cos t]_0^\pi = k(\pi - 2)$$

From Equations 4 we have

$$\begin{aligned} \bar{y} &= \frac{1}{m} \int_C y \rho(x, y) \, ds = \frac{1}{k(\pi - 2)} \int_C y k(1 - y) \, ds \\ &= \frac{1}{\pi - 2} \int_0^\pi (\sin t - \sin^2 t) \, dt = \frac{1}{\pi - 2} \left[-\cos t - \frac{1}{2}t + \frac{1}{4} \sin 2t \right]_0^\pi \\ &= \frac{4 - \pi}{2(\pi - 2)} \end{aligned}$$

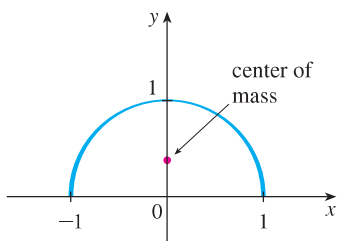


FIGURE 6

By symmetry we see that $\bar{x} = 0$, so the center of mass is

$$\left(0, \frac{4 - \pi}{2(\pi - 2)} \right) \approx (0, 0.38)$$

See Figure 6. ■

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2. They are called the **line integrals of f along C with respect to x and y** :

$$\boxed{5} \quad \int_C f(x, y) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i$$

$$\boxed{6} \quad \int_C f(x, y) \, dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i$$

When we want to distinguish the original line integral $\int_C f(x, y) \, ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t : $x = x(t)$, $y = y(t)$, $dx = x'(t) \, dt$, $dy = y'(t) \, dt$.

$$\boxed{7} \quad \int_C f(x, y) \, dx = \int_a^b f(x(t), y(t)) x'(t) \, dt$$

$$\int_C f(x, y) \, dy = \int_a^b f(x(t), y(t)) y'(t) \, dt$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y) \, dx + \int_C Q(x, y) \, dy = \int_C P(x, y) \, dx + Q(x, y) \, dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given. In particular, we often need to parametrize a line segment, so it's useful to remember that a

vector representation of the line segment that starts at \mathbf{r}_0 and ends at \mathbf{r}_1 is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

(See Equation 12.5.4.)

EXAMPLE 4 Evaluate $\int_C y^2 dx + x dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

SOLUTION

(a) A parametric representation for the line segment is

$$x = 5t - 5 \quad y = 5t - 3 \quad 0 \leq t \leq 1$$

(Use Equation 8 with $\mathbf{r}_0 = \langle -5, -3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.) Then $dx = 5 dt$, $dy = 5 dt$, and Formulas 7 give

$$\begin{aligned} \int_{C_1} y^2 dx + x dy &= \int_0^1 (5t - 3)^2(5 dt) + (5t - 5)(5 dt) \\ &= 5 \int_0^1 (25t^2 - 25t + 4) dt \\ &= 5 \left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t \right]_0^1 = -\frac{5}{6} \end{aligned}$$

(b) Since the parabola is given as a function of y , let's take y as the parameter and write C_2 as

$$x = 4 - y^2 \quad y = y \quad -3 \leq y \leq 2$$

Then $dx = -2y dy$ and by Formulas 7 we have

$$\begin{aligned} \int_{C_2} y^2 dx + x dy &= \int_{-3}^2 y^2(-2y) dy + (4 - y^2) dy \\ &= \int_{-3}^2 (-2y^3 - y^2 + 4) dy \\ &= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y \right]_{-3}^2 = 40\frac{5}{6} \end{aligned}$$

Notice that we got different answers in parts (a) and (b) of Example 4 even though the two curves had the same endpoints. Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path. (But see Section 16.3 for conditions under which the integral is independent of the path.)

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve. If $-C_1$ denotes the line segment from $(0, 2)$ to $(-5, -3)$, you can verify, using the parametrization

$$x = -5t \quad y = 2 - 5t \quad 0 \leq t \leq 1$$

that

$$\int_{-C_1} y^2 dx + x dy = \frac{5}{6}$$

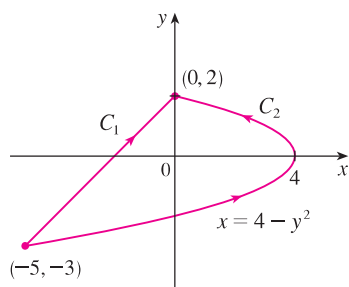


FIGURE 7

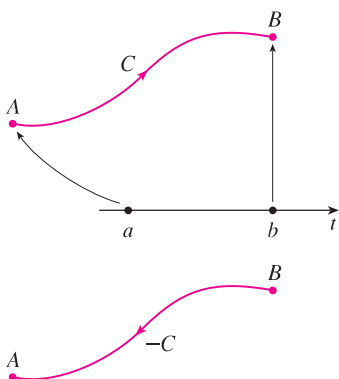


FIGURE 8

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, determines an **orientation** of a curve C , with the positive direction corresponding to increasing values of the parameter t . (See Figure 8, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to $t = b$.)

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y) dx = -\int_C f(x, y) dx \quad \int_{-C} f(x, y) dy = -\int_C f(x, y) dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y) ds = \int_C f(x, y) ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C .

Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If f is a function of three variables that is continuous on some region containing C , then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

We evaluate it using a formula similar to Formula 3:

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)| dt = L$$

where L is the length of the curve C (see Formula 13.3.3).

Line integrals along C with respect to x , y , and z can also be defined. For example,

$$\begin{aligned}\int_C f(x, y, z) dz &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta z_i \\ &= \int_a^b f(x(t), y(t), z(t)) z'(t) dt\end{aligned}$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\boxed{10} \quad \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

by expressing everything (x , y , z , dx , dy , dz) in terms of the parameter t .

EXAMPLE 5 Evaluate $\int_C y \sin z ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$. (See Figure 9.)

SOLUTION Formula 9 gives

$$\begin{aligned}\int_C y \sin z ds &= \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} dt = \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t) dt \\ &= \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin 2t \right]_0^{2\pi} = \sqrt{2} \pi\end{aligned}$$

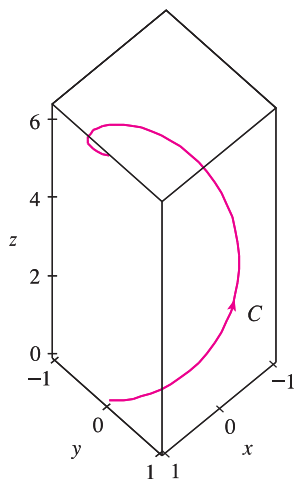


FIGURE 9

EXAMPLE 6 Evaluate $\int_C y dx + z dy + x dz$, where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$, followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.

SOLUTION The curve C is shown in Figure 10. Using Equation 8, we write C_1 as

$$\mathbf{r}(t) = (1 - t) \langle 2, 0, 0 \rangle + t \langle 3, 4, 5 \rangle = \langle 2 + t, 4t, 5t \rangle$$

or, in parametric form, as

$$x = 2 + t \quad y = 4t \quad z = 5t \quad 0 \leq t \leq 1$$

Thus

$$\begin{aligned}\int_{C_1} y dx + z dy + x dz &= \int_0^1 (4t) dt + (5t)4 dt + (2 + t)5 dt \\ &= \int_0^1 (10 + 29t) dt = 10t + 29 \frac{t^2}{2} \Big|_0^1 = 24.5\end{aligned}$$

Likewise, C_2 can be written in the form

$$\mathbf{r}(t) = (1 - t) \langle 3, 4, 5 \rangle + t \langle 3, 4, 0 \rangle = \langle 3, 4, 5 - 5t \rangle$$

$$\text{or} \quad x = 3 \quad y = 4 \quad z = 5 - 5t \quad 0 \leq t \leq 1$$

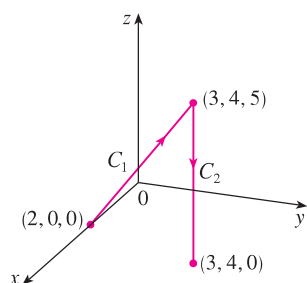


FIGURE 10

Then $dx = 0 = dy$, so

$$\int_{C_2} y \, dx + z \, dy + x \, dz = \int_0^1 3(-5) \, dt = -15$$

Adding the values of these integrals, we obtain

$$\int_C y \, dx + z \, dy + x \, dz = 24.5 - 15 = 9.5 \quad \blacksquare$$

Line Integrals of Vector Fields

Recall from Section 6.4 that the work done by a variable force $f(x)$ in moving a particle from a to b along the x -axis is $W = \int_a^b f(x) \, dx$. Then in Section 12.3 we found that the work done by a constant force \mathbf{F} in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is a continuous force field on \mathbb{R}^3 , such as the gravitational field of Example 16.1.4 or the electric force field of Example 16.1.5. (A force field on \mathbb{R}^2 could be regarded as a special case where $R = 0$ and P and Q depend only on x and y .) We wish to compute the work done by this force in moving a particle along a smooth curve C .

We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 11 for the three-dimensional case.) Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the i th subarc corresponding to the parameter value t_i^* . If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^* . Thus the work done by the force \mathbf{F} in moving the particle from P_{i-1} to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \Delta s_i$$

and the total work done in moving the particle along C is approximately

$$(11) \quad \sum_{i=1}^n [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \Delta s_i$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C . Intuitively, we see that these approximations ought to become better as n becomes larger. Therefore we define the **work** W done by the force field \mathbf{F} as the limit of the Riemann sums in (11), namely,

$$(12) \quad W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force*.

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

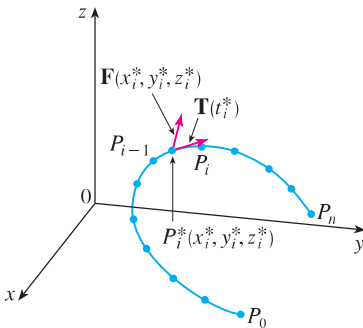


FIGURE 11

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well. Therefore we make the following definition for the line integral of *any* continuous vector field.

13 Definition Let \mathbf{F} be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of \mathbf{F} along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

When using Definition 13, bear in mind that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for the vector field $\mathbf{F}(x(t), y(t), z(t))$, so we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$. Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t) dt$.

Figure 12 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.

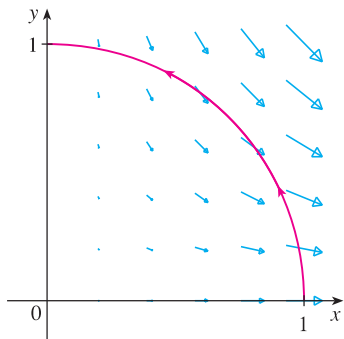


FIGURE 12

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2 \mathbf{i} - xy \mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$, $0 \leq t \leq \pi/2$.

SOLUTION Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t \mathbf{i} - \cos t \sin t \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

Therefore the work done is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{\pi/2} (-2 \cos^2 t \sin t) dt \\ &= 2 \left. \frac{\cos^3 t}{3} \right|_0^{\pi/2} = -\frac{2}{3} \end{aligned}$$

NOTE Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector \mathbf{T} is replaced by its negative when C is replaced by $-C$.

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$ and C is the twisted cubic given by

$$x = t \quad y = t^2 \quad z = t^3 \quad 0 \leq t \leq 1$$

SOLUTION We have

$$\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3 \mathbf{i} + t^5 \mathbf{j} + t^4 \mathbf{k}$$

Figure 13 shows the twisted cubic C in Example 8 and some typical vectors acting at three points on C .

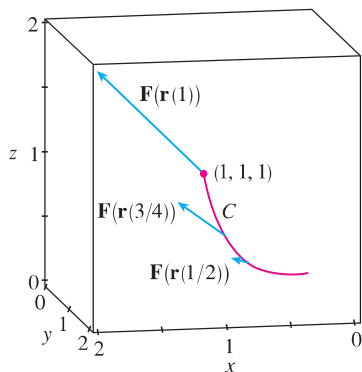


FIGURE 13

$$\begin{aligned}\text{Thus} \quad \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 (t^3 + 5t^6) dt = \left. \frac{t^4}{4} + \frac{5t^7}{7} \right|_0^1 = \frac{27}{28}\end{aligned}$$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field \mathbf{F} on \mathbb{R}^3 is given in component form by the equation $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. We use Definition 13 to compute its line integral along C :

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}) dt \\ &= \int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)] dt\end{aligned}$$

But this last integral is precisely the line integral in (10). Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz \quad \text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

For example, the integral $\int_C y dx + z dy + x dz$ in Example 6 could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

16.2 EXERCISES

1–16 Evaluate the line integral, where C is the given curve.

1. $\int_C y ds$, $C: x = t^2, y = 2t, 0 \leq t \leq 3$

2. $\int_C (x/y) ds$, $C: x = t^3, y = t^4, 1 \leq t \leq 2$

3. $\int_C xy^4 ds$, C is the right half of the circle $x^2 + y^2 = 16$

4. $\int_C xe^y ds$, C is the line segment from $(2, 0)$ to $(5, 4)$

5. $\int_C (x^2y + \sin x) dy$,
 C is the arc of the parabola $y = x^2$ from $(0, 0)$ to (π, π^2)

6. $\int_C e^x dx$,
 C is the arc of the curve $x = y^3$ from $(-1, -1)$ to $(1, 1)$

7. $\int_C (x + 2y) dx + x^2 dy$, C consists of line segments from $(0, 0)$ to $(2, 1)$ and from $(2, 1)$ to $(3, 0)$

8. $\int_C x^2 dx + y^2 dy$, C consists of the arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$ followed by the line segment from $(0, 2)$ to $(4, 3)$

9. $\int_C x^2y ds$,
 $C: x = \cos t, y = \sin t, z = t, 0 \leq t \leq \pi/2$

10. $\int_C y^2z ds$,
 C is the line segment from $(3, 1, 2)$ to $(1, 2, 5)$

11. $\int_C xe^{yz} ds$,
 C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$

12. $\int_C (x^2 + y^2 + z^2) ds$,
 $C: x = t, y = \cos 2t, z = \sin 2t, 0 \leq t \leq 2\pi$

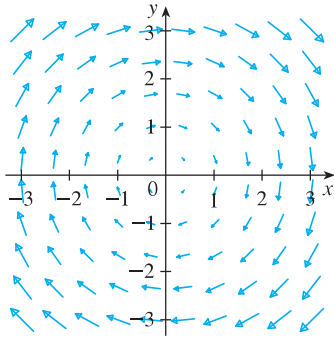
13. $\int_C xye^{yz} dy$, $C: x = t, y = t^2, z = t^3, 0 \leq t \leq 1$

14. $\int_C y dx + z dy + x dz$,
 $C: x = \sqrt{t}, y = t, z = t^2, 1 \leq t \leq 4$

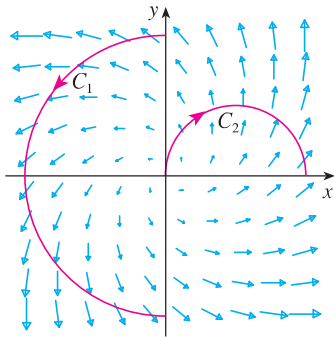
15. $\int_C z^2 dx + x^2 dy + y^2 dz$,
 C is the line segment from $(1, 0, 0)$ to $(4, 1, 2)$

16. $\int_C (y + z) dx + (x + z) dy + (x + y) dz$,
 C consists of line segments from $(0, 0, 0)$ to $(1, 0, 1)$ and from $(1, 0, 1)$ to $(0, 1, 2)$

17. Let \mathbf{F} be the vector field shown in the figure.
- (a) If C_1 is the vertical line segment from $(-3, -3)$ to $(-3, 3)$, determine whether $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.
- (b) If C_2 is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.



18. The figure shows a vector field \mathbf{F} and two curves C_1 and C_2 . Are the line integrals of \mathbf{F} over C_1 and C_2 positive, negative, or zero? Explain.



19–22 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

19. $\mathbf{F}(x, y) = xy^2\mathbf{i} - x^2\mathbf{j}$,
 $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$, $0 \leq t \leq 1$
20. $\mathbf{F}(x, y, z) = (x + y^2)\mathbf{i} + xz\mathbf{j} + (y + z)\mathbf{k}$,
 $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} - 2t\mathbf{k}$, $0 \leq t \leq 2$
21. $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$,
 $\mathbf{r}(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 1$
22. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$,
 $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq \pi$

23–26 Use a calculator to evaluate the line integral correct to four decimal places.

23. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \sqrt{x + y}\mathbf{i} + (y/x)\mathbf{j}$ and
 $\mathbf{r}(t) = \sin^2 t\mathbf{i} + \sin t \cos t\mathbf{j}$, $\pi/6 \leq t \leq \pi/3$


24. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = yze^x\mathbf{i} + zxe^y\mathbf{j} + xye^z\mathbf{k}$ and
 $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \tan t\mathbf{k}$, $0 \leq t \leq \pi/4$
25. $\int_C xy \arctan z \, ds$, where C has parametric equations
 $x = t^2$, $y = t^3$, $z = \sqrt{t}$, $1 \leq t \leq 2$
26. $\int_C z \ln(x + y) \, ds$, where C has parametric equations
 $x = 1 + 3t$, $y = 2 + t^2$, $z = t^4$, $-1 \leq t \leq 1$

CAS 27–28 Use a graph of the vector field \mathbf{F} and the curve C to guess whether the line integral of \mathbf{F} over C is positive, negative, or zero. Then evaluate the line integral.


27. $\mathbf{F}(x, y) = (x - y)\mathbf{i} + xy\mathbf{j}$,
 C is the arc of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(0, -2)$

28. $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$,
 C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

29. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where
 $\mathbf{F}(x, y) = e^{x-1}\mathbf{i} + xy\mathbf{j}$ and C is given by
 $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, $0 \leq t \leq 1$.

 (b) Illustrate part (a) by using a graphing calculator or computer to graph C and the vectors from the vector field corresponding to $t = 0, 1/\sqrt{2}$, and 1 (as in Figure 13).

30. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where
 $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ and C is given by
 $\mathbf{r}(t) = 2t\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k}$, $-1 \leq t \leq 1$.

 (b) Illustrate part (a) by using a computer to graph C and the vectors from the vector field corresponding to $t = \pm 1$ and $\pm \frac{1}{2}$ (as in Figure 13).

CAS 31. Find the exact value of $\int_C x^3 y^2 z \, ds$, where C is the curve with parametric equations $x = e^{-t} \cos 4t$, $y = e^{-t} \sin 4t$, $z = e^{-t}$, $0 \leq t \leq 2\pi$.

32. (a) Find the work done by the force field
 $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.

CAS (b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

33. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \geq 0$. If the linear density is a constant k , find the mass and center of mass of the wire.

34. A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius a . If the density function is $\rho(x, y) = kxy$, find the mass and center of mass of the wire.

35. (a) Write the formulas similar to Equations 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire in the shape of a space curve C if the wire has density function $\rho(x, y, z)$.

(b) Find the center of mass of a wire in the shape of the helix $x = 2 \sin t, y = 2 \cos t, z = 3t, 0 \leq t \leq 2\pi$, if the density is a constant k .

36. Find the mass and center of mass of a wire in the shape of the helix $x = t, y = \cos t, z = \sin t, 0 \leq t \leq 2\pi$, if the density at any point is equal to the square of the distance from the origin.

37. If a wire with linear density $\rho(x, y)$ lies along a plane curve C , its **moments of inertia** about the x - and y -axes are defined as

$$I_x = \int_C y^2 \rho(x, y) \, ds \quad I_y = \int_C x^2 \rho(x, y) \, ds$$

Find the moments of inertia for the wire in Example 3.

38. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C , its **moments of inertia** about the x -, y -, and z -axes are defined as

$$I_x = \int_C (y^2 + z^2) \rho(x, y, z) \, ds$$

$$I_y = \int_C (x^2 + z^2) \rho(x, y, z) \, ds$$

$$I_z = \int_C (x^2 + y^2) \rho(x, y, z) \, ds$$

Find the moments of inertia for the wire in Exercise 35.

39. Find the work done by the force field

$$\mathbf{F}(x, y) = x \mathbf{i} + (y + 2) \mathbf{j}$$

in moving an object along an arch of the cycloid

$$\mathbf{r}(t) = (t - \sin t) \mathbf{i} + (1 - \cos t) \mathbf{j} \quad 0 \leq t \leq 2\pi$$

40. Find the work done by the force field $\mathbf{F}(x, y) = x^2 \mathbf{i} + ye^x \mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.

41. Find the work done by the force field

$$\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.

42. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. (See Example 16.1.5.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

43. The position of an object with mass m at time t is $\mathbf{r}(t) = at^2 \mathbf{i} + bt^3 \mathbf{j}, 0 \leq t \leq 1$.

- What is the force acting on the object at time t ?
- What is the work done by the force during the time interval $0 \leq t \leq 1$?

44. An object with mass m moves with position function $\mathbf{r}(t) = a \sin t \mathbf{i} + b \cos t \mathbf{j} + ct \mathbf{k}, 0 \leq t \leq \pi/2$. Find the work done on the object during this time period.

45. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?

46. Suppose there is a hole in the can of paint in Exercise 45 and 9 lb of paint leaks steadily out of the can during the man's ascent. How much work is done?

47. (a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$.

(b) Is this also true for a force field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$, where k is a constant and $\mathbf{x} = \langle x, y \rangle$?

48. The base of a circular fence with radius 10 m is given by $x = 10 \cos t, y = 10 \sin t$. The height of the fence at position (x, y) is given by the function $h(x, y) = 4 + 0.01(x^2 - y^2)$, so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers 100 m². Sketch the fence and determine how much paint you will need if you paint both sides of the fence.

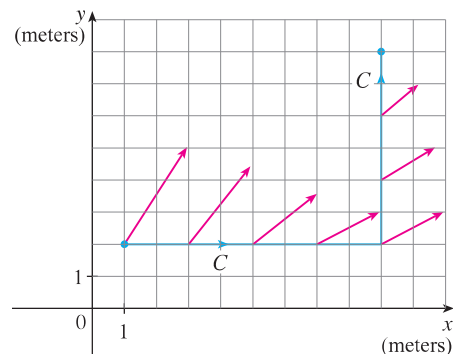
49. If C is a smooth curve given by a vector function $\mathbf{r}(t), a \leq t \leq b$, and \mathbf{v} is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

50. If C is a smooth curve given by a vector function $\mathbf{r}(t), a \leq t \leq b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \frac{1}{2} [|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2]$$

51. An object moves along the curve C shown in the figure from $(1, 2)$ to $(9, 8)$. The lengths of the vectors in the force field \mathbf{F} are measured in newtons by the scales on the axes. Estimate the work done by \mathbf{F} on the object.



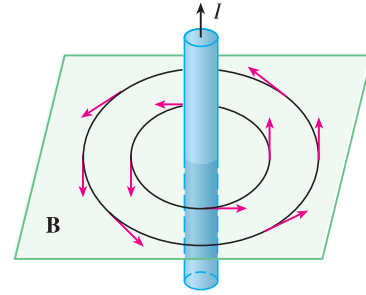
52. Experiments show that a steady current I in a long wire produces a magnetic field \mathbf{B} that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric

current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where I is the net current that passes through any surface bounded by a closed curve C , and μ_0 is a constant called the permeability of free space. By taking C to be a circle with radius r , show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance r from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



16.3 The Fundamental Theorem for Line Integrals

Recall from Section 5.3 that Part 2 of the Fundamental Theorem of Calculus can be written as

$$\boxed{1} \quad \int_a^b F'(x) dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$. We also called Equation 1 the Net Change Theorem: The integral of a rate of change is the net change.

If we think of the gradient vector ∇f of a function f of two or three variables as a sort of derivative of f , then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C . Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

NOTE Theorem 2 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) simply by knowing the value of f at the endpoints of C . In fact, Theorem 2 says that the line integral of ∇f is the net change in f . If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, as in Figure 1(a), then Theorem 2 becomes

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, as in Figure 1(b), then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

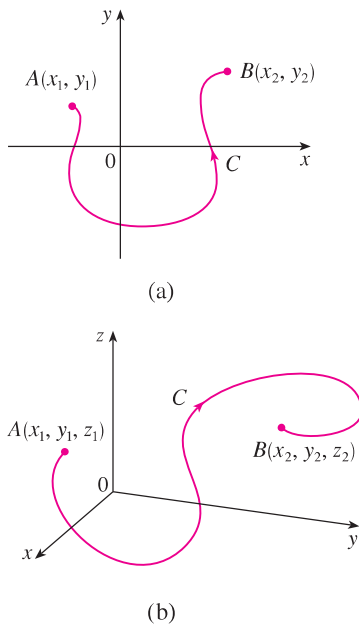


FIGURE 1

Let's prove Theorem 2 for this case.

PROOF OF THEOREM 2 Using Definition 16.2.13, we have

$$\begin{aligned} \int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt \quad (\text{by the Chain Rule}) \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \end{aligned}$$

The last step follows from the Fundamental Theorem of Calculus (Equation 1). ■

Although we have proved Theorem 2 for smooth curves, it is also true for piecewise-smooth curves. This can be seen by subdividing C into a finite number of smooth curves and adding the resulting integrals.

EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3} \mathbf{x}$$

in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C . (See Example 16.1.4.)

SOLUTION From Section 16.1 we know that \mathbf{F} is a conservative vector field and, in fact, $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore, by Theorem 2, the work done is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \\ &= f(2, 2, 0) - f(3, 4, 12) \\ &= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG \left(\frac{1}{2\sqrt{2}} - \frac{1}{13} \right) \quad \blacksquare \end{aligned}$$

■ Independence of Path

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and terminal point B . We know from Example 16.2.4 that, in general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. But one implication of Theorem 2 is that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever ∇f is continuous. In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

In general, if \mathbf{F} is a continuous vector field with domain D , we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points. With this terminology we can say that *line integrals of conservative vector fields are independent of path*.

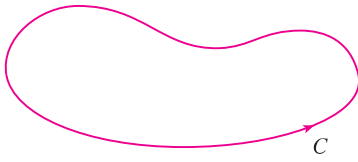


FIGURE 2

A closed curve

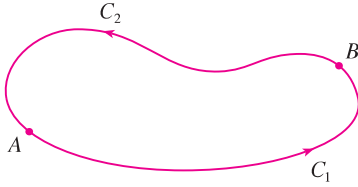


FIGURE 3

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$. (See Figure 2.) If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D , we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A . (See Figure 3.) Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.

Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D , then we demonstrate independence of path as follows. Take any two paths C_1 and C_2 from A to B in D and define C to be the curve consisting of C_1 followed by $-C_2$. Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Thus we have proved the following theorem.

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D .

Since we know that the line integral of any conservative vector field \mathbf{F} is independent of path, it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path. The physical interpretation is that the work done by a conservative force field (such as the gravitational or electric field in Section 16.1) as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves. We assume that D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D . (So D doesn't contain any of its boundary points.) In addition, we assume that D is **connected**: this means that any two points in D can be joined by a path that lies in D .

4 Theorem Suppose \mathbf{F} is a vector field that is continuous on an open connected region D . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D , then \mathbf{F} is a conservative vector field on D ; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

PROOF Let $A(a, b)$ be a fixed point in D . We construct the desired potential function f by defining

$$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

for any point (x, y) in D . Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, it does not matter which path C from (a, b) to (x, y) is used to evaluate $f(x, y)$. Since D is open, there exists a disk contained in D with center (x, y) . Choose any point (x_1, y) in the disk with $x_1 < x$ and let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal line segment C_2 from (x_1, y) to (x, y) . (See Figure 4.) Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a,b)}^{(x_1,y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Notice that the first of these integrals does not depend on x , so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

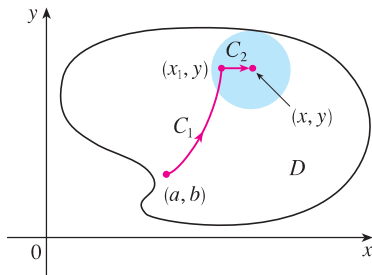


FIGURE 4

If we write $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$, then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P dx + Q dy$$

On C_2 , y is constant, so $dy = 0$. Using t as the parameter, where $x_1 \leq t \leq x$, we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P dx + Q dy = \frac{\partial}{\partial x} \int_{x_1}^x P(t, y) dt = P(x, y)$$

by Part 1 of the Fundamental Theorem of Calculus (see Section 5.3). A similar argument, using a vertical line segment (see Figure 5), shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P dx + Q dy = \frac{\partial}{\partial y} \int_{y_1}^y Q(x, t) dt = Q(x, y)$$

Thus
$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = \nabla f$$

which says that \mathbf{F} is conservative. ■

The question remains: how is it possible to determine whether or not a vector field \mathbf{F} is conservative? Suppose it is known that $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ is conservative, where P and Q have continuous first-order partial derivatives. Then there is a function f such that $\mathbf{F} = \nabla f$, that is,

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

5 Theorem If $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D , then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse of Theorem 5 is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 6; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.]

In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D . Notice from Figure 7 that, intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces.

In terms of simply-connected regions, we can now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on \mathbb{R}^2 is conservative. The proof will be sketched in the next section as a consequence of Green's Theorem.

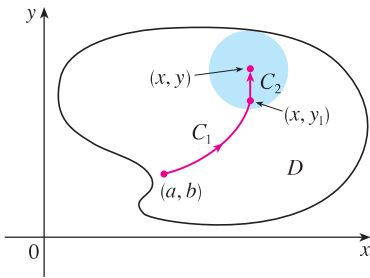


FIGURE 5

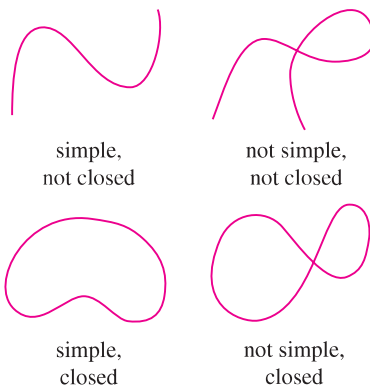


FIGURE 6
Types of curves

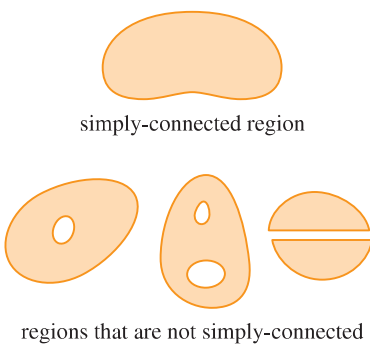


FIGURE 7

6 Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D . Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then \mathbf{F} is conservative.

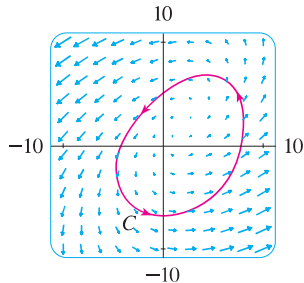


FIGURE 8

Figures 8 and 9 show the vector fields in Examples 2 and 3, respectively. The vectors in Figure 8 that start on the closed curve C all appear to point in roughly the same direction as C . So it looks as if $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$ and therefore \mathbf{F} is not conservative. The calculation in Example 2 confirms this impression. Some of the vectors near the curves C_1 and C_2 in Figure 9 point in approximately the same direction as the curves, whereas others point in the opposite direction. So it appears plausible that line integrals around all closed paths are 0. Example 3 shows that \mathbf{F} is indeed conservative.

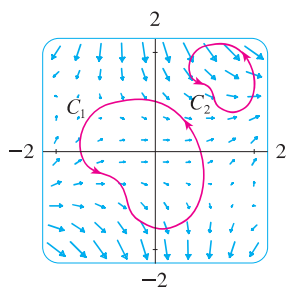


FIGURE 9

EXAMPLE 2 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$$

is conservative.

SOLUTION Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$. Then

$$\frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = 1$$

Since $\partial P/\partial y \neq \partial Q/\partial x$, \mathbf{F} is not conservative by Theorem 5. ■

EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$$

is conservative.

SOLUTION Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

Also, the domain of \mathbf{F} is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected. Therefore we can apply Theorem 6 and conclude that \mathbf{F} is conservative. ■

In Example 3, Theorem 6 told us that \mathbf{F} is conservative, but it did not tell us how to find the (potential) function f such that $\mathbf{F} = \nabla f$. The proof of Theorem 4 gives us a clue as to how to find f . We use “partial integration” as in the following example.

EXAMPLE 4

- (a) If $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.
 (b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} \quad 0 \leq t \leq \pi$$

SOLUTION

- (a) From Example 3 we know that \mathbf{F} is conservative and so there exists a function f with $\nabla f = \mathbf{F}$, that is,

$$\text{7} \quad f_x(x, y) = 3 + 2xy$$

$$\text{8} \quad f_y(x, y) = x^2 - 3y^2$$

Integrating (7) with respect to x , we obtain

$$\boxed{9} \quad f(x, y) = 3x + x^2y + g(y)$$

Notice that the constant of integration is a constant with respect to x , that is, a function of y , which we have called $g(y)$. Next we differentiate both sides of (9) with respect to y :

$$\boxed{10} \quad f_y(x, y) = x^2 + g'(y)$$

Comparing (8) and (10), we see that

$$g'(y) = -3y^2$$

Integrating with respect to y , we have

$$g(y) = -y^3 + K$$

where K is a constant. Putting this in (9), we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

(b) To use Theorem 2 all we have to know are the initial and terminal points of C , namely, $\mathbf{r}(0) = (0, 1)$ and $\mathbf{r}(\pi) = (0, -e^\pi)$. In the expression for $f(x, y)$ in part (a), any value of the constant K will do, so let's choose $K = 0$. Then we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^\pi) - f(0, 1) = e^{3\pi} - (-1) = e^{3\pi} + 1$$

This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 16.2. ■

A criterion for determining whether or not a vector field \mathbf{F} on \mathbb{R}^3 is conservative is given in Section 16.5. Meanwhile, the next example shows that the technique for finding the potential function is much the same as for vector fields on \mathbb{R}^2 .

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2 \mathbf{i} + (2xy + e^{3z}) \mathbf{j} + 3ye^{3z} \mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f , then

$$\boxed{11} \quad f_x(x, y, z) = y^2$$

$$\boxed{12} \quad f_y(x, y, z) = 2xy + e^{3z}$$

$$\boxed{13} \quad f_z(x, y, z) = 3ye^{3z}$$

Integrating (11) with respect to x , we get

$$\boxed{14} \quad f(x, y, z) = xy^2 + g(y, z)$$

where $g(y, z)$ is a constant with respect to x . Then differentiating (14) with respect to y , we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with (12) gives

$$g_y(y, z) = e^{3z}$$

Thus $g(y, z) = ye^{3z} + h(z)$ and we rewrite (14) as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with (13), we obtain $h'(z) = 0$ and therefore $h(z) = K$, a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$. ■

■ Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field \mathbf{F} that moves an object along a path C given by $\mathbf{r}(t)$, $a \leq t \leq b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C . According to Newton's Second Law of Motion (see Section 13.4), the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t) dt \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)] dt && \text{(Theorem 13.2.3, Formula 4)} \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2 dt = \frac{m}{2} [|\mathbf{r}'(t)|^2]_a^b && \text{(Fundamental Theorem of Calculus)} \\ &= \frac{m}{2} (|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2) \end{aligned}$$

Therefore

$$\boxed{15} \quad W = \frac{1}{2}m |\mathbf{v}(b)|^2 - \frac{1}{2}m |\mathbf{v}(a)|^2$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

The quantity $\frac{1}{2}m |\mathbf{v}(t)|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

$$\boxed{16} \quad W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C .

Now let's further assume that \mathbf{F} is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then by Theorem 2 we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

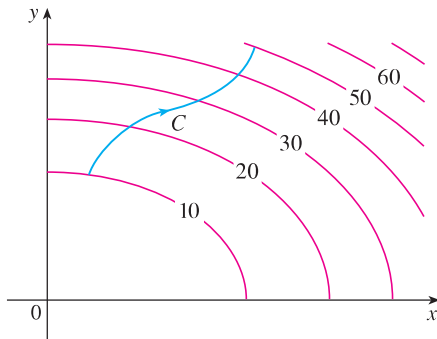
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

16.3 EXERCISES

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations

$$x = t^2 + 1 \quad y = t^3 + t \quad 0 \leq t \leq 1$$

$x \backslash y$	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

- 3–10 Determine whether or not \mathbf{F} is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

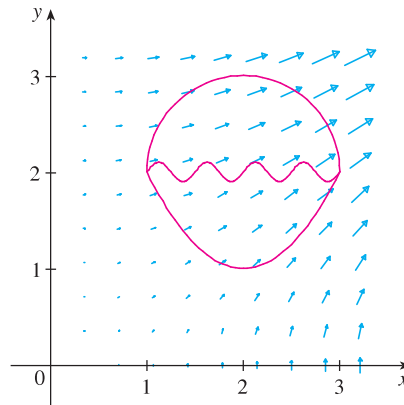
3. $\mathbf{F}(x, y) = (xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$
4. $\mathbf{F}(x, y) = (y^2 - 2x)\mathbf{i} + 2xy\mathbf{j}$
5. $\mathbf{F}(x, y) = y^2e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$
6. $\mathbf{F}(x, y) = ye^x\mathbf{i} + (e^x + e^y)\mathbf{j}$
7. $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x \cos y)\mathbf{j}$
8. $\mathbf{F}(x, y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}, \quad y > 0$

9. $\mathbf{F}(x, y) = (y^2 \cos x + \cos y)\mathbf{i} + (2y \sin x - x \sin y)\mathbf{j}$

10. $\mathbf{F}(x, y) = (\ln y + y/x)\mathbf{i} + (\ln x + x/y)\mathbf{j}$

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at $(1, 2)$ and end at $(3, 2)$.

- (a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.
- (b) What is this common value?



- 12–18 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C .

12. $\mathbf{F}(x, y) = (3 + 2xy^2)\mathbf{i} + 2x^2y\mathbf{j}$,
 C is the arc of the hyperbola $y = 1/x$ from $(1, 1)$ to $(4, \frac{1}{4})$

13. $\mathbf{F}(x, y) = x^2y^3\mathbf{i} + x^3y^2\mathbf{j}$,
 $C: \mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t \rangle, \quad 0 \leq t \leq 1$

14. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$,
 $C: \mathbf{r}(t) = \cos t\mathbf{i} + 2 \sin t\mathbf{j}, \quad 0 \leq t \leq \pi/2$

15. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$,
 C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

16. $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$,
 $C: x = \sqrt{t}, y = t + 1, z = t^2, 0 \leq t \leq 1$

17. $\mathbf{F}(x, y, z) = yze^{xz}\mathbf{i} + e^{xz}\mathbf{j} + xye^{xz}\mathbf{k}$,
 $C: \mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + (t^2 - 2t)\mathbf{k}$,
 $0 \leq t \leq 2$

18. $\mathbf{F}(x, y, z) = \sin y\mathbf{i} + (x \cos y + \cos z)\mathbf{j} - y \sin z\mathbf{k}$,
 $C: \mathbf{r}(t) = \sin t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq \pi/2$

19–20 Show that the line integral is independent of path and evaluate the integral.

19. $\int_C 2xe^{-y} dx + (2y - x^2e^{-y}) dy$,
 C is any path from $(1, 0)$ to $(2, 1)$

20. $\int_C \sin y dx + (x \cos y - \sin y) dy$,
 C is any path from $(2, 0)$ to $(1, \pi)$

21. Suppose you're asked to determine the curve that requires the least work for a force field \mathbf{F} to move a particle from one point to another point. You decide to check first whether \mathbf{F} is conservative, and indeed it turns out that it is. How would you reply to the request?

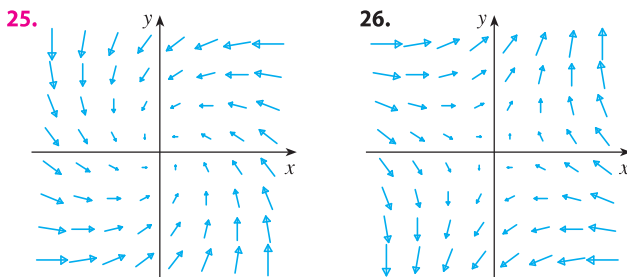
22. Suppose an experiment determines that the amount of work required for a force field \mathbf{F} to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by \mathbf{F} in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about \mathbf{F} ? Why?

23–24 Find the work done by the force field \mathbf{F} in moving an object from P to Q .

23. $\mathbf{F}(x, y) = x^3\mathbf{i} + y^3\mathbf{j}; P(1, 0), Q(2, 2)$

24. $\mathbf{F}(x, y) = (2x + y)\mathbf{i} + x\mathbf{j}; P(1, 1), Q(4, 3)$

25–26 Is the vector field shown in the figure conservative? Explain.



CAS 27. If $\mathbf{F}(x, y) = \sin y\mathbf{i} + (1 + x \cos y)\mathbf{j}$, use a plot to guess whether \mathbf{F} is conservative. Then determine whether your guess is correct.

28. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ (b) $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

29. Show that if the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

30. Use Exercise 29 to show that the line integral $\int_C y dx + x dy + xyz dz$ is not independent of path.

31–34 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

31. $\{(x, y) \mid 0 < y < 3\}$

32. $\{(x, y) \mid 1 < |x| < 2\}$

33. $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$

34. $\{(x, y) \mid (x, y) \neq (2, 3)\}$

35. Let $\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [Hint: Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

36. (a) Suppose that \mathbf{F} is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c , where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Find the work done by \mathbf{F} in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 16.1.4. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}$ N·m²/kg².)

(c) Another example of an inverse square field is the electric force field $\mathbf{F} = \epsilon qQ\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 16.1.5. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value $\epsilon = 8.985 \times 10^9$.)

16.4 Green's Theorem

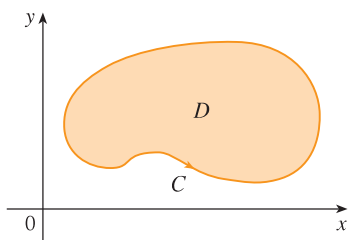


FIGURE 1

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C . (See Figure 1. We assume that D consists of all points inside C as well as all points on C .) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C . Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C . (See Figure 2.)

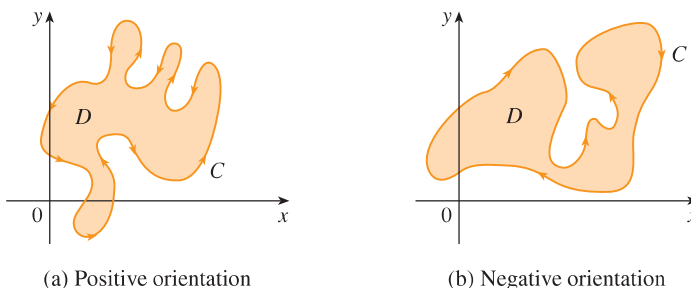


FIGURE 2

Recall that the left side of this equation is another way of writing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

NOTE The notation

$$\oint_C P dx + Q dy \quad \text{or} \quad \oint_C P dx + Q dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C . Another notation for the positively oriented boundary curve of D is ∂D , so the equation in Green's Theorem can be written as

$$\boxed{1} \quad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P dx + Q dy$$

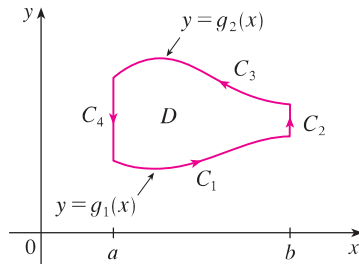
Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x) dx = F(b) - F(a)$$

In both cases there is an integral involving derivatives (F' , $\partial Q/\partial x$, and $\partial P/\partial y$) on the left side of the equation. And in both cases the right side involves the values of the original functions (F , Q , and P) only on the *boundary* of the domain. (In the one-dimensional case, the domain is an interval $[a, b]$ whose boundary consists of just two points, a and b .)

George Green

Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

**FIGURE 3**

Green's Theorem is not easy to prove in general, but we can give a proof for the special case where the region is both type I and type II (see Section 15.2). Let's call such regions **simple regions**.

PROOF OF GREEN'S THEOREM FOR THE CASE IN WHICH D IS A SIMPLE REGION Notice that Green's Theorem will be proved if we can show that

$$\boxed{2} \quad \int_C P \, dx = - \iint_D \frac{\partial P}{\partial y} \, dA$$

and

$$\boxed{3} \quad \int_C Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA$$

We prove Equation 2 by expressing D as a type I region:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

$$\boxed{4} \quad \iint_D \frac{\partial P}{\partial y} \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y) \, dy \, dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))] \, dx$$

where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Equation 2 by breaking up C as the union of the four curves C_1 , C_2 , C_3 , and C_4 shown in Figure 3. On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \leq x \leq b$. Thus

$$\int_{C_1} P(x, y) \, dx = \int_a^b P(x, g_1(x)) \, dx$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \leq x \leq b$. Therefore

$$\int_{C_3} P(x, y) \, dx = - \int_{-C_3} P(x, y) \, dx = - \int_a^b P(x, g_2(x)) \, dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y) \, dx = 0 = \int_{C_4} P(x, y) \, dx$$

Hence

$$\begin{aligned} \int_C P(x, y) \, dx &= \int_{C_1} P(x, y) \, dx + \int_{C_2} P(x, y) \, dx + \int_{C_3} P(x, y) \, dx + \int_{C_4} P(x, y) \, dx \\ &= \int_a^b P(x, g_1(x)) \, dx - \int_a^b P(x, g_2(x)) \, dx \end{aligned}$$

Comparing this expression with the one in Equation 4, we see that

$$\int_C P(x, y) \, dx = - \iint_D \frac{\partial P}{\partial y} \, dA$$

Equation 3 can be proved in much the same way by expressing D as a type II region (see Exercise 30). Then, by adding Equations 2 and 3, we obtain Green's Theorem. ■

EXAMPLE 1 Evaluate $\int_C x^4 \, dx + xy \, dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

SOLUTION Although the given line integral could be evaluated as usual by the methods of Section 16.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4). If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

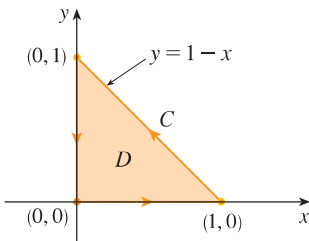


FIGURE 4

$$\begin{aligned} \int_C x^4 \, dx + xy \, dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_0^1 \int_0^{1-x} (y - 0) \, dy \, dx \\ &= \int_0^1 \left[\frac{1}{2} y^2 \right]_{y=0}^{y=1-x} \, dx = \frac{1}{2} \int_0^1 (1-x)^2 \, dx \\ &= -\frac{1}{6} (1-x)^3 \Big|_0^1 = \frac{1}{6} \end{aligned}$$

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy$, where C is the circle $x^2 + y^2 = 9$.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \leq 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\begin{aligned} \oint_C (3y - e^{\sin x}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy &= \iint_D \left[\frac{\partial}{\partial x} (7x + \sqrt{y^4 + 1}) - \frac{\partial}{\partial y} (3y - e^{\sin x}) \right] \, dA \\ &= \int_0^{2\pi} \int_0^3 (7 - 3) r \, dr \, d\theta = 4 \int_0^{2\pi} d\theta \int_0^3 r \, dr = 36\pi \end{aligned}$$

Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4 \, dA = 4 \cdot \pi(3)^2 = 36\pi$$

In Examples 1 and 2 we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C , then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_C P \, dx + Q \, dy = 0$$

no matter what values P and Q assume in the region D .

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1 \, dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$